Asymptotic completeness, global existence and the infrared problem for the Maxwell-Dirac equations

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Abstract: In this monograph we prove that the nonlinear Lie algebra representation given by the manifestly covariant Maxwell-Dirac (M-D) equations is integrable to a global nonlinear representation U of the Poincaré group $\mathcal{P}_0$ on a differentiable manifold $\mathcal{U}_\infty$ of small initial conditions for the M-D equations. This solves, in particular, the Cauchy problem for the M-D equations, namely existence of global solutions for initial data in $\mathcal{U}_\infty$ at $t = 0$. The existence of modified wave operators Ω_+ and Ω_- and asymptotic completeness is proved. The asymptotic representations $U_g^{(\varepsilon)} = \Omega_\varepsilon^{-1} \circ U_g \circ \Omega_\varepsilon$, $\varepsilon = \pm$, $g \in \mathcal{P}_0$, turn out to be nonlinear. A cohomological interpretation of the results in the spirit of nonlinear representation theory and its connection to the infrared tail of the electron is given.

*THIS MONOGRAPH IS DEDICATED TO THE MEMORY
OF THE CHIEF CREATOR OF QUANTUM ELECTRODYNAMICS,
A GIANT IN CONTEMPORARY PHYSICS,
A GREAT HUMAN BEING AND UNFORGETTABLE FRIEND – JULIAN SCHWINGER*

1. Introduction

It is well-known that the construction of the observables on the Fock space of QED (Quantum Electrodynamics) requires infrared corrections to eliminate the infrared divergencies in the perturbative expression of the quantum scattering operator. These corrections are introduced by hand with the purpose to give a posteriori a finite theory. In this monograph we shall prove rigorous results which we have obtained concerning the infrared problem for the (classical) Maxwell-Dirac equations. Our belief is that such results can a priori be of interest for QED, especially for the infrared regime and combined with the deformation-quantization approach [1], [2], [4]. Our results show in particular that, also in the classical case one obtains infrared divergencies, if one requires free asymptotic fields as it is needed in QED. But before continuing the physical motivation of the paper, we shall describe the mathematical context.

1.1 The Mathematical Framework

1.1.a *The Equations.* We shall use conventional notations: electron charge $e = 1$; Dirac matrices γ^μ, $0 \leq \mu \leq 3$; metric tensor $g^{\mu\nu}$, $g^{00} = 1$, $g^{ii} = -1$ for $1 \leq i \leq 3$ and $g^{\mu\nu} = 0$ for $\mu \neq \nu$; $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$; $\partial_\mu = \partial/(\partial y^\mu)$; $\square = \partial_\mu\partial^\mu$ (with the Einstein summation convention and the index raising convention). Then the classical Maxwell-Dirac (M-D) equations read:

$$\square A_\mu = \overline{\psi}\gamma_\mu\psi, \quad 0 \leq \mu \leq 3, \tag{1.1a}$$

$$(i\gamma^\mu\partial_\mu + m)\psi = A_\mu\gamma^\mu\psi, \quad m > 0, \tag{1.1b}$$

$$\partial_\mu A^\mu = 0, \tag{1.1c}$$

where $\overline{\psi} = \psi^+\gamma_0$, ψ^+ being the Hermitian conjugate of the Dirac spinor ψ. We write equations (1.1a) and (1.1b) as an evolution equation:

$$\frac{d}{dt}(A_\mu(t), \dot{A}_\mu(t)) = (\dot{A}_\mu(t), \Delta A_\mu(t)) + (0, \overline{\psi}(t)\gamma_\mu\psi(t)), \tag{1.2a}$$

$$\frac{d}{dt}\psi(t) = \mathcal{D}\psi(t) - iA_\mu(t)\gamma^0\gamma^\mu\psi(t), \tag{1.2b}$$

where $\mathcal{D} = -\sum_{j=1}^3 \gamma^0\gamma^j\partial_j + i\gamma^0 m$, $\Delta = \sum_{j=1}^3 \partial_j^2$, $t \in \mathbb{R}$ and where $A_\mu(t)$, $\dot{A}_\mu(t)\colon \mathbb{R}^3 \to \mathbb{R}$, $\psi(t)\colon \mathbb{R}^3 \to \mathbb{C}^4$. The Lorentz gauge condition (1.1c) takes on initial conditions $A_\mu(t_0)$, $\dot{A}_\mu(t_0)$, $0 \leq \mu \leq 3$, and $\psi(t_0)$ at $t = t_0$ the form (cf. [10], [8])

$$\dot{A}^0(t_0) + \sum_{i=1}^3 \partial_i A^i(t_0) = 0, \tag{1.3a}$$

$$\Delta A^0(t_0) + |\psi(t_0)|^2 + \sum_{i=1}^3 \partial_i \dot{A}^i(t_0) = 0, \tag{1.3b}$$

where $A^\mu = g^{\mu\nu}A_\nu$.

1.1.b *Poincaré covariance: linear part and representation spaces.* Since equations (1.1a), (1.1b) and (1.1c) are manifestly covariant under the action of the universal covering group $\mathcal{P}_0 = \mathbb{R}^4 \otimes SL(2,\mathbb{C})$ of the Poincaré group, it is easy to complete the time translation generator, formally defined by (1.2a) and (1.2b), to a nonlinear representation of the whole Lie algebra $\mathfrak{p} = \mathbb{R}^4 \oplus \mathfrak{sl}(2,\mathbb{C})$ of $\mathcal{P}_0$. To do this we first introduce topological vector spaces on which the representations will be defined. Let M^ρ, $-1/2 < \rho < \infty$ be the completion of $S(\mathbb{R}^3,\mathbb{R}^4) \oplus S(\mathbb{R}^3,\mathbb{R}^4)$ with respect to the norm

$$\|(f,\dot f)\|_{M^\rho} = \big(\||\nabla|^\rho f\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^4)} + \||\nabla|^{\rho-1}\dot f\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^4)}\big)^{1/2}, \tag{1.4a}$$

where $S(\mathbb{R}^3,\mathbb{R}^4)$ is the Schwartz space of test functions from $\mathbb{R}^3$ to $\mathbb{R}^4$ and where $|\nabla| = (-\Delta)^{1/2}$. Let $D = L^2(\mathbb{R}^3,\mathbb{C}^4)$ and let $E^\rho = M^\rho \oplus D$, $-1/2 < \rho < \infty$, be the Hilbert space with norm

$$\|(f,\dot f,\alpha)\|_{E^\rho} = \big(\|(f,\dot f)\|^2_{M^\rho} + \|\alpha\|^2_D\big)^{1/2}. \tag{1.4b}$$

When there is no possibility of confusion we write E (resp. M) instead of E^ρ (resp. M^ρ) for $1/2 < \rho < 1$. $M^{\circ\rho}$ is the closed subspace of elements $(f,\dot f) \in M^\rho$, $-1/2 < \rho < \infty$ such that

$$\dot f_0 = \sum_{1\le i\le 3} \partial_i f_i, \tag{1.4$_r mc$}$$

$$f_0 = -\sum_{1\le i\le 3} |\nabla|^{-2}\partial_i \dot f_i,$$

where $f = (f_0,f_1,f_2,f_3)$ and $\dot f = (\dot f_0,\dot f_1,\dot f_2,\dot f_3)$. A solution B_μ of $\square B_\mu = 0$, $0 \le \mu \le 3$, with initial conditions $(f,\dot f) \in M^\rho$ satisfies the gauge condition $\partial_\mu B^\mu = 0$ if and only if $(f,\dot f) \in M^{\circ\rho}$. We define $E^{\circ\rho} = M^{\circ\rho} \oplus D$.

Let $\Pi = \{P_\mu, M_{\alpha\beta} \big| 0 \le \mu \le 3, 0 \le \alpha < \beta \le 3\}$ be a standard basis of the Poincaré Lie algebra $\mathfrak{p} = \mathbb{R}^4 \oplus \mathfrak{sl}(2,\mathbb{C})$, where P_0 is the time translation generator, P_i, $1 \le i \le 3$ the space translation generators, M_{ij}, $1 \le i < j \le 3$, are the space rotation generators and M_{0j}, $1 \le j \le 3$, are the boost generators. We define $M_{\alpha\beta} = -M_{\beta\alpha}$ for $0 \le \beta \le \alpha \le 3$. There is a linear (strongly) continuous representation U^1 of $\mathcal{P}_0$ in E^ρ, $-1/2 < \rho < \infty$, (see Lemma 2.1) with space of differentiable vectors E^ρ_∞, the differential of which is the following linear representation T^1 of $\mathfrak{p}$ in E^ρ_∞:

$$T^1_{P_0}(f,\dot f,\alpha) = (\dot f, \Delta f, \mathcal{D}\alpha), \tag{1.5a}$$

$$T^1_{P_i}(f,\dot f,\alpha) = \partial_i(f,\dot f,\alpha), \quad 1 \le i \le 3, \tag{1.5b}$$

$$(T^1_{M_{ij}}(f,\dot f,\alpha))(x) = -(x_i\partial_j - x_j\partial_i)(f,\dot f,\alpha)(x) + (n_{ij}f, n_{ij}\dot f, \sigma_{ij}\alpha)(x) \tag{1.5c}$$
$$1 \le i < j \le 3, \sigma_{ij} = 1/2\gamma_i\gamma_j \in \mathfrak{su}(2), n_{ij} \in \mathfrak{so}(3),$$

$$(T^1_{M_{0i}}(f,\dot f,\alpha))(x) = \big(x_i \dot f(x), \sum_{j=0}^{3} \partial_j\big(x_i\partial_j f(x)\big), x_i\mathcal{D}\alpha(x)\big) + (n_{0i}f, n_{0i}\dot f, \sigma_{0i}\alpha)(x), \tag{1.5d}$$
$$1 \le i \le 3, \sigma_{0i} = 1/2\gamma_0\gamma_i, n_{0i} \in \mathfrak{so}(3,1),$$

where $x = (x_1, x_2, x_3)$ are related to contravariant coordinates y^μ by $x_i = y^i$ (we use this notation to avoid writing components of vectors in $\mathbb{R}^3$ with uppper indices that could be confused with powers). The explicit form of $n_{\alpha\beta}$, $0 \leq \alpha < \beta \leq 3$, defining a vector representation of $\mathfrak{p}$ in $\mathbb{R}^4$, is not important here. We remark that U^1 leaves $E^{\circ\rho}$ invariant.

Following closely [20], we introduce the graded sequence of Hilbert spaces E^ρ_i, $i \geq 0$, $E^\rho_0 = E^\rho$, where $E^\rho_\infty \subset E^\rho_j \subset E^\rho_i$ for $i \leq j$. E^ρ_i is the space of C^i-vectors of the representation U^1. Suppose given an ordering $X_1 < X_2 < \cdots < X_{10}$ on Π. Then in the universal enveloping algebra $U(\mathfrak{p})$ of $\mathfrak{p}$, the subset of all products $X_1^{\alpha_1} \cdots X_{10}^{\alpha_{10}}$, $0 \leq \alpha_i$, $1 \leq i \leq 10$, is a basis Π' of $U(\mathfrak{p})$. If $Y = X_1^{\alpha_1} \cdots X_{10}^{\alpha_{10}}$, then we define $|Y| = |\alpha| = \sum_{0 \leq i \leq 10} \alpha_i$. Let E^ρ_i, $i \in \mathbb{N}$ be the completion of E^ρ_∞ with respect to the norm

$$\|u\|_{E^\rho_i} = \Big(\sum_{\substack{Y \in \Pi' \\ |Y| \leq i}} \|T^1_Y(u)\|^2_{E^\rho} \Big)^{1/2}, \tag{1.6a}$$

where T^1_Y, $Y \in U(\mathfrak{p})$ is defined by the canonical extension of T^1 to the enveloping algebra $U(\mathfrak{p})$ of $\mathfrak{p}$. Let $M^{\circ\rho}_i = M^\rho_i \cap M^{\circ\rho}$, $E^{\circ\rho}_i = E^\rho_i \cap E^{\circ\rho}$, $M^{\circ\rho}_\infty = M^\rho_\infty \cap M^{\circ\rho}$ and $E^{\circ\rho}_\infty = E^\rho_\infty \cap E^{\circ\rho}$, where M^ρ_i (resp. M^ρ_∞) is the image of E^ρ_i (resp. E^ρ_∞) in M^ρ under the canonical projection of E^ρ on M^ρ. Let D_i (resp. D_∞) be the image of E^ρ_i (resp. E^ρ_∞) in D under the canonical projection of E^ρ on D. To understand better what the elements of the spaces E^ρ_i are, we introduce the seminorms q_n, $n \geq 0$, on E^ρ_∞, where

$$\begin{aligned}
q_n(u) &= (q^M_n(v)^2 + q^D_n(\alpha)^2)^{1/2}, \quad u = (v, \alpha) \in E^\rho_\infty, v \in M^\rho_\infty, \alpha \in D_\infty, \\
q^M_n(v) &= \Big(\sum_{|\mu| \leq |\nu| \leq n} \|M_\mu \partial^\nu v\|^2_{M^\rho} \Big)^{1/2}, \\
q^D_n(\alpha) &= \Big(\sum_{\substack{|\mu| \leq n \\ |\nu| \leq n}} \|M_\mu \partial^\nu \alpha\|^2_D \Big)^{1/2},
\end{aligned}$$

where $\mu = (\mu_1, \mu_2, \mu_3)$, $\nu = (\nu_1, \nu_2, \nu_3)$ are multi-indices, $\partial^\mu = (\partial_1)^{\mu_1} (\partial_2)^{\mu_2} (\partial_3)^{\mu_3}$ and $M_\mu(x) = x_1^{\mu_1} x_2^{\mu_2} x_3^{\mu_3}$. The norms $\| \cdot \|_{E^\rho_n}$ and q_n are equivalent (see Theorem 2.9). This shows in particular that $D_\infty = S(\mathbb{R}^3, \mathbb{C}^4)$. Moreover, if $1/2 < \rho < 1$ and $(f, \dot{f}) \in M^\rho_\infty$, then (see Theorem 2.12 and Theorem 2.13)

$$(1 + |x|)^{3/2 - \rho} |f(x)| \leq C \|(f, 0)\|_{M^\rho_1}, \tag{1.6b}$$

$$(1 + |x|)^{5/2 - \rho + |\nu|} (|\partial^\nu \partial_i f(x)| + |\partial^\nu \dot{f}(x)|) \leq C_{|\nu|} \|(f, \dot{f})\|_{M^\rho_{|\nu|+2}}, \tag{1.6c}$$

for $|\nu| \geq 0$, $1 \leq i \leq 3$ and if $f \in L^q$, $q = 6/(3 - 2\rho)$, $x^\alpha \partial^\beta \partial_i f \in L^p$ and $x^\alpha \partial^\beta \dot{f} \in L^p$, $p = 6/(5 - 2\rho)$, $|\alpha| \leq |\beta| \leq n$, $1 \leq i \leq 3$, then

$$\|(f, \dot{f})\|_{M^\rho_n} \leq C_n \sum_{0 \leq |\alpha| \leq |\beta| \leq n} \Big(\sum_{0 \leq i \leq 3} \|x^\alpha \partial^\beta \partial_i f\|_{L^p} + \|x^\alpha \partial^\beta \dot{f}\|_{L^p} \Big). \tag{1.6d}$$

In particular, it follows that, if

$$|\partial^\nu f(x)| \le C_{|\nu|}(1+|x|)^{\rho-3/2-|\nu|-\varepsilon}, \qquad \text{(1.6e)}$$
$$|\partial^\nu \dot f(x)| \le C_{|\nu|}(1+|x|)^{\rho-5/2-|\nu|-\varepsilon},$$

for each $|\nu| \ge 0$ and some $\varepsilon > 0$, then $(f,\dot f) \in M^\rho_\infty$, $1/2 < \rho < 1$. Thus M^ρ_∞ contains long-range potentials.

1.1.c *Nonlinear Poincaré covariance.* A nonlinear representation (see Lemma 2.17) T of $\mathfrak{p}$ in E^ρ_∞, in the sense of [5], is obtained by the fact that the M-D equations are manifestly covariant:

$$T_X = T^1_X + T^2_X, \quad X \in \mathfrak{p}, \qquad \text{(1.7)}$$

where T^1 is given by (1.5a)–(1.5d) and, for $u = (f,\dot f,\alpha) \in E^\rho_\infty$,

$$T^2_{P_0}(u) = (0, \overline{\alpha}\gamma\alpha, -if_\mu\gamma^0\gamma^\mu\alpha), \quad \gamma = (\gamma_0,\gamma_1,\gamma_2,\gamma_3), \qquad \text{(1.8a)}$$
$$T^2_{P_i} = 0 \quad \text{for } 1 \le i \le 3, \quad T^2_{M_{ij}} = 0 \quad \text{for } 1 \le i < j \le 3, \qquad \text{(1.8b)}$$
$$(T^2_{M_{0j}}(u))(x) = x_j(T^2_{P_0}(u))(x), \quad 1 \le j \le 3, x = (x_1,x_2,x_3). \qquad \text{(1.8c)}$$

The gauge condition (1.1c) takes on initial data $u = (f,\dot f,\alpha)$ the form

$$\dot f_0 = \sum_{1\le i\le 3} \partial_i f_i, \qquad \Delta f_0 = \sum_{1\le i\le 3} \partial_i \dot f_i - |\alpha|^2. \qquad \text{(1.1c}'\text{)}$$

The subspace (topological) V^ρ_∞, $1/2 < \rho < 1$, of elements in E^ρ_∞ which satisfy these gauge conditions, is diffeomorphic to $E^{\circ\rho}_\infty$ (see Theorem 6.11). The problem to integrate globally the nonlinear Lie algebra representation T therefore consists of proving the existence of an open neighbourhood $\mathcal{U}_\infty$ of zero in V^ρ_∞ and a group action $U\colon \mathcal{P}_0 \times \mathcal{U}_\infty \to \mathcal{U}_\infty$, which is C^∞ and such that $U_g(0) = 0$ for $g \in \mathcal{P}_0$ and $\frac{d}{dt}\, U_{\exp(tX)}(u)\,|_{t=0} = T_X\, u, X \in \mathfrak{p}$.

Continuing to follow [20], we extend the linear map $X \mapsto T_X$, from $\mathfrak{p}$ to the vector space of all differentiable maps from E^ρ_∞ to E^ρ_∞, to the enveloping algebra $U(\mathfrak{p})$ by defining inductively: $T_{\mathbb{I}} = I$, where $\mathbb{I}$ is the identity element in $U(\mathfrak{p})$, and

$$T_{YX} = DT_Y.T_X, \quad Y \in U(\mathfrak{p}), X \in \mathfrak{p}. \qquad \text{(1.9)}$$

Here $(DA.B)(f)$ is the Fréchet derivative of A at the point f in the direction $B(f)$.[1)] It was proved in [20] that this is a linear map from $U(\mathfrak{p})$ to the vector space of differentiable maps from E^ρ_∞ to E^ρ_∞ (see formula (1.10) of [20] and the sequel). For completeness we recall the proof in [20]. The vector field T_X, $X \in \mathfrak{p}$, defines a linear differential operator d_X of degree at most one operating on the space $C^\infty(E^\rho_\infty)$ by $d_X F = DF.T_X$, $F \in C^\infty(E_\infty)$. The fact that $X \mapsto T_X$ is a nonlinear representation of $\mathfrak{p}$ on E^ρ_∞ implies that $X \mapsto d_X$ is a

[1)] We also use the notation $(D^nA)(f; f_1,\dots,f_n)$ for the n-th derivative of A at f in the directions $f_1,\dots,f_n$ and the notation $D^nA.(f_1,\dots,f_n)$ for the function $f \mapsto (D^nA)(f; f_1,\dots,f_n)$.

linear representation of $\mathfrak{p}$ on linear differential operators of degree at most one. This linear continuous representation has a canonical extension $Y \mapsto d_Y$ to $U(\mathfrak{p})$ on linear differential operators of arbitrary order operating on $C^\infty(E^\rho_\infty)$. If e_Y, $Y \in U(\mathfrak{p})$, is the part of d_Y of degree not higher than one, then $Y \mapsto e_Y$ is a linear map of $U(\mathfrak{p})$ into the space of linear partial differential operators of degree at most one. Let $Y \in U(\mathfrak{p})$. We write $Y = Z + a$, where $a \in \mathbb{C} \cdot \mathbb{I}$ and Z has no component on $\mathbb{C} \cdot \mathbb{I}$ (relative to the natural graduation of $U(\mathfrak{p})$). Then the previous definition of T_Y gives $e_Y F = DF.T_Z + aF$, which proves that $Y \mapsto T_Y$ is a linear map on $U(\mathfrak{p})$. Moreover, it was proved in [20] that

$$\frac{d}{dt} T_{Y(t)}(u(t)) = T_{P_0 Y(t)}(u(t)), \quad Y \in U(\mathfrak{p},) \tag{1.10}$$

where

$$Y(t) = \exp(t \operatorname{ad}_{P_0}) Y, \quad \operatorname{ad}_{P_0} Z = [P_0, Z], \tag{1.11}$$

if

$$\frac{d}{dt} u(t) = T_{P_0}(u(t)). \tag{1.12}$$

We note that definition (1.11) makes sense since $(\operatorname{ad}_{P_0})^n$, $n \geq 0$, is a linear map from $U(\mathfrak{p})$ to $U(\mathfrak{p})$ leaving invariant the subspace of elements of degree at most l, $l \geq 0$, in $U(\mathfrak{p})$ and since then $\exp(t \operatorname{ad}_{P_0}) Y = \sum_{n \geq 0} (n!)^{-1} (t \operatorname{ad}_{P_0})^n Y$ converges absolutely. For completeness we also recall the proof of (1.10). Since $\frac{d}{dt} Y(t) = \operatorname{ad}_{P_0} Y(t)$, it follows from (1.9) and (1.12), that

$$\begin{aligned}
\frac{d}{dt} T_{Y(t)}(u(t)) &= T_{\frac{d}{dt} Y(t)}(u(t)) + (DT_{Y(t)}.T_{P_0})(u(t)) \\
&= T_{[P_0, Y(t)]}(u(t)) + T_{Y(t) P_0}(u(t)) = T_{P_0 Y(t)}.
\end{aligned}$$

T^n_Y, $Y \in U(\mathfrak{p})$, denotes the n-homogeneous part of T_Y and we shall identify T^n_Y with a n-linear symmetric map and also with a continuous linear map from $\hat{\otimes}^n E^\rho_\infty$ into E^ρ_∞, where $\hat{\otimes}^n$ is the n-fold complete tensor product endowed with the projective tensor product topology. $T^M_Y(u)$, $T^{Mn}_Y(u)$ and $U^M_g(u)$ (resp. $T^D_Y(u)$, $T^{Dn}_Y(u)$ and $U^D_g(u)$) is the projection of $T_Y(u)$, $T^n_Y(u)$ and $U_g(u)$ on M^ρ (resp. D). We also define $\tilde{T}_Y$, $Y \in U(\mathfrak{p})$ by

$$T_Y = T^1_Y + \tilde{T}_Y. \tag{1.13}$$

Since it does not bring any contradiction, we shall also denote by T^{M1} or T^{1M} (resp. T^{D1} or T^{1D}) the linear representation of $\mathfrak{p}$ which is the restriction of the linear representation T^1 to M^ρ (resp. D). Similarly, U^{M1}_g or U^{1M}_g (resp. U^{D1}_g or U^{1D}_g) denotes the restriction of U^1_g to M^ρ (resp. D).

1.2 The Infrared Problem

On the classical level the infrared problem consists of determining to which extent the long-range interaction created by the coupling $A^\mu j_\mu$ between the electromagnetic potential A_μ and the current $j_\mu = \overline{\psi} \gamma_\mu \psi$ is an obstruction for the separation, when $|t| \to \infty$, of the nonlinear relativistic system into two asymptotic isolated relativistic systems, one for the electromagnetic potential A_μ and one for the Dirac field ψ. It will be proved here that there

is such an obstruction, which in particular implies that *asymptotic in and out states do not transform according to a linear representation of the Poincaré group.* This constitutes a serious problem for the second quantization of the asymptotic (in and out going) fields, since the particle interpretation usually requires free relativistic fields, i.e. at least a linear representation of the Poincaré group (U^1 in our case). Therefore we have to introduce nonlinear representations $U^{(-)}$ and $U^{(+)}$ of $\mathcal{P}_0$, in the sense of [5], which can give the transformation of the asymptotic in and out states under $\mathcal{P}_0$ and which can permit a particle interpretation.

There are two reasons which permit to determine the class of asymptotic representations $U^{(\varepsilon)}$, $\varepsilon = \pm$. First, the classical observables, 4-current density, 4-momentum and 4-angular momemtum are invariant under gauge transformations. Second, if the evolution equations become linear after a gauge transformation one can use the freedom of gauge in the second quantization of the fields. It is therefore reasonable to postulate that the asymptotic representations $g \mapsto U_g^{(\varepsilon)}$ are linear modulo a nonlinear gauge transformation depending on g and respecting the Lorentz gauge condition. We shall make precise the meaning of this statement at the end of this introduction.

Moreover to determine the class of admissible modified wave operators it is reasonable to postulate that solutions of the M-D equations (1.1a)–(1.1c) should converge when $t \to \pm\infty$ in E^ρ to free solutions (i.e. solutions of equations (1.1a)–(1.1c) but with vanishing right-hand side) modulo a gauge transformation, not even respecting the Lorentz gauge condition; such transformations are admissible since they leave invariant the observables.

In mathematical terms the infrared problem of the M-D equations then consists of determining two diffeomorphisms $\Omega_\varepsilon \colon \mathcal{O}_\infty^{(\varepsilon)} \to \mathcal{U}_\infty$, $\varepsilon = \pm$, the modified wave operators, where $\mathcal{O}_\infty^{(\varepsilon)}$ is an open neighbourhood of zero in $E_\infty^{\circ\rho}$ and where $\mathcal{U}_\infty$ is a neighbourhood of zero in V_∞^ρ, satisfying

$$U_g^{(\varepsilon)} = \Omega_\varepsilon^{-1} \circ U_g \circ \Omega_\varepsilon, \quad g \in \mathcal{P}_0, \varepsilon = \pm, \tag{1.14}$$

where the asymptotic representations are C^∞ functions $U^{(\varepsilon)} \colon \mathcal{P}_0 \times E_\infty^{\circ\rho} \to E_\infty^{\circ\rho}$. In order to satisfy the two preceding postulates, we impose supplementary conditions on $U^{(\varepsilon)}$ and Ω_ε, which we shall justify at the end of this introduction. Let the Fourier transformation $f \mapsto \hat{f}$ be defined by

$$\hat{f}(k) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ikx} f(x) dx. \tag{1.15}$$

The orthogonal projections $P_\varepsilon(-i\partial)$ in D on initial data with energy sign ε, $\varepsilon = \pm$, for the Dirac equation are given by:

$$(P_\varepsilon(-i\partial)\alpha)^\wedge(k) = P_\varepsilon(k)\hat{\alpha}(k) = \frac{1}{2}\Big(I + \varepsilon\Big(-\sum_{j=1}^{3} \gamma^0\gamma^j k_j + m\gamma^0\Big)\omega(k)^{-1}\Big)\hat{\alpha}(k), \tag{1.16}$$

where $\omega(k) = (m^2 + |k|^2)^{1/2}$. We postulate that the asymptotic representations have the following form:

$$U_g^{(+)}(u) = (U_g^{(+)M}(u), U_g^{(+)D}(u)), \quad U^{(+)M} = U^{1M}, \tag{1.17a}$$

$$(U_g^{(+)D}(u))^\wedge(k) = \sum_{\varepsilon=\pm} e^{i\varphi_g(u,-\varepsilon k)} P_\varepsilon(k)(U_g^{1D}\alpha)^\wedge(k), \quad g \in \mathcal{P}_0, \tag{1.17b}$$

where $u=(f,\dot f,\alpha)\in E^{\circ\rho}_\infty$, the function $(g,u,k)\mapsto\varphi_g(u,-\varepsilon k)$ from $\mathcal{P}_0\times E^{\circ\rho}_\infty\times\mathbb{R}^3$ to $\mathbb{R}$ is C^∞ and if $(h_g(u))(t,x)=\varphi_g(u,mx(t^2-|x|^2)^{-1/2})$, $t>0$, $|x|<t$, then $\Box h_g(u)=0$. Moreover growth conditions on φ_g should be satisfied, so that the main contribution to the phase and its derivatives in g and k comes from U^{1D}_g, the restriction of U^1 to D. Finally we impose the asymptotic condition

$$\begin{aligned}&\|U^M_{\exp(tP_0)}(\Omega_+(u))-U^{1M}_{\exp(tP_0)}(f,\dot f)\|_{M^\rho}\\&\quad+\|U^D_{\exp(tP_0)}(\Omega_+(u))-\sum_{\varepsilon=\pm}e^{is^{(+)}_\varepsilon(u,t,-i\partial)}P_\varepsilon(-i\partial)U^{1D}_{\exp(tP_0)}\alpha\|_D\to 0,\end{aligned}\tag{1.17c}$$

when $t\to\infty$, where $u=(f,\dot f,\alpha)\in\mathcal{U}_\infty$, $(u,t,k)\mapsto s^{(+)}_\varepsilon(u,t,k)$ is a C^∞ function from $\mathcal{U}_\infty\times\mathbb{R}\times\mathbb{R}^3$ to $\mathbb{R}$, $s^{(+)}_\varepsilon(u,t,-i\partial)$ is the operator defined by inverse Fourier transform of the multiplication operator $k\mapsto s^{(+)}_\varepsilon(u,t,k)$. Moreover $s^{(+)}_\varepsilon(u,t,k)$ should satisfy growth conditions so that its major contribution comes from $U^{1D}_{\exp(tP_0)}(u)$. There are similar conditions for $U^{(-)}$, Ω_- and $s^{(-)}_\varepsilon$, $\varepsilon=\pm$. We note that if $s^{(+)}_\varepsilon$ is determined, then Ω_+ is determined and so is $U^{(+)}$. It was proved in [8], that on a set of asymptotic states $(f,\dot f,\alpha)$ such that $\hat f,\hat{\dot f}\in C^\infty_0(\mathbb{R}^3-\{0\})$ and $\hat\alpha\in C^\infty_0(\mathbb{R}^3)$[2)], it is possible to choose (formula (3.33a) of [8])

$$s^{(+)}_\varepsilon(u,t,k)=-\vartheta\big(A^{(+)}(u),(t,-\varepsilon tk/\omega(k))\big),\tag{1.18}$$

where $A^{(+)}(u)$ is a certain approximate solution of the M-D equations absorbing the long-range part of A for a solution (A,ψ), and where

$$\vartheta(H,y)=\int_{L(y)}H_\mu(z)dz^\mu,\quad y\in\mathbb{R}^4,\tag{1.19}$$

where $H:\mathbb{R}^4\to\mathbb{R}^4$ and $L(y)=\{z\in\mathbb{R}^4\big|z=sy,0\le s\le 1\}$. The function $(t,k)\mapsto s^{(+)}_\varepsilon(u,t,k)$ was determined by the fact, that has been proved in [8], that $S_\varepsilon(u,t,k)=\varepsilon\omega(k)t+s^{(+)}_\varepsilon(u,t,k)$ has to be in a certain sense an approximate solution of the Hamilton-Jacobi equation for a relativistic electron in an external electromagnetic potential:

$$\begin{aligned}&\big(\frac{\partial}{\partial t}\mathcal{S}_\varepsilon(u,t,k)+A_0(t,-\nabla_k\mathcal{S}_\varepsilon(u,t,k))\big)^2\\&\quad-\sum_{i=1}^3\big(k_i+A_i(t,-\nabla_k\mathcal{S}_\varepsilon(u,t,k))\big)^2=m^2.\end{aligned}\tag{1.20a}$$

We proved and used the fact that

$$y^\mu\partial_\mu\vartheta(H,y)=y^\mu H_\mu(y),\tag{1.20b}$$

to establish that $S_\varepsilon(u,t,k)$ is an approximate solution of the Hamilton-Jacobi equation.

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2) $C^k_0(X)$, $k\ge 0$, denotes the space of k times continuously differentiable functions on X with compact support.

1.3 Presentation of the main results

1.3.a *Some notations.* Let $u_+ = (f, \dot{f}, \alpha) \in E_\infty^{\circ\rho}$ and let (we do not write explicitely the multiplication sign $\times$ in formulas extending over more than one line at cut at a multiplication)

$$J_\mu^{(+)}(t,x) = (m/t)^3 \big(\omega(q(t,x))/m\big)^5 \tag{1.21}$$
$$\sum_{\varepsilon=\pm} ((P_\varepsilon(-i\partial)\alpha)^\wedge(\varepsilon q(t,x)/m))^+ \gamma^0 \gamma_\mu ((P_\varepsilon(-i\partial))\alpha)^\wedge(\varepsilon q(t,x)/m),$$

for $t > 0$, $|x| < t$, where $q(t,x) = -mx/(t^2 - |x|^2)^{1/2}$ and, for $|x| > 0$, $0 \leq t \leq |x|$, let $J_\mu^{(+)}(t,x) = 0$. Since $\alpha \in S(\mathbb{R}^3, \mathbb{C}^4)$, it follows that $J_\mu^{(+)} \in C^\infty((\mathbb{R}^+ \times \mathbb{R}^3) - \{0\})$ and that its support is contained in the forward light cone. We choose $\chi \in C^\infty(\mathbb{R})$, $\chi(\tau) = 0$ for $\tau \leq 1$, $\chi(\tau) = 1$ for $\tau \geq 2$, $0 \leq \chi(\tau) \leq 1$ for $\tau \in \mathbb{R}$, $\chi_0 \in C^\infty([0,\infty[)$, $\chi_0(\tau) = 1$ for $\tau \geq 2$, $0 \leq \chi_0(\tau) \leq 1$ for $\tau \in [0, \infty[$, and introduce $A^{(+)} = A^{(+)1} + A^{(+)2}$ by

$$\left.\begin{aligned} (A^{(+)1}(t))(x) &= \chi_0((t^2 - |x|^2)^{1/2})(B^{(+)1}(t))(x) \\ (A^{(+)2}(t))(x) &= \chi((t^2 - |x|^2)^{1/2})(B^{(+)2}(t))(x), \end{aligned}\right\} \quad \text{for } t \geq |x|,$$

and (1.22a)

$$\left.\begin{aligned} (A^{(+)1}(t))(x) &= \chi_0(0)(B^{(+)1}(t))(x), \\ (A^{(+)2}(t))(x) &= 0, \end{aligned}\right\} \quad \text{for } t < |x|,$$

where

$$B_\mu^{(+)1}(t) = \cos(|\nabla| t) f_\mu + |\nabla|^{-1} \sin(|\nabla| t) \dot{f}_\mu, \quad t \in \mathbb{R}, \tag{1.22b}$$

$$B_\mu^{(+)2}(t) = -\int_t^\infty |\nabla|^{-1} \sin(|\nabla|(t-s)) J_\mu^{(+)}(s, \cdot) ds, \quad t > 0. \tag{1.22c}$$

The cut-off function χ has been introduced to exclude in a Lorentz invariant way the points $(0, x)$, $x \in \mathbb{R}^3$ from the support of $A^{(+)2}$. $U^{(+)}$ is defined by formulas (1.17a) and (1.17b) and by $\varphi_g = 0$ for $g \in SL(2, \mathbb{C})$ and

$$\varphi_g(u, -\varepsilon k) = \vartheta^\infty\big((\chi_0 \circ \rho)(B^{(+)1}(\cdot + a) - B^{(+)1}(\cdot)), (\omega(k), -\varepsilon k)\big), \tag{1.23a}$$

for $g = \exp(a^\mu P_\mu)$, $k \in \mathbb{R}^3$, where

$$\vartheta^\infty(H, y) = \int_0^\infty y^\mu H_\mu(sy) ds, \quad y \in \mathbb{R}^4, \text{ and } \rho(t,x) = (t^2 - |x|)^{1/2}. \tag{1.23b}$$

We have made the identification convention that $B^{(+)1}(t,x) = (B^{(+)1}(t))(x)$. We note that the function $y \mapsto \vartheta^\infty(H, y)$ is homogeneous of degree zero, so we could also have taken the argument $(1, -\varepsilon k/\omega(k))$ instead of $(\omega(k), -\varepsilon k)$, which corresponds to the choice in [9]. The phase function $s_\varepsilon^{(+)}$ in (1.17c), which determines Ω_+, is defined by formula (1.18) and the choice of $A^{(+)} = A^{(+)1} + A^{(+)2}$ given by (1.22a). With the choice $\chi_0 = 1$ the function

h_g introduced after (1.17b) is given by $(h_g(u))(y) = \vartheta^\infty(B^{(+)1}(\cdot + a) - B^{(+)1}(a), y)$ and satisfies $\Box h_g(u) = 0$ if $(f, \dot{f}) \in E^{\circ\rho}$.

1.3.b *Statements.* We state the main results of this article for the case where $t \to +\infty$. There are analog results for $t \to -\infty$.

Theorem I. *Let $1/2 < \rho < 1$. If $n \geq 4$ then $U^{(+)} \colon \mathcal{P}_0 \times E_n^{\circ\rho} \to E_n^{\circ\rho}$ is a continuous nonlinear representation of $\mathcal{P}_0$ in $E_n^{\circ\rho}$ and, in addition, the function $U^{(+)} \colon \mathcal{P}_0 \times E_\infty^{\circ\rho} \to E_\infty^{\circ\rho}$ is C^∞. Moreover $U^{(+)}$ is not equivalent by a C^2 map to a linear representation on $E_\infty^{\circ\rho}$: If $U_1^{(+)}$ and $U_2^{(+)}$ are defined via (1.23a) by two choices of the function χ_0, they are equivalent.*

Theorem I partially sums up Theorems 3.12–3.14.

Theorem II. *Let $1/2 < \rho < 1$. There exist an open neighbourhood $\mathcal{U}_\infty$ (resp. $\mathcal{O}_\infty^{(+)}$) of zero in V_∞^ρ (resp. $E_\infty^{\circ\rho}$), a diffeomorphism $\Omega_+ \colon \mathcal{O}_\infty^{(+)} \to \mathcal{U}_\infty$ and a C^∞ function $U \colon \mathcal{P}_0 \times \mathcal{U}_\infty \to \mathcal{U}_\infty$, defining a nonlinear representation of $\mathcal{P}_0$, such that:*

i) $\dfrac{d}{dt} U_{\exp(tX)}(u) = T_X(U_{\exp(tX)}(u)), \quad X \in \mathfrak{p}, t \in \mathbb{R}, u \in \mathcal{U}_\infty,$

ii) $\Omega_+ \circ U_g^{(+)} = U_g \circ \Omega_+$

iii)
$$\lim_{t\to\infty} \Big(\|U^M_{\exp(tP_0)}(\Omega_+(u)) - U^{M1}_{\exp(tP_0)}(f, \dot{f})\|_{M^\rho} + \|U^D_{\exp(tP_0)}(\Omega_+(u)) - \sum_{\varepsilon=\pm} e^{is_\varepsilon^{(+)}(u,t,-i\partial)} P_\varepsilon(-i\partial) U^{D1}_{\exp(tP_0)} \alpha\|_D \Big) = 0,$$

for $u = (f, \dot{f}, \alpha) \in \mathcal{O}_\infty^{(+)}$.

This theorem (see Theorem 6.19) solves in particular *the Cauchy problem* for small initial data and proves *asymptotic completeness.* By the construction of the wave operator Ω_+ in chapter 6, the solution $(A(t,\cdot), \dot{A}(t,\cdot), \psi(t,\cdot)) = U_{\exp(tP_0)}(u)$ of the Cauchy problem satisfies

$$\sup_{\substack{x\in\mathbb{R}^3 \\ t\geq 0}} \Big((1+|x|+t)^{3/2-\rho} |A_\mu(t,x)| + (1+|x|+t) |\partial_\nu A_\mu(t,x)| + (1+|x|+t)^{3/2} |\psi(t,x)| \Big) < \infty.$$

1.3.c *Cohomology.* These results and conditions (1.14) and (1.17c) have a natural *cohomological interpretation.* We only consider the case where $t \to \infty$, and since the representations $U^{(+)}$ defined for different χ_0 via (1.23a) are equivalent, we only consider the case where $\chi_0 = 1$. A necessary condition for $U^{(+)}$ and Ω_+ to be a solution of equation (1.14) is that the formal power series development of $U_g^{(+)}$, U_g and Ω_+ in the initial conditions satisfy the cohomological equations defined in [5]and [6]. In particular (after trivial transformations) the second order terms $U_g^{(+)2}$, U_g^2 and Ω_+^2 must satisfy

$$\delta\Omega_+^2 = C^2, \quad \delta R^{(+)2} = 0, \tag{1.24}$$

where δ is the coboundary operator defined by the representation $(g, A) \mapsto U^1_g A^2(\otimes^2 U_{g-1})$ of the Poincaré group $\mathcal{P}_0$ on bilinear symmetric maps A^2 from $E^\rho_\infty \to E^\rho_\infty$ and where $C^2 = R^{(+)2} - R^2$ is the cocycle defined by $R^2_g = U^2_g(U^1_{g-1} \otimes U^1_{g-1})$ and $R^{(+)2}_g = U^{(+)2}_g(U^1_{g-1} \otimes U^1_{g-1})$, $g \in \mathcal{P}_0$. Equation (1.24) shows that the cochain $R^{(+)2}$ has to be a cocycle and then that the cocycle C^2 has to be a coboundary. This is equivalent to the existence of a solution $U^{(+)}$, Ω_+ of equation (1.14) modulo terms of order at least three. There are similar equations for higher order terms:

$$\delta \Omega^n_+ = C^n, \quad \delta R^{(+)n} = 0, \tag{1.25}$$

where C^n and $R^{(+)n}$ are functions of $\Omega^2_+, \ldots, \Omega^{n-1}_+$ and $U^{(+)2}, \ldots, U^{(+)n}$.

In a previous article [8], we proved that there exist a modified wave operator and global solutions of the M-D equations for a set of scattering data $(f, \dot{f}, \alpha)$, which is a subset of the spaces $E^{\circ\rho}_\infty$ introduced in the present paper, satisfying $\hat{f}_\mu(k) = \hat{\dot{f}}_\mu(k) = 0$ for k in a neighbourhood of zero, i.e. f_μ and $\dot{f}_\mu$ have no low frequencies component. It follows from that paper that the usual wave operator (i.e. $U^{(+)} = U^1, s^{(+)}_\varepsilon = 0$ in (1.17c)) does not exist, even in this case where there are no low frequencies. In fact there is an obstruction to the existence of such a solution of equation (1.14) for $n = 3$ due to the self-coupling of ψ with the electromagnetic potential created by the current $\overline{\psi}\gamma_\mu\psi$. However, as was proved in [8] (see also Lemma 3.2 of the present paper) there exists a modified wave operator satisfying (1.25) for $n \geq 2$, with $U^{(+)} = U^1$. Moreover, as we have already pointed out, the phase function $s^{(+)}_\varepsilon$ is given by formula (1.18) (see formulas (1.5), (3.28) and (3.33) of [8]).

Thus in the absence of low frequencies in the scattering data, for the electromagnetic potential, we can choose Ω_+ such that it intertwines the linear representation U^1 and the nonlinear representation U of $\mathcal{P}_0$. Now if $(f, \dot{f}) \in M^{\circ\rho}_\infty$, $1/2 < \rho < 1$, have nontrivial low frequency part, as is the case for the Coulomb potential ($\hat{f}_\mu(k) \simeq |k|^{-2}$) and which is necessary to assume in order to have asymptotic completeness, then there is a cohomological obstruction already for $n = 2$ if we want to obtain $U^{(+)2} = 0$. The essential point now is that the cocycle R^2 can be split into a trivializable part $-C^2$ (a coboundary) and a nontrivializable part $R^{(+)2}$, which defines $U^{(+)2}$ and therefore the whole representation $U^{(+)}$. We shall call this nontrivial part the *infrared cocycle*. According to the definition of $R^{(+)2}$ and formula (1.23a) it follows that $R^{(+)2}_g = 0$ for $g \in SL(2, \mathbb{C})$, $R^{(+)2}_g = (R^{(+)2M}_g, R^{(+)2D}_g)$, $R^{(+)2M} = 0$ and

$$\begin{aligned}(R^{(+)2D}_g&(u_1 \otimes u_2))^\wedge(k) \\ &= \frac{i}{2} \sum_{\varepsilon = \pm} \big(\vartheta^\infty(B^{(+)1}_1(\cdot) - B^{(+)1}_1(\cdot - a), (\omega(k), -\varepsilon k)\big) P_\varepsilon(k) \hat{\alpha}_2(k) \\ &\qquad + \vartheta^\infty\big(B^{(+)1}_2(\cdot) - B^{(+)1}_2(\cdot - a), (\omega(k), -\varepsilon k)\big) P_\varepsilon(k) \hat{\alpha}_1(k)),\end{aligned} \tag{1.26}$$

for $g = \exp(a^\mu P_\mu)$, $u_i = (f_i, \dot{f}_i, \alpha_i) \in E^{\circ\rho}_\infty$, $i \in \{1, 2\}$, and where $B^{(+)1}_{i\mu}$ is the corresponding free field given by (1.22b).

When the limit $\vartheta(B^{(+)1},(t,-t\varepsilon k/\omega(k)) \to \vartheta^{\infty}(B^{(+)1},(\omega(k),-\varepsilon k))$ exists for $t \to \infty$, then the infrared cocycle is in fact a coboundary. In particular, this is the case when $\hat{f}_\mu$, $\hat{\dot{f}}_\mu$ are equal to zero in a neighbourhood of zero.

1.4 Physical Remarks

1.4.a *Physical motivation of the asymptotic condition.* Next, we return to the physical justification of conditions (1.17a)–(1.17c), and to be more specific, as this is the main case of this paper, we choose $s_\varepsilon^{(+)}$ to be given by (1.18) and $A^{(+)}$ to be given by (1.22a)–(1.22c) with $\chi_0 = 1$. According to statement iv) of Theorem 6.16 and Theorem 6.19, the asymptotic condition in statement iii) of Theorem II is equivalent to

$$\|(A(t),\dot{A}(t)) - (A_L(t),\dot{A}_L(t))\|_{M^\rho} + \|\psi(t) - (e^{-i\varphi}\psi_L)(t)\|_D \to 0, \tag{1.27}$$

when $t \to \infty$, where $(A(t),\dot{A}(t),\psi(t)) = U_{\exp(tP_0)}(\Omega_+(u))$, $(A_L(t),\dot{A}_L(t),\psi_L(t)) = U^1_{\exp(tP_0)}u$ and where for a fixed $u = (f,\dot{f},\alpha) \in \mathcal{O}^{(+)}_\infty$, $(t,x) \mapsto \varphi(t,x)$, $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$ is a C^∞ function given in Theorem 6.16. When needed we shall write $(u,(t,x)) \mapsto \varphi(u,(t,x))$ to stress the dependence of φ on u. It follows from Theorem A.1 that φ in (1.27) can be replaced by $\vartheta(A^{(+)},\cdot)$, but for technical reasons we keep φ. In fact, from a heuristic point of view, since $\vartheta(A^{(+)},(t,x)) = -s_\varepsilon(u,t,-\varepsilon x/(t^2-|x|^2)^{1/2})$, (1.27) with φ replaced by $\vartheta(A^{(+)},\cdot)$ is obtained from the leading term in the stationary phase expansion of $\sum_\varepsilon e^{is_\varepsilon(u,t,-i\partial)} U^{1D}_{\exp(tP_0)}\alpha$.

The energy-momentum tensor for the M–D system is, in one of its forms, given by

$$t^{\mu\nu} = -F^{\mu\alpha}F^{\nu}{}_{\alpha} + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}(\overline{\psi}\gamma^\mu(i\partial^\nu - A^\nu)\psi + (\overline{(i\partial^\nu - A^\nu)\psi})\gamma^\mu\psi), \tag{1.28}$$

where $0 \le \mu \le 3$, $0 \le \nu \le 3$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The current density vector is given by

$$j^\mu = \overline{\psi}\gamma^\mu\psi, \quad 0 \le \mu \le 3. \tag{1.29}$$

$t^{\mu\nu}$ and j^μ are invariant under gauge transformations: $\psi' = e^{i\lambda}\psi$, $A'_\mu = A_\mu - \partial_\mu\lambda$ (not necessarily respecting the Lorentz gauge condition that gives here $\Box\lambda = 0$). We also introduce the energy-momentum tensor and the current density vector for the system of free fields A_L and ψ_L by

$$t_L^{\mu\nu} = -F_L^{\mu\alpha}F^{\nu}_{L\alpha} + \frac{1}{4}g^{\mu\nu}F_{L\alpha\beta}F_L^{\alpha\beta} + \frac{1}{2}(\overline{\psi}_L\gamma^\mu(i\partial^\nu)\psi_L + (\overline{(i\partial^\nu)\psi_L})\gamma^\mu\psi_L), \tag{1.30}$$

and

$$j_L^\mu = \overline{\psi}_L\gamma^\mu\psi_L, \tag{1.31}$$

where $F_{L\mu\nu} = \partial_\mu A_{L\nu} - \partial_\nu A_{L\mu}$. Since $t^{\mu\nu}$ can be written as

$$t^{\mu\nu} = -F^{\mu\alpha}F^{\nu}{}_{\alpha} + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}(\overline{\psi}'\gamma^\mu(i\partial^\nu - A'^\nu)\psi' + (\overline{(i\partial^\nu - A'^\nu)\psi})\gamma^\mu\psi'),$$

where $\psi' = e^{i\varphi}\psi$, $A'_\mu = A_\mu - \partial_\mu\varphi$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, it follows from statement iii) of Theorem 6.16 that

$$\lim_{t\to\infty}\int_{\mathbb{R}^3} |t^{\mu\nu}(t,x) - t_L^{\mu\nu}(t,x)|\, dx = \lim_{t\to\infty}\|A'(t)\|_{L^\infty}\|\psi_L(t)\|_D^2.$$

By unitarity of U^{1D}, $\|\psi_L(t)\|_D^2 = \|\alpha\|_D^2$. Using statement iii) of Theorem 6.16 for $\|A(t)\|_{L^\infty}$ and using statement ii) of Lemma 4.4 for $\partial_\mu \vartheta(A^{(+)1}, \cdot)$ and estimate (6.246) for $\partial_\mu(\varphi - \vartheta(A^{(+)1}, \cdot))$, it follows that $\|A'(t)\|_{L^\infty} \to 0$, when $t \to \infty$. Therefore

$$\lim_{t\to\infty} \int_{\mathbb{R}^3} |t^{\mu\nu}(t,x) - t_L^{\mu\nu}(t,x)|\, dx = 0. \tag{1.32}$$

Similarly

$$\lim_{t\to\infty} \int_{\mathbb{R}^3} |j^{\mu}(t,x) - j_L^{\mu}(t,x)|\, dx = 0. \tag{1.33}$$

Limits (1.32) and (1.33) show that, as far as one measures energy-momentum and current, the solutions of the M-D system are asymptotically indistinguishable from free solutions because of gauge invariance. Condition (1.17c) is therefore natural. A similar discussion can be made for the angular momentum tensor; however it is technically more involved, so we omit it.

1.4.b *Gauge transformations.* A gauge transformation $\psi' = e^{i\lambda}\psi$, $A'_\mu = A_\mu - \partial_\mu \lambda$, respecting the Lorentz gauge condition, i.e. $\Box\lambda = 0$, transforms a solution (A,ψ) of the M–D equations (1.1a)–(1.1c) into a solution (A',ψ') of the M–D equations. Let $v \in \mathcal{U}_\infty$. v is the initial data for the solution (A,ψ) of the M–D equations, and if λ is sufficiently small and regular, the initial data v' for the solution (A',ψ') is in $\mathcal{U}_\infty$, since $\mathcal{U}_\infty$ is open. Let $u = \Omega_+^{-1}(v)$ and $u' = \Omega_+^{-1}(v')$. Since $\lambda(y) = \lambda(0) + \vartheta(\Lambda, y)$, with $\Lambda_\mu = \partial_\mu\lambda$, it follows from (1.27) that

$$\|(A'(t), \dot{A}'(t)) - (A'_L(t), \dot{A}'_L(t))\|_{M^\rho} + \|\psi'(t) - e^{-i\varphi(u',(t,\cdot))}\psi'_L(t)\|_D \to 0, \tag{1.34}$$

when $t \to \infty$, where $(A'_L(t), \dot{A}'_L(t), \psi'_L(t)) = U^1_{\exp(tP_0)}u'$ and $u' = (f', \dot{f}', \alpha')$, $f'_\mu(x) = f_\mu(x) - (\partial_\mu\lambda)(0,x)$, $\dot{f}'_\mu(x) = \dot{f}_\mu(x) - (\partial_\mu\partial_0\lambda)(0,x)$, $\alpha' = e^{i\lambda(0)}\alpha$. Since we can choose the constant $\lambda(0)$ arbitrarily, it follows that the admissible gauge transformations $u \mapsto u'$ of the scattering data are given by

$$A'_{L\mu} = A_{L\mu} - \partial_\mu\lambda, \quad \psi'_L = e^{ic}\psi_L, \tag{1.35}$$

where $c \in \mathbb{R}$ and λ is such that $\Box\lambda = 0$, and such that $(\Lambda(0,\cdot), \dot{\Lambda}(0,\cdot)) \in E_\infty^{\circ\rho}$, $\Lambda_\mu(t,x) = (\partial_\mu\lambda)(t,x)$, $\dot{\Lambda}_\mu(t,x) = (\partial_0\partial_\mu\lambda)(t,x)$.

We now introduce the notion of *gauge-projective map*. Let $Q\colon E_\infty^{\circ\rho} \to E_\infty^{\circ\rho}$ be a C^∞ map leaving $\mathcal{O}_\infty^{(+)}$ invariant. More general situations are possible, but to fix the ideas we make this hypothesis. If there exists a gauge transformation G of the form (1.35) such that

$$\begin{aligned}
&\|U^M_{\exp(tP_0)}\big(\Omega_+(Q(u))\big) - U^{1M}_{\exp(tP_0)}(f,\dot{f})\|_{M^\rho} \\
&\quad + \|U^D_{\exp(tP_0)}\big(\Omega_+(Q(u))\big) - \sum_\varepsilon e^{is_\varepsilon^{(+)}(G(u),t)} P_\varepsilon(-i\partial) U^{1D}_{\exp(tP_0)}\alpha\|_D \to 0,
\end{aligned} \tag{1.36}$$

where $u = (f, \dot{f}, \alpha)$, then we say that Q is a gauge-projective map.

To show that the *asymptotic representation* $g \mapsto U_g^{(+)}$ *is equal to* U^1 *modulo a gauge-projective map* depending on g, we write the asymptotic condition (1.17c), with $U_g^{(+)}(u)$ instead of u:

$$\begin{aligned}&\|U^M_{\exp(tP_0)}\big(\Omega_+(U_g^{(+)}(u))\big) - U^{1M}_{\exp(tP_0)}(U_g^{1M}(f,\dot f))\|_{M^\rho} \\ &\quad + \|U^D_{\exp(tP_0)}\big(\Omega_+(U_g^{(+)}(u))\big) - \sum_\varepsilon e^{is_\varepsilon^{(+)}(U_g^{(+)}(u),t,-i\partial)} P_\varepsilon(-i\partial) U^{1D}_{\exp(tP_0)} U_g^{(+)D}(u)\|_D \to 0,\end{aligned} \tag{1.37}$$

when $t \to \infty$. Definition (1.23a) of the function φ_g defining $U_g^{(+)}$ by (1.17a) and (1.17b), shows that

$$\varphi_g(u, -\varepsilon k) - \vartheta\big(B^{(+)1}(\cdot + a) - B^{(+)1}(\cdot), (t, -\varepsilon t k/\omega(k))\big) \to 0, \tag{1.38}$$

when $t \to \infty$, for $g = \exp(a^\mu P_\mu)$. We recall that $\varphi_g(u) = 0$ for $g \in SL(2, \mathbb{C})$ and we note that $\square\vartheta(H, \cdot) = 0$ for $H(y) = B^{(+)1}(y + a) - B^{(+)1}(y)$. It follows from (1.37) and (1.38) that, if $\lambda = \vartheta(H, \cdot)$, then

$$\begin{aligned}&\|U^M_{\exp(tP_0)}\big((\Omega_+(U_g^{(+)}u))\big) - U^{1M}_{\exp(tP_0)}(U_g^{1M}(f,\dot f))\|_{M^\rho} \\ &+ \|U^D_{\exp(tP_0)}\big((\Omega_+(U_g^{(+)}u))\big) - \sum_\varepsilon e^{is_\varepsilon^{(+)}(U_g^{(+)}(u),t,-i\partial)} e^{-iq(t,-i\varepsilon t\partial)} P_\varepsilon(-i\partial) U^{1D}_{\exp(tP_0)} U_g^{1D}\alpha\|_D \to 0,\end{aligned} \tag{1.39}$$

when $t \to \infty$, where $q(t, k) = \lambda(t, -tk/\omega(k))$. The definition of $A^{(+)}$ and $\varphi(u, (t, x))$ give that $s_\varepsilon^{(+)}(U_g^{(+)}(u), t, k) = s_\varepsilon^{(+)}(U_g^1(u), t, k)$. Since $\lambda = \vartheta(\Lambda, \cdot)$, where $\Lambda_\mu = \partial_\mu\lambda$, it follows from (1.39) that

$$\begin{aligned}&\|U^M_{\exp(tP_0)}\big((\Omega_+(U_g^{(+)}u))\big) - U^{1M}_{\exp(tP_0)} U_g^{1M}(f,\dot f)\|_{M^\rho} \\ &\quad + \|U^D_{\exp(tP_0)}\big((\Omega_+(U_g^{(+)}u))\big) - \sum_\varepsilon e^{is_\varepsilon^{(+)}(G(U_g^1 u),t,-i\partial)} P_\varepsilon(-i\partial) U^{1D}_{\exp(tP_0)} U_g^{1D}\alpha\|_D \to 0,\end{aligned} \tag{1.40}$$

when $t \to \infty$, where G is the gauge transformation given by λ (as in (1.35) with $c = 0$). This shows that $Q_g = U_g^{(+)} U^1_{g^{-1}}$ is a gauge-projective transformation according to (1.36).

1.4.c *Organization of the monograph.*

In chapter 2, we prove that the sequence E_n^ρ of Hilbert spaces admits a family of smoothing operators (Theorem 2.5) in the sense of [17], in order to use an implicit function theorem in the Fréchet space E_∞^ρ.

In chapter 3, we prove that $U^{(+)}$ is a nonlinear representation (Theorem 3.12) which is nonlinearizable (Theorem 3.13).

In chapter 4, we construct approximate solutions (A_n, ϕ_n), $n \geq 0$, (see formulas (4.135a)–(4.135b)) of the M–D system. They are obtained, essentially by using stationary phase methods and by iterating formulas (4.137b)–(4.137c). Their decrease properties are given by Theorem 4.9 and Theorem 4.10. They are approximate solutions in the sense that a certain remainder term (Δ_n^M, Δ_n^D) (see (4.140a)–(4.140b)) satisfies Theorem 4.11. In particular the remainder term for the electromagnetic potential Δ_n^M is short-range (it belongs to $L^2(\mathbb{R}^3, \mathbb{R}^4)$ for each fixed time).

In chapter 5, we prove equal time weighted L^2–L^2 and L^2–L^∞ estimates for the linear inhomogeneous Dirac equation $(i\gamma^\mu\partial_\mu + m - \gamma^\mu G_\mu)h = g$, with external field G (Theorem 5.5 and Theorem 5.8). Combination of these estimates with an energy estimate (Corollary 5.2) leads to existence of h and to estimates on h, adapted to the nonlinear problems treated in chapter 6.

In chapter 6, we end the construction of an approximate solution (A^*, ϕ^*) (formulas (6.1a)–(6.2b) and (6.30)) satisfying the Lorentz gauge condition (Proposition 6.2). The existence of the rest term (K, Φ) (see formulas (6.30–(6.31c)) follows by the construction of a contraction mapping (formula (6.33)) in a Banach space $\mathcal{F}_N$ (Corollary 6.8). In particular $K(t) \in L^2(\mathbb{R}^3, \mathbb{R}^4)$. The existence of solutions of the M–D equations (Theorem 6.10) and the existence of a modified wave operator, denoted temporarily by Ω_1 (formula (6.172), Theorem 6.12) are then proved. The existence of Ω_1^{-1} is shown (Theorem 6.13) by using an implicit functions theorem in the Fréchet space E^ρ_∞. The disadvantage of Ω_1 is that it gives a nonexplicit expression for the the asymptotic representation because it requires the use of the solutions of the M–D systems. To overcome this problem, we introduce the final modified wave operator $\Omega^{(+)}$ (formulas (6.227a), (6.227b) and (6.276)) and its extension Ω_+ to a Poincaré invariant domain. Ω_+ has the advantage of giving a simple expression (see (1.17a) and (1.17b)) for the asymptotic representation $U^{(+)}$.

This monograph (at least in so far as the M-D equations are concerned) is largely self-contained. A few technical results are borrowed from quoted references. It may be read with little or no knowledge of nonlinear group representation theory, but the latter is crucial to a real understanding of the methods and choice of spaces.

1.4.d *Final remarks.*

i) It can be natural to think of the underlying classical theory of QED not as the M-D equations with c-number spinor components but as a theory with anticommuting spinor components. Since the second order terms in the spinor component of the wave operator originates in the coupling $A_\mu\psi$ of a c-number free electromagnetic potential A_μ and a free Dirac field, we believe that the infrared cocycle will remain the same for a theory with anticommuting spinor components. Further for higher order terms, the nilpotency property $\psi_\alpha(x)^2 = 0$ can ameliorate the infrared problems arising from the self-coupling.

ii) The observables, 4-current, 4-momentum and 4-angular momentum, defined for the asymptotic representation $U^{(+)}$, which is "gauge projectively linear", converge when $t \to \infty$ to the usual free field observables. Therefore there should be no observable phenomena which distinguish $U^{(+)}$ from U^1, at least as far as these observables are concerned. Since the representation U is equivalent to $U^{(+)}$ by Ω_+, we shall call U a "gauge projectively linearizable" nonlinear representation. Of course if the Dirac field or the electromagnetic potential itself was an observable, then this would no longer be true, i.e. it would then be possible to distinguish $U^{(+)}$ and U^1 by the observables.

iii) One should note that the phase factor (3.33a of [8]), which looks as a very familiar factor in abelian and non abelian gauge theories, was obtained in [8] in the different context of the Hamilton-Jacobi equation associated with the full Maxwell-Dirac equations. Would one have taken initial data for the A_μ field decreasing slower than $r^{-1/2}$ and suitable initial data for $\frac{d}{dt}A_\mu$, one would obtain phase factors with higher powers of A_μ.

iv) The same methods can be used for nonabelian gauge theories (of the Yang-Mills type) coupled with fermions. The aim here is to separate asymptotically the linear (modulo an infrared problem that can be a lot worse in the nonabelian case) equation for the spinors from the pure Yang-Mills equation (the A_μ part). The next step would then be to linearize analytically the pure Yang-Mills equation (that is known [7] to be formally linearizable), and then to combine all this with the deformation-quantization approach to deal rigorously with the corresponding quantum field theories.

2. The nonlinear representation T and spaces of differentiable vectors.

In this chapter we prove properties of T_Y, $Y \in U(\mathfrak{p})$, and of spaces E_i, $i \geq 0$, in order be able to use an *implicit functions theorem in Fréchet spaces.*

To begin with, we introduce the linear representation of $\mathcal{P}_0$, with differential T^1, given by equation (1.5). We shall denote in the same way the operators $T^1_X \in E^\rho$, $X \in g$, and its closure. It follows that $T^1_{P_\mu}$ are skew-adjoint, so $\exp(tT^1_{P_\mu})$, $t \in \mathbb{R}$, is unitary on E^ρ. Let b_μ (resp. β) be the solutions of equation $\Box b_\mu = 0$ (resp. $(i\gamma^\mu\partial_\mu + m)\beta = 0$) with initial conditions $a_\mu, \dot{a}_\mu$ (resp. α), where $(a, \dot{a}, \alpha) \in E^\rho$. We define the representation U^1 of $\mathcal{P}_0$ by $U^1_g(a, \dot{a}, \alpha) = (a_g, \dot{a}_g, \alpha_g)$, $(a, \dot{a}, \alpha) \in E^\rho$, $g = (n, L) \in \mathbb{R}^4 \otimes SL(2, \mathbb{C})$, and

$$a_g(\vec{y}) = \Lambda_L b(\Lambda_L^{-1}((0, \vec{y}) - n)), \tag{2.1a}$$

$$\dot{a}_g(\vec{y}) = \Lambda_L\big(\frac{\partial}{\partial y^0} b(\Lambda_L^{-1}((y^0, \vec{y}) - n)\big)_{y^0=0}, \tag{2.1b}$$

$$\alpha_g(\vec{y}) = \Gamma(L)\beta(\Lambda_L^{-1}(0, \vec{y}) - n), \tag{2.1c}$$

where $L \mapsto \Lambda_L$ is the canonical projection of $SL(2, \mathbb{C})$ onto $SO(3, 1)$ and Γ is the Dirac spinor representation of $SL(2, \mathbb{C})$. We define $\Lambda_g = \Lambda_L$.

As $T^1_{M_{0i}}$, $1 \leq i \leq 3$, is not in general skew-adjoint on E^ρ, we state the following result of which we omit the proof as it is straightforward by using Fourier decomposition:

Lemma 2.1. *$g \mapsto U^1_g$ is a strongly continuous linear representation of $\mathcal{P}_0$ on E^ρ, with $\rho > -1/2$, and $g \mapsto (\Lambda_g^{-1} \oplus \Lambda_g^{-1} \oplus I)U^1_g$ is unitary in $E^{1/2}$.*

In order prove properties of the space of differentiable vectors E^ρ_∞ of U^1, it will be useful to know that the Fourier transform of U^1 is a unitary representation after a multiplication by a C^∞-multiplier.

Theorem 2.2. *Let $h_g(p_0, \vec{p}) = (p_0/(\Lambda_g^{-1}p)_0)^{-\rho+1/2}$, $(p_0, \vec{p}) \in \mathbb{R}^4$, $g \in \mathcal{P}_0$ and $u = (a, \dot{a}, \alpha) \in E^\rho$, $\rho > -1/2$. Let,*

$$V_g u = \big(\Lambda_g^{-1} h_g(|\nabla|, -i\nabla)a_g, \Lambda_g^{-1} h_g(|\nabla|, -i\nabla)\dot{a}_g, \alpha_g\big),$$

where $(a_g, \dot{a}_g, \alpha_g) = U^1_g u$. Then $g \mapsto V_g$ is a unitary representation of $\mathcal{P}_0$ on E^ρ.The representations $(\Lambda^{-1} \oplus \Lambda^{-1} \oplus I)U^1$ on $E^{1/2}$ and V on E^ρ are unitarily equivalent. Moreover the representations U^1 on E^ρ and V on E^ρ have the same Hilbert space of C^n-vectors, namely E^ρ_n, $n \geq 0$.

Proof. The map $g \mapsto H_g$ from $\mathcal{P}_0$ to $L^\infty(\mathbb{R}^3, GL(4, \mathbb{R}))$, defined by $H_g(\vec{p}) = \Lambda_g^{-1} h_g(|\vec{p}|, \vec{p})$ is C^∞. It follows then from the definition of E^ρ that the map $g \mapsto V_g u$ is C^n if and only if this is the case for the map $g \mapsto U^1_g u$. By construction, the space of C^n-vectors for U^1 is E^ρ_n. This proves the last part of the proposition. By direct calculation one finds that for $u \in S(\mathbb{R}^3, \mathbb{R}^4) \oplus S(\mathbb{R}^3, \mathbb{R}^4) \oplus S(\mathbb{R}^3, \mathbb{C}^4)$

$$V_g u = (|\nabla|^{-\rho+1/2} \oplus |\nabla|^{-\rho+1/2} \oplus I)(\Lambda_g^{-1} \oplus \Lambda_g^{-1} \oplus I)U^1_g(|\nabla|^{\rho-1/2} \oplus |\nabla|^{\rho-1/2} \oplus I)u.$$

As $|\nabla|^{\rho-1/2} \oplus |\nabla|^{\rho-1/2} \oplus I \colon E^\rho \to E^{1/2}$ is an isomorphism and as the representation $(\Lambda^{-1} \oplus \Lambda^{-1} \oplus I)\, U^1$ is a unitary representation on $E^{1/2}$, it follows that V is unitary in E^ρ. This proves the theorem.

To be able to use the implicit functions theorem in the Fréchet space E_∞, we shall establish the existence of smoothing operators (cf. [17]). As this can be done in a general context of unitary representations, in the following lemma and theorem, V will be an arbitrary unitary representation of a Lie group $\mathcal{G}$ on a Hilbert space H and S will be the corresponding representation on H_∞ (the space of C^∞-vectors) of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ by differentiation and Π is a general basis of $\mathfrak{g}$. The only fact which is really used is that the Laplace operator for the representation V, $\mathbf{\Delta} = \sum_{X \in \Pi} (S_X)^2$ in H with domain H_∞ is essentially self-adjoint (see [22] Theorem 4.4.4.3). Let $\overline{\mathbf{\Delta}}$ denote the closure of $\mathbf{\Delta}$.

Lemma 2.3. *The domain of $(1 - \overline{\mathbf{\Delta}})^{n/2}$ is H_n (the space of C^n-vectors of V), and the norms $\|\cdot\|_{H_n}$ and $\|(1 - \overline{\mathbf{\Delta}})^{n/2} \cdot \|_H$ are equivalent, i.e.*

$$C_n^{-1} \|f\|_{H_n} \leq \|(1 - \overline{\mathbf{\Delta}})^{n/2} f\|_H \leq C_n \|f\|_{H_n}, \quad C_n > 0, f \in H_n, n \geq 0.$$

Proof. Let $\Pi = \{X_1, \ldots, X_r\}$, $r = \dim \mathfrak{g}$. The statement is true for $n = 0$. Let $n \geq 1$ and let $f \in H_\infty$. Then

$$\begin{aligned} \|f\|^2_{H_n} &\leq \|f\|^2_{H_{n-1}} + \sum_{1 \leq i_1, \ldots, i_n \leq r} \|S_{X_{i_1} \cdots X_{i_n}} f\|^2 \\ &= \|f\|^2_{H_{n-1}} + \sum_{1 \leq i_1, \ldots, i_n \leq r} (-1)^n (f, S_{X_{i_n} \cdots X_{i_1} X_{i_1} \cdots X_{i_n}} f), \end{aligned} \tag{2.2}$$

as S_X, $X \in \mathfrak{g}$, is skew-symmetric on H_∞. By successive commutations we obtain

$$\begin{aligned} (-1)^n \sum_{1 \leq i_1, \ldots, i_n \leq r} & X_{i_n} \cdots X_{i_1} X_{i_1} \cdots X_{i_n} \\ &= (-1)^n \sum_{1 \leq i_1, \ldots, i_n \leq r} (X_{i_1})^2 \cdots (X_{i_n})^2 + A_{2n-1} + \cdots + A_1 \\ &= (-\mathbf{\Delta})^n + A_{2n-1} + \cdots + A_1, \end{aligned} \tag{2.3}$$

where A_l is a (noncommutative) polynomial of degree l in $X_j \in \Pi$, $1 \leq j \leq r$. As S_{X_j} is a skew-symmetric operator on H_∞, we have

$$|(f, S_{A_l} f)| \leq C \|f\|_{H_q} \|f\|_{H_{l-q}}, \quad 0 \leq q \leq l, \tag{2.4}$$

where C depends on A_l.

Equality (2.3) and inequality (2.4) give

$$\begin{aligned} \sum_{1 \leq i_1, \ldots, i_n \leq r} \|S_{X_{i_1} \cdots X_{i_n}} f\|^2 &\leq (f, (-\mathbf{\Delta})^n f) + C_n \|f\|_{H_n} \|f\|_{H_{n-1}} \\ &\leq (f, (1 - \mathbf{\Delta})^n f) + C_n \|f\|_{H_n} \|f\|_{H_{n-1}}. \end{aligned}$$

It follows from the last inequality and (2.2) that

$$\begin{aligned}\|f\|^2_{H_n} &\le (f,(1-\mathbf{\Delta})^n f) + C_n\|f\|_{H_n}\|f\|_{H_{n-1}} + \|f\|^2_{H_{n-1}} \\ &\le (f,(1-\mathbf{\Delta})^n f) + 2C'_n\|f\|_{H_n}\|f\|_{H_{n-1}}.\end{aligned} \tag{2.5}$$

Assuming that we have proved that

$$\|f\|_{H_l} \le C_l\|(1-\overline{\mathbf{\Delta}})^{l/2}f\|_H, \quad f\in H_\infty, 0\le l\le n-1,$$

which is true for $l=0$, we get by induction using (2.5) that

$$\|f\|^2_{H_n} \le \|(1-\overline{\mathbf{\Delta}})^{n/2}f\|^2_H + 2C_n\|f\|_{H_n}\|(1-\overline{\mathbf{\Delta}})^{\frac{n-1}{2}}f\|_H$$

for some C_n. This inequality implies

$$\|f\|_{H_n} \le \big(\|(1-\overline{\mathbf{\Delta}})^{n/2}f\|^2_H + C_n^2\|(1-\overline{\mathbf{\Delta}})^{\frac{n-1}{2}}f\|^2_H)^{1/2} + C_n\|(1-\overline{\mathbf{\Delta}})^{\frac{n-1}{2}}f\|_H,$$

which by induction proves that (redefining C_n)

$$\|f\|_{H_n} \le C_n\|(1-\overline{\mathbf{\Delta}})^{n/2}f\|_H, \quad f\in H_\infty, n\ge 0. \tag{2.6}$$

To prove the second inequality in the theorem for $f\in H_\infty$ we observe that $(1-\mathbf{\Delta})^n = S_{A'_{2n}}$, where A'_{2n} is a noncommutative polynomial of degree $2n$ in $X_j\in\Pi$, $1\le j\le r$. It follows now as in (2.4) that

$$\|(1-\overline{\mathbf{\Delta}})^{n/2}f\|^2_H = (f,(1-\mathbf{\Delta})^n f) = (f, S_{A'_{2n}}f) \le C_n^2\|f\|^2_{H_n},$$

for some C_n.

This inequality and (2.6) give

$$C_n^{-1}\|f\|_{H_n} \le \|(1-\overline{\mathbf{\Delta}})^{n/2}f\|_H \le C_n\|f\|_{H_n}, \quad f\in H_\infty, n\ge 0. \tag{2.7}$$

As $(1-\overline{\mathbf{\Delta}})^{n/2}$ is a maximal closed operator, it follows from (2.7) that the domain of $(1-\overline{\mathbf{\Delta}})^{n/2}$ is H_n. By continuity (2.7) is then true for $f\in H_n$, which proves the lemma.

We can now easily prove the existence of smoothing operators for the sequence H_i, $i\ge 0$, by using the spectral decomposition of $(1-\overline{\mathbf{\Delta}})^{1/2}$.

Theorem 2.4. *We denote by $L_b(H,H_\infty)$ the space of linear continuous mappings from H to H_∞ endowed with the topology of uniform convergence on bounded sets. There exists a C^∞ one-parameter family $\Gamma_\tau\in L_b(H,H_\infty)$, $\tau>0$, such that if $f\in H_l$, $l\ge 0$, then*

i) $\|\Gamma_\tau f\|_{H_n} \le C_{n,l}\|f\|_{H_l}, \quad n\le l,$

ii) $\|\Gamma_\tau f\|_{H_n} \le C_{n,l}\ \tau^{n-l}\|f\|_{H_l}, \quad n\ge l,$

iii) $\|(1-\Gamma_\tau)f\|_{H_n} \le C_{n,l}\ \tau^{n-l}\|f\|_{H_l}, \quad n\le l,$

iv) $\|\frac{d}{d\tau}\Gamma_\tau f\|_{H_n} \le C_{n,l}\ \tau^{n-l-1}\|f\|_{H_l}, \quad n\ge 0, l\ge 0.$

Proof. Let $(1-\overline{\mathbf{\Delta}})^{1/2}$ denote the positive self-adjoint square root of the positive self-adjoint operator $(1-\overline{\mathbf{\Delta}}) \geq 1$. Since, according to Lemma 2.3, the norms $\|\cdot\|_{H_n}$ and $\|(1-\overline{\mathbf{\Delta}})^{n/2}\cdot\|_H$ are equivalent, it is enough to prove i)–iv) with $\|\cdot\|_{H_n}$ replaced by $\|(1-\overline{\mathbf{\Delta}})^{n/2}\cdot\|_H$.

Let

$$(1-\overline{\mathbf{\Delta}})^{1/2} = \int_1^\infty \lambda \, de_\lambda, \tag{2.8}$$

where $\lambda \mapsto e_\lambda$ is the spectral resolution of $(1-\overline{\mathbf{\Delta}})^{1/2}$ and let $\varphi \in C_0^\infty(\mathbb{R})$, where $\varphi(s) = 1$ for $|s| \leq 1$, $\varphi(s) = 0$ for $|s| \geq 2$ and $0 \leq \varphi(s) \leq 1$ for $s \in \mathbb{R}$. We define Γ_τ, $\tau > 0$, by

$$\Gamma_\tau f = \int_1^\infty \varphi(\lambda/\tau) d(e_\lambda f), \quad f \in H, \tau > 0. \tag{2.9}$$

Since the map $\tau \mapsto \varphi_\tau$, $\varphi_\tau(\lambda) = \varphi(\lambda/\tau)$, is C^∞ from $]0,\infty[$ to $L^\infty(\mathbb{R})$, it follows using (2.9) that the map $\tau \mapsto \Gamma_\tau$ is C^∞ from $]0,\infty[$ to $L_b(H, H_\infty)$.

We have for f belonging to the domain of $(1-\overline{\mathbf{\Delta}})^{l/2}$,

$$\begin{aligned} \|(1-\overline{\mathbf{\Delta}})^{n/2}\Gamma_\tau f\|_H^2 &= \int_1^\infty \lambda^{2n} (\varphi(\lambda/\tau))^2 d\|e_\lambda f\|_H^2 \\ &= \int_1^\infty \lambda^{2(n-l)} (\varphi(\lambda/\tau))^2 \lambda^{2l} d\|e_\lambda f\|_H^2 \\ &\leq \sup_{\lambda \geq 1} \big(\lambda^{2(n-l)} (\varphi(\lambda/\tau))^2\big) \|(1-\overline{\mathbf{\Delta}})^{l/2} f\|_H, \quad n, l \geq 0. \end{aligned} \tag{2.10}$$

If $n \leq l$, then

$$\sup_{\lambda \geq 1} \big(\lambda^{2(n-l)} (\varphi(\lambda/\tau))^2\big) \leq 1, \quad \tau > 0,$$

which together with (2.10) proves statement i). If $n \geq l$, then

$$\begin{aligned} \sup_{\lambda \geq 1} \big(\lambda^{2(n-l)} (\varphi(\lambda/\tau))^2\big) &= \tau^{2(n-l)} \sup_{s \geq \tau^{-1}} s^{2(n-l)} (\varphi(s))^2 \\ &\leq \tau^{2(n-l)} \sup_{s \in \mathbb{R}} s^{2(n-l)} (\varphi(s))^2 \\ &\leq 2^{2(n-l)} \tau^{2(n-l)}, \quad \tau > 0, \end{aligned}$$

which together with (2.10) proves statement ii). We have for f belonging to the domain of $(1-\overline{\mathbf{\Delta}})^{l/2}$ and $n \leq l$,

$$\begin{aligned} \|(1-\overline{\mathbf{\Delta}})^{n/2}(1-\Gamma_\tau) f\|_H^2 &= \int_1^\infty \lambda^{2n} (1-\varphi(\lambda/\tau))^2 d\|e_\lambda f\|_H^2 \\ &= \tau^{2(n-l)} \int_1^\infty (\lambda/\tau)^{2(n-l)} (1-\varphi(\lambda/\tau))^2 \lambda^{2l} d\|e_\lambda f\|_H^2 \\ &\leq \tau^{2(n-l)} \sup_{s \in \mathbb{R}} s^{2(n-l)} (1-\varphi(s))^2 \|(1-\overline{\Delta})^{l/2} f\|_H^2. \end{aligned}$$

This inequality gives using that $\sup_{s\in\mathbb{R}} s^{2(n-l)}(1-\varphi(s))^2 \le 1$, $n \le l$,

$$\|(1-\overline{\mathbf{\Delta}})^{n/2}(1-\Gamma_\tau)f\|_H^2 \le \tau^{2(n-l)}\|(1-\overline{\mathbf{\Delta}}^{l/2}f\|_H, \quad n \le l, \tau > 0. \tag{2.11}$$

Inequality (2.11) proves statement iii).

Finally we have, since $\tau \mapsto \varphi_\tau \in L^\infty(\mathbb{R})$ is differentiable,

$$(1-\overline{\mathbf{\Delta}})^{n/2}\frac{d}{d\tau}\Gamma_\tau f = -\int_1^\infty \lambda^n \lambda/\tau^2 \varphi'(\lambda/\tau)d(e_\lambda f),$$

where φ' is the derivative of φ. If $n \ge 0$, $l \ge 0$ and f belongs to the domain of $(1-\overline{\mathbf{\Delta}})^{l/2}$, it then follows that

$$\begin{aligned}\|(1-\overline{\mathbf{\Delta}})^{n/2}\frac{d}{d\tau}\Gamma_\tau f\|_H^2 &= \tau^{-2}\int_1^\infty \lambda^{2(n-l)}(\lambda/\tau)^2(\varphi'(\lambda/\tau))^2\lambda^{2l}d\|e_\lambda f\|_H^2 \\ &\le \tau^{2(n-l-1)}\sup_{s\in\mathbb{R}}\big(s^{2(n-l+1)}(\varphi'(s))^2\big)\|(1-\overline{\mathbf{\Delta}})^{l/2}f\|_H^2.\end{aligned}$$

By the definition of φ we have $\varphi'(s) = 0$ for $|s| \le 1$ and $|s| \ge 2$. The last inequality then gives

$$\|(1-\overline{\mathbf{\Delta}})^{n/2}\frac{d}{d\tau}\Gamma_\tau f\|_H^2 \le C_{n,l}\,\tau^{2(n-l-1)}\|(1-\overline{\mathbf{\Delta}})^{l/2}f\|_H^2, \quad n,l \ge 0,$$

where $C_{n,l} = \sup_{s\in\mathbb{R}}\big(s^{2(n-l+1)}(\varphi'(s))^2\big) < \infty$. This proves the statement iv).

We are now prepared to prove the existence of a smoothing operator for the sequence of Hilbert spaces defined by (1.6).

Theorem 2.5. *There exists a C^∞ one-parameter family $\Gamma_\tau \in L_b(E, E_\infty)$, $\tau > 0$, such that if $f \in E_l$, $l \ge 0$, then statements i)–iv) of Theorem 2.4 hold with H_n and H_l replaced by E_n and E_l respectively.*

Proof. According to Theorem 2.2, E_n, $n \ge 0$, is the Hilbert space of n-differentiable vectors of the unitary representation $g \mapsto V_g$ of $\mathcal{P}_0$ in E. It follows then from Theorem 2.4 that there exists a C^∞ one-parameter family with the announced properties.

The existence of a smoothing operator guarantees that the norms $\|\cdot\|_{E_n}$ satisfy a convexity property, which we make explicit now:

Corollary 2.6. *Let $0 \le n_2 \le n \le n_1$, $n_1 \ne n_2$. Then*

$$\|f\|_{E_n} \le C_{n_1,n_2}\|f\|_{E_{n_1}}^{\frac{n-n_2}{n_1-n_2}}\|f\|_{E_{n_2}}^{\frac{n_1-n}{n_1-n_2}}, \quad f \in E_{n_1}.$$

Moreover if $N_0 \le n_i \le N$, $i = 1,2$, and $n_1 + n_2 \le N_0 + N$, then

$$\|f\|_{E_{n_1}}\|g\|_{E_{n_2}} \le C_N\big(\|f\|_{E_{N_0}}\|g\|_{E_N} + \|f\|_{E_N}\|g\|_{E_{N_0}}\big), \quad f,g \in E_N.$$

Proof. The first statement follows from Theorem 2.2.2 of [17] and statements ii) and iii) of Theorem 2.4. To prove the second statement let $n_1+n_2=N'+N$, where $N'\leq N_0$ and let $n_1=aN+(1-a)N'$, where $0\leq a\leq 1$. Then $n_2=(1-a)N+aN'$. It follows from the first statement of the corollary that

$$\|f\|_{E_{n_1}}\leq C'_N\|f\|^a_{E_N}\|f\|^{1-a}_{E_{N'}},\quad \|g\|_{E_{n_2}}\leq C'_N\|g\|^{1-a}_{E_N}\|g\|^a_{E_{N'}}.$$

This gives

$$\begin{aligned}\|f\|_{E_{n_1}}\|g\|_{E_{n_2}}&\leq C''_N(\|f\|_{E_N}\|g\|_{E_{N'}})^a(\|f\|_{E_{N'}}\|g\|_{E_N})^{1-a}\\&\leq C''_N\big(a\|f\|_{E_N}\|g\|_{E_{N'}}+(1-a)\|f\|_{E_{N'}}\|g\|_{E_N}\big).\end{aligned}$$

Since $\|f\|_{E_{N'}}\leq\|f\|_{E_{N_0}}$, $\|g\|_{E_{N'}}\leq\|g\|_{E_{N_0}}$, this proves the corollary.

To establish estimates it will be useful to have more explicit expressions for the norms $\|\cdot\|_{E_n}$, than those given in (1.6).

For $Y=X_1X_2\cdots X_L$, $L\geq 1$, and $X_1,\dots,X_L\in\Pi$, let $|Y|=L$ and $\mathcal{L}(Y)$ be the number of factors equal to $M_{\mu\nu}$, $0\leq\mu<\nu\leq 3$, in Y. Let $\mathcal{L}(Y)=0$ and $|Y|=0$, for $Y=\mathbb{I}$.

Lemma 2.7. *There exist linear maps $Y\mapsto Q_i(Y)$, $Y\mapsto R_i(Y)$, $i=1,2$, and $Y\mapsto\Gamma(Y)$ from $U(\mathfrak{p})$ into the space of differential operators on $\mathbb{R}^3$ with polynomial coefficients such that*

$$T^1_Y(u)=\big(Q_1(Y)f+R_1(Y)\dot f,R_2(Y)f+Q_2(Y)\dot f,\Gamma(Y)\alpha\big),\tag{2.12}$$

for $u=(f,\dot f,\alpha)\in E_\infty$, $Y\in U(\mathfrak{p})$. If $Y=\mathbb{I}$, then $Q_i(Y)=I$, $R_i(Y)=0$, $i=1,2$, and $\Gamma(Y)=I$, where I is the identity mapping. If $Y=X_1\cdots X_L$, $X_i\in\Pi$, $1\leq i\leq L$, $L\geq 1$, then the polynomials $Q_i(Y,x,\xi)$, $R_i(Y,x,\xi)$ and $\Gamma(Y,x,\xi)$, $x,\xi\in\mathbb{R}^3$, associated to $Q_i(Y,x,-i\nabla)$, $R_i(Y,x,-i\nabla)$ and $\Gamma(Y,x,-i\nabla)$ respectively, satisfy for $a\neq 0$:

$$Q_i(Y,a^{-1}x,a\xi)=a^{|Y|-\mathcal{L}(Y)}Q_i(Y,x,\xi),\qquad \deg Q_i(Y)\leq|Y|+\mathcal{L}(Y),\tag{2.13a}$$
$$R_1(Y,a^{-1}x,a\xi)=a^{|Y|-\mathcal{L}(Y)-1}R_1(Y,x,\xi),\quad \deg R_1(Y)\leq|Y|+\mathcal{L}(Y)-1,\tag{2.13b}$$
$$R_2(Y,a^{-1}x,a\xi)=a^{|Y|-\mathcal{L}(Y)+1}R_2(Y,x,\xi),\quad \deg R_2(Y)\leq|Y|+\mathcal{L}(Y)+1,\tag{2.13c}$$

The degree of a polynomial, denoted by deg*, is the total degree in (x,ξ). If $\deg_x$ (resp. $\deg_\xi$) denotes the degree relative to the variable x (resp. ξ), then*

$$\deg_\xi\Gamma(Y,x,\xi)\leq|Y|,\quad \deg_x\Gamma(Y,x,\xi)\leq\mathcal{L}(Y).\tag{2.14}$$

Proof. Since $T^1_{\mathbb{I}}$ is the identity operator in E, we have $Q_i(\mathbb{I})=I$, $R_i(\mathbb{I})=0$, $i=1,2$, and $\Gamma(\mathbb{I})=I$. For $Y=X_1\cdots X_L$, $X_i\in\Pi$, $L\geq 1$, we prove the lemma by induction in $|Y|=L$. For $|Y|=1$, it follows from definition (1.5) of T^1_X, $X\in\Pi$, that formula (2.12), properties (2.13) and (2.14) are satisfied. Suppose (2.12), (2.13) and (2.14) are true for $1\leq|Y|=L$. If $Y'=YX$, $X\in\Pi$, then $T^1_{Y'}=T^1_YT^1_X$, $|Y'|=|Y|+1$ and $\mathcal{L}(Y')=\mathcal{L}(Y)+\mathcal{L}(X)$.

It follows from the induction hypothesis and (2.12) that

$$
\begin{aligned}
Q_1(Y', x, -i\nabla) &= Q_1(Y, x, -i\nabla) Q_1(X, x, -i\nabla) + R_1(Y, x, -i\nabla) R_2(X, x, -i\nabla), && \text{(2.15a)}\\
Q_2(Y', x, -i\nabla) &= R_2(Y, x, -i\nabla) R_1(X, x, -i\nabla) + Q_2(Y, x, -i\nabla) Q_2(X, x, -i\nabla), && \text{(2.15b)}\\
R_1(Y', x, -i\nabla) &= Q_1(Y, x, -i\nabla) R_1(X, x, -i\nabla) + R_1(Y, x, -i\nabla) Q_2(X, x, -i\nabla), && \text{(2.15c)}\\
R_2(Y', x, -i\nabla) &= R_2(Y, x, -i\nabla) Q_1(X, x, -i\nabla) + Q_2(Y, x, -i\nabla) R_2(X, x, -i\nabla), && \text{(2.15d)}\\
\Gamma(Y', x, -i\nabla) &= \Gamma(Y, x, -i\nabla)\ \Gamma(X, x, -i\nabla). && \text{(2.15e)}
\end{aligned}
$$

Let $D_1(x,\xi)$, $D_2(x,\xi)$ be two polynomials in $x, \xi \in \mathbb{R}^3$ with $\deg D_i = d_i$, $i = 1,2$, and let

$$D_i(a^{-1}x, a\xi) = a^{\nu_i} D_i(x,\xi), \quad a \neq 0, \nu_i \in \mathbb{Z}.$$

It then follows using the Leibniz rule that there is a polynomial D in x and ξ such that $D(x, -i\nabla) = D_1(x, -i\nabla) D_2(x, -i\nabla)$ and

$$\deg D = d_1 + d_2, D(a^{-1}x, a\xi) = a^{\nu_1+\nu_2} D(x,\xi). \tag{2.16}$$

We apply (2.16) to $Q_1(Y')$ in (2.15a), which gives according to (2.13a)

$$
\begin{aligned}
\deg Q_1(Y') &\leq \max\big(\deg Q_1(Y) + \deg Q_1(X), \deg R_1(Y) + \deg R_2(X)\big)\\
&\leq \max\big(|Y| + \mathcal{L}(Y) + |X| + \mathcal{L}(X), |Y| + \mathcal{L}(Y) - 1 + |X| + \mathcal{L}(X) + 1\big)\\
&= |Y| + |X| + \mathcal{L}(Y) + \mathcal{L}(X) = |Y'| + \mathcal{L}(Y')
\end{aligned}
$$

and

$$
\begin{aligned}
Q_1(Y', a^{-1}x, a\xi) &= a^{|Y| - \mathcal{L}(Y) + |X| - \mathcal{L}(X)} Q_1(Y', x, \xi)\\
&= a^{|Y'| - \mathcal{L}(Y')} Q_1(Y', x, \xi).
\end{aligned}
$$

This proves that (2.13a) is true for $L+1$ and $i = 1$. Application of (2.16) to $Q_2(Y')$, $R_1(Y')$, $R_2(Y')$, $\Gamma(Y')$ in (2.15b)–(2.15e) proves similarly that (2.13a)–(2.13c) are true for $L+1$.

Let $M_i(x,\xi)$, $i = 1,2$, be two polynomials in $x, \xi \in \mathbb{R}^3$. Using Liebniz rule it follows that there is a polynomial $M(x,\xi)$ such that $M(x, -i\nabla) = M_1(x, -i\nabla) M_2(x, -i\nabla)$ and that

$$\deg_\xi M = \deg_\xi M_1 + \deg_\xi M_2, \quad \deg_x M = \deg_x M_1 + \deg_x M_2. \tag{2.17}$$

Formula (2.15e) and relation (2.17) prove (2.14) for $L+1$. This proves the lemma.

Lemma 2.8. *If $u = (f, \dot{f}, \alpha) \in E_1^\rho$, $\frac{1}{2} < \rho < 1$, then*

$$\Big(\sum_{0 \leq |\beta| \leq |\delta| \leq 1} \|M_\beta \partial^\delta (f, \dot{f})\|^2_{M^\rho} + \sum_{\substack{|\beta| \leq 1\\ |\delta| \leq 1}} \|M_\beta \partial^\delta \alpha\|^2_{L^2}\Big)^{\frac{1}{2}} \leq C \|u\|_{E_1},$$

for some constant C depending only on ρ. Here $M_\beta(x) = x^\beta = x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}$.

Proof. Let $g \mapsto V_g$ be the unitary representation of $\mathcal{P}_0$ in E^ρ defined in Theorem 2.2. Let $X \mapsto \xi_X$, be the corresponding Lie algebra representation of $\mathfrak{p}$ in the space of differential vectors of V, which according to Theorem 2.2 is E_∞. We write

$$\xi_X u = (\xi^M_X(f,\dot f), \xi^D_X \alpha), \quad u = (f, \dot f, \alpha) \in E_\infty. \tag{2.18}$$

Since, by Theorem 2.2, the Hilbert space of C^1-vectors of V is E_1, we have

$$\Big(\|u\|^2_E + \sum_{X \in \Pi} \|\xi_X u\|^2_E\Big)^{\frac{1}{2}} \leq C\|u\|_{E_1}, \quad u \in E_\infty, \tag{2.19}$$

for some constant C.

We shall prove the lemma by giving an upper bound for the Laplacian $\mathbf{\Delta}_E$ of the representation V:

$$\mathbf{\Delta}_E u = \sum_{X \in \Pi} \xi^2_X u = (\mathbf{\Delta}_M(f,\dot f), \mathbf{\Delta}_D \alpha), \tag{2.20a}$$

where

$$\mathbf{\Delta}_M = \sum_{X \in \Pi} (\xi^M_X)^2, \quad \mathbf{\Delta}_D = \sum_{X \in \Pi} (\xi^D_X)^2. \tag{2.20b}$$

According to Theorem 2.2, the "Dirac part" of T^1 is identical to ξ^D. Expressions (1.5a)–(1.5d) give after rearrangement of the terms

$$\begin{aligned}
\mathbf{\Delta}_D &= \mathcal{D}^2 + \sum_{1 \leq i \leq 3} \partial_i^2 + \sum_{1 \leq i \leq j \leq 3} (x_i\partial_j - x_j\partial_i + \sigma_{ij})^2 + \sum_{1 \leq i \leq 3} (x_i\mathcal{D} + \sigma_{0i})^2 \\
&= 2\Delta - m^2 + \sum_{i<j} \big((x_i\partial_j - x_j\partial_i)^2 + 2(x_i\partial_j - x_j\partial_i)\sigma_{ij} + \sigma^2_{ij}\big) \\
&\quad + \sum_i \big(x_i\mathcal{D}^2 x_i + x_i\mathcal{D}[x_i,\mathcal{D}] + x_i\mathcal{D}\sigma_{0i} + x_i\sigma_{0i}\mathcal{D} + \sigma^2_{0i}\big).
\end{aligned}$$

Using the fact that

$$[x_i, \mathcal{D}] = -\gamma_0\gamma_i = -2\sigma_{0i}, \quad (\sigma_{ij})^2 = -\frac{1}{4}, \quad (\sigma_{0i})^2 = \frac{1}{4},$$

we obtain

$$\begin{aligned}
\mathbf{\Delta}_D &= 2\Delta - m^2 + \sum_{i<j} \big((x_i\partial_j - x_j\partial_i)^2 + 2(x_i\partial_j - x_j\partial_i)\sigma_{ij} - \frac{1}{4}\big) \\
&\quad + \sum_i \big(x_i(\Delta - m^2)x_i + x_i[\sigma_{0i}, \mathcal{D}] + \frac{1}{4}\big).
\end{aligned}$$

A direct calculation gives

$$\begin{aligned}
[\sigma_{0j}, \mathcal{D}] &= im\gamma_j + \sum_l \frac{1}{2}(\gamma_j\gamma_l - \gamma_l\gamma_j)\partial_l \\
&= im\gamma_j - 2\sum_{l \neq j} \sigma_{jl}\partial_l,
\end{aligned}$$

which shows that

$$\sum_j x_j[\sigma_{0j}, \mathcal{D}] = im \sum_j x_j \gamma_j - 2 \sum_{j<l} \sigma_{jl}(x_j \partial_l - x_l \partial_j).$$

Therefore

$$\mathbf{\Delta}_D = 2\Delta - m^2 + \sum_{i<j} (x_i \partial_j - x_j \partial_i)^2 + \sum_i x_i(\Delta - m^2) x_i + im \sum_l x_l \gamma_l. \tag{2.21}$$

Using the fact that

$$\sum_{i<j} (x_i \partial_j - x_j \partial_i)^2 = \sum_{i,j} (\partial_j x_i x_i \partial_j - \partial_i x_i x_j \partial_j), \tag{2.22a}$$

and that

$$\sum_i x_i \Delta x_i = \sum_{i,j} \partial_j x_i x_i \partial_j, \tag{2.22b}$$

we obtain

$$\mathbf{\Delta}_D = -m^2 + 2\Delta - m^2 |x|^2 + 3 \sum_{i,j} \partial_j x_i x_i \partial_j - \sum_{i,j} \partial_i x_i x_j \partial_j + im \sum_l x_l \gamma_l. \tag{2.23}$$

It follows from (2.23) that

$$\begin{aligned}(\alpha, -\mathbf{\Delta}_D \alpha) &= m^2 \|\alpha\|^2_{L^2} + 2 \sum_i \|\partial_i \alpha\|^2_{L^2} + m^2 \sum_i \|x_i \alpha\|^2_{L^2} \\ &\quad + 3 \sum_{i,j} \|x_i \partial_j \alpha\|^2_{L^2} - \| \sum_i x_i \partial_i \alpha\|^2_{L^2} - m \sum_j (\alpha, i\gamma_j x_j \alpha) \\ &\geq m^2 \Big(\|\alpha\|^2_{L^2} + \sum_i \|x_i \alpha\|^2_{L^2} \Big) + 2 \sum_i \|\partial_i \alpha\|^2_{L^2} \\ &\quad + 2 \sum_{i,j} \|x_i \partial_j \alpha\|^2_{L^2} - m \sum_j \|x_j \alpha\|_{L^2} \|\alpha\|_{L^2}.\end{aligned} \tag{2.24}$$

As pointed out in the proof of Theorem 2.2, the unitary representation V in E^ρ is equivalent to a unitary representation V' in $E^{1/2}$ by the isomorphism $|\nabla|^{\rho - \frac{1}{2}} \oplus I \colon M^\rho \oplus D \to M^{1/2} \oplus D$. V is a direct sum: $V = V^{M^\rho} \oplus V^D$ and $V' = V^{M^{1/2}} \oplus V^D$. The generators of $V^{M^{1/2}}$ are given by

$$\begin{aligned}\xi^{M^{1/2}}_{P_0}(f, \dot f) &= (\dot f, \Delta f), \\ \xi^{M^{1/2}}_{P_i} &= \partial_i, \quad 1 \leq i \leq 3, \\ \xi^{M^{1/2}}_{M_{ij}} &= m_{ij}, \quad 1 \leq i \leq j \leq 3, \\ m_{ij} &= -x_i \partial_j + x_j \partial_i, \\ \xi^{M^{1/2}}_{M_{0i}}(f, \dot f) &= \Big(x_i \dot f, \sum_{j=1}^3 \partial_j x_i \partial_j f \Big), \quad 1 \leq i \leq 3.\end{aligned} \tag{2.25}$$

The Laplacian $\mathbf{\Delta}_{M^{1/2}}$ is diagonal. Let $\mathbf{\Delta}_{M^{1/2}}(f,\dot{f}) = (\mathbf{\Delta}^{(1)}_{M^{1/2}}f, \mathbf{\Delta}^{(2)}_{M^{1/2}}\dot{f})$. Using (2.25) a direct calculation gives:

$$\mathbf{\Delta}^{(1)}_{M^{1/2}} = 2\Delta + \sum_{i,j}(2\partial_j x_i x_i \partial_j - \partial_i x_i x_j \partial_j) - \sum_i x_i \partial_i, \tag{2.26a}$$

$$\mathbf{\Delta}^{(2)}_{M^{1/2}} = 2\Delta + \sum_{i,j}(2\partial_j x_i x_i \partial_j - \partial_i x_i x_j \partial_j) + \sum_i x_i \partial_i. \tag{2.26b}$$

The equivalence of V^{M^ρ} on M^ρ and $V^{M^{1/2}}$ on $M^{1/2}$ shows that $\mathbf{\Delta}_{M^\rho} = (\mathbf{\Delta}^{(1)}_{M^\rho}, \mathbf{\Delta}^{(2)}_{M^\rho}) = |\nabla|^{\frac{1}{2}-\rho}\mathbf{\Delta}_{M^{1/2}}|\nabla|^{\rho-\frac{1}{2}}$. This fact shows that if $v=(f,\dot{f})\in M^\rho_\infty$:

$$(v, \mathbf{\Delta}_{M^\rho}v)_{M^\rho} = (|\nabla|^{\rho+\frac{1}{2}}f, \mathbf{\Delta}^{(1)}_{M^{1/2}}|\nabla|^{\rho-\frac{1}{2}}f) + (|\nabla|^{\rho-\frac{3}{2}}\dot{f}, \mathbf{\Delta}^{(2)}_{M^{1/2}}|\nabla|^{\rho-\frac{1}{2}}\dot{f}) \tag{2.27}$$

Since M^ρ is a real vector space and $[x_i\partial_j, |\nabla|^a] = a|\nabla|^{a-2}\partial_i\partial_j$, it follows from (2.26) and (2.27) that

$$\begin{aligned}(v, -\mathbf{\Delta}_{M^\rho}v)_{M^\rho} = 2\|\nabla v\|^2_{M^\rho} + 2\sum_{i,j}\|x_i\partial_j v\|^2_{M^\rho} - \|\sum_i x_i\partial_i v\|^2_{M^\rho} \\ + C_1(\rho)\||\nabla|^\rho f\|^2_{L^2} + C_2(\rho)\||\nabla|^{\rho-1}\dot{f}\|^2_{L^2},\end{aligned} \tag{2.28}$$

where $C_1(\rho)$ and $C_2(\rho)$ are two real numbers. This gives the inequality

$$(v, -\mathbf{\Delta}_{M^\rho}v)_{M^\rho} \geq 2\|\nabla v\|^2_{M^\rho} + \sum_{i,j}\|x_i\partial_j v\|^2_{M^\rho} + C(\rho)\|v\|^2_{M^\rho}, \tag{2.29}$$

where $C(\rho) = \min(C_1(\rho), C_2(\rho))$. We define the norm q_1 by

$$q_1(u) = \Big(\sum_{|\beta|\leq|\delta|\leq 1}\|M_\beta\partial^\delta(f,\dot{f})\|^2_{M^\rho} + \sum_{\substack{|\beta|\leq 1\\ |\delta|\leq 1}}\|M_\beta\partial^\delta\alpha\|^2_{L^2}\Big)^{\frac{1}{2}} \tag{2.30}$$

for $u=(f,\dot{f},\alpha)\in E^\rho_\infty$.

It follows from inequalities (2.24) and (2.29) that

$$(u, -\mathbf{\Delta}_E u)_{E^\rho} \geq \mu(\rho)^2(q_1(u))^2 - 2\nu_1(\rho)q_1(u)\|u\|_E - \nu_2(\rho)^2\|u\|^2_E$$

where $\mu(\rho)>0$ and $\nu_i(\rho)\geq 0$, $\nu_2(\rho)\geq 0$. This inequality gives that

$$q_1(u) \leq \frac{1}{\mu}\big((u, -\mathbf{\Delta}_E u) + (\nu_1^2+\nu_2^2)\|u\|^2_E\big) + \nu_1\|u\|_E, \quad \mu>0,$$

where we have omitted to indicate the dependence on ρ of the constants. ξ_X, $X\in\mathfrak{p}$, is skew-symmetric with domain E_∞, so $\sum_{X\in\Pi}\|\xi_X u\|^2_E = (u, -\mathbf{\Delta}_E u)$, $u\in E_\infty$. This fact and the last inequality show that

$$q_1(u) \leq C\Big(\sum_{X\in\Pi}\|\xi_X u\|^2_E + \|u\|^2_E\Big)^{\frac{1}{2}}, \quad u\in E_\infty, \tag{2.31}$$

for some constant C. It follows from inequalities (2.19) and (2.31) that (with a new constant C)

$$q_1(u) \leq C\|u\|_{E_1}, \quad u \in E_\infty.$$

By continuity this inequality is true for $u \in E_1$, which proves the lemma.

To state the next theorem we introduce the following *seminorms* q_n, $n \geq 0$, on E_∞:

$$(q_n(u))^2 = (q_n^M(v))^2 + (q_n^D(\alpha))^2, \quad u = (v, \alpha) \in E_\infty, v \in M_\infty, \alpha \in D_\infty, \tag{2.32a}$$

$$q_n^M(v) = \Big(\sum_{|\mu| \leq |\nu| \leq n} \|M^\mu \partial^\nu v\|_M^2 \Big)^{\frac{1}{2}}$$

and

$$q_n^D(\alpha) = \Big(\sum_{\substack{|\mu| \leq n \\ |\nu| \leq n}} \|M^\mu \partial^\nu \alpha\|_D^2 \Big)^{\frac{1}{2}}, \tag{2.32b}$$

where μ and ν are multiindices and M^μ is defined as in Lemma 2.8. As will be proved, q_n has a continuous extension to E_n.

Theorem 2.9. *There exist constants $C_n > 0$ such that*

$$C_n^{-1}\|u\|_{E_n} \leq q_n(u) \leq C_n\|u\|_{E_n}, \quad n \geq 0.$$

Moreover the linear space

$$E_c = M_c \oplus D_c = \{(f, \dot{f}, \alpha) \in E_\infty \big| \hat{f}, \hat{\dot{f}} \in C_0^\infty(\mathbb{R}^3 - \{0\}, \mathbb{C}^4), \hat{\alpha} \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)\}$$

is dense in E_∞.

Proof. According to definition (1.6a) of $\|\cdot\|_{E_n}$ and expression (2.12) for $T_Y^1 u$, $Y \in \Pi'$, $u = (f, \dot{f}, \alpha) \in E_\infty$ of Lemma 2.7 we have

$$\begin{aligned}\|u\|_{E_n}^2 = \sum_{\substack{Y \in \Pi' \\ |Y| \leq n}} \Big(&\|\Gamma(Y)\alpha\|_{L^2}^2 + \||\nabla|^\rho (Q_1(Y)f + R_1(Y)\dot{f})\|_{L^2}^2 \\ &+ \||\nabla|^{\rho-1}(R_2(Y)f + Q_2(Y)\dot{f})\|_{L^2}^2 \Big).\end{aligned} \tag{2.33}$$

We shall estimate the terms on the right-hand side of (2.33). It follows at once from inequality (2.14) in Lemma 2.7 that

$$\|\Gamma(Y)\alpha\|_{L^2} \leq C_n q_n^D(\alpha), \quad Y \in \Pi', |Y| \leq n, \tag{2.34}$$

and from (2.13a) that

$$\big(\||\nabla|^\rho Q_1(Y)f\|_{L^2}^2 + \||\nabla|^{\rho-1} Q_2(Y)\dot{f}\|_{L^2}^2\big)^{\frac{1}{2}} \leq C_n q_n^M(f, \dot{f}), \quad Y \in \Pi', |Y| \leq n. \tag{2.35}$$

We observe that
$$\||\nabla|^\rho R_1(Y)\dot f\|^2_{L^2} = \sum_{1\le j\le 3} \||\nabla|^{\rho-1}\partial_j R_1(Y)\dot f\|^2_{L^2}.$$
We have $\partial_j R_1(Y)\dot f = R_{1j}(Y)\dot f + P_{1j}(Y)\dot f$, where
$$R_{1j}(Y,x,\xi) = \frac{\partial}{\partial x_j} R_1(Y,x,\xi), P_{1j}(Y,x,\xi) = R_1(Y,x,\xi) i\xi_j.$$
Since R_1 satisfies (2.13b), the polynomials R_{1i} and P_{1i} satisfy condition (2.13a) of Q_i. This shows that
$$\||\nabla|^\rho R_1(Y)\dot f\|^2_{L^2} \le C_n \sum_{|\mu|\le|\nu|\le n} \||\nabla|^{\rho-1} M_\mu \partial^\nu \dot f\|^2_{L^2}, \quad Y\in\Pi', |Y|\le n. \tag{2.36}$$

It follows from (2.13c) that $R_2(Y,x,\xi) = \sum_{i\le j\le 3} r_j^{(1)}(Y,x,\xi) i\xi_j$, where $r_j^{(1)}$ satisfies condition (2.13a) of Q_i. Then $R_2(Y)f = \sum_{1\le i\le 3} r_j^{(1)}(Y)\partial_j f = \sum_{1\le i\le 3} \partial_j r_j^{(1)}(Y) f + R_2^{(1)}(Y)f$, where $R_2^{(1)}(Y,x,\xi), r_j^{(1)}(Y,x,\xi)$, once more satisfies the homogeneity condition in (2.13c) and $\deg R_2^{(1)} \le |Y| + \mathcal{L}(Y) - 1$, with $R_2^{(1)} = 0$ if this number is strictly smaller than one. We now repeat this argument with $R_2^{(l+1)}$ instead of $R_2^{(l)}, R_2^{(0)} = R_2$ until we have $R_2^{(L)} = 0$ for some $L \le \mathcal{L}(Y)+1$. The corresponding sequence $r^{(1)}, \dots, r^{(L)}$, gives
$$R_2(Y)f = \sum_{1\le j\le 3} \partial_j r_j(Y)f, r_j(Y,x,\xi) = \sum_{l=1}^{L} r_j^{(l)}(Y,x,\xi), \tag{2.37}$$
where r_j satisfies (2.13a). Equalities (2.37) and (2.13a) show that
$$\begin{aligned}\||\nabla|^{\rho-1} R_2(Y)f\|_{L^2} &\le \sum_j \||\nabla|^{\rho-1}\partial_j r_j(Y)f\|_{L^2} \\ &\le \sum_j \||\nabla|^\rho r_j(Y)f\|_{L^2} \\ &\le C_n\Big(\sum_{|\mu|\le|\nu|\le n} \||\nabla|^\rho M_\mu\partial^\nu f\|^2_{L^2}\Big)^{\frac12}, \quad Y\in\Pi', |Y|\le n.\end{aligned} \tag{2.38}$$

It follows from expression (2.33) of $\|u\|^2_{E_n}$ and inequalities (2.34), (2.35), (2.36) and (2.38) that $\|u\|_{E_n} \le C_n\, q_n(u)$, $u\in E_\infty$, $n\ge 0$, for some constants C_n. This proves the first inequality of the theorem.

To prove the second inequality of the theorem, by induction we note that $q_0(u) = \|u\|_{E_0} = \|u\|_E$ by definition and suppose that $q_i(u) \le C_i\|u\|_{E_i}$ for $0\le i\le n$. It follows from definition (2.32a) of q_n^M that
$$(q^M_{n+1}(v))^2 = (q^M_n(v))^2 + \sum_{\substack{|\mu|\le n\\ |\nu|=n+1}} \|M_\mu\partial^\nu v\|^2_M + \sum_{\substack{|\mu|=n+1\\ |\nu|=n+1}} \|M_\mu\partial^\nu v\|^2_M, \tag{2.39}$$

where $v \in M_\infty$. We estimate the two last two terms on the right-hand side of this equality. According to the induction hypothesis

$$\begin{aligned}\sum_{\substack{|\mu|\leq n\\|\nu|=n+1}} \|M_\mu \partial^\nu v\|_M^2 &= C_n'' \sum_{1\leq i\leq 3} \sum_{\substack{|\mu|\leq n\\|\nu|=n}} \|M_\mu \partial^\nu \partial_i v\|_M^2 \\ &\leq C_n'' \sum_{1\leq i\leq 3} (q_n^M(\partial_i v))^2 \\ &\leq C_n'' C_n^2 \sum_{1\leq i\leq 3} \|T_{P_i}^{1M} v\|_{M_n}^2 \\ &\leq C_n' \|v\|_{M_{n+1}}^2,\end{aligned} \tag{2.40}$$

for some constants C_n' and C_n''. For the term with $|\mu| = n+1$, and $|\nu| = n+1$ it follows, using the same argument which led to (2.37), that

$$M_\mu \partial^\nu = \sum_{l,k} x_l \partial_k r_{lk}^{\mu\nu}, \quad r_{lk}^{\mu\nu}(x,\xi) = r_{lk}^{\mu\nu}(a^{-1}x, a\xi), \quad \deg r_{lk} \leq 2n. \tag{2.41}$$

This and Lemma 2.8, with $\alpha = 0$, show that, for $|\mu| = |\nu| = n+1$,

$$\|M_\mu \partial^\nu v\|_M \leq \sum_{l,k} \|x_l \partial_k r_{lk}^{\mu\nu} v\|_M \leq C \sum_{l,k} \|r_{lk}^{\mu\nu} v\|_{M_1}, \tag{2.42}$$

$|\mu| = |\nu| = n+1$, where $r_{lk}^{\mu\nu} = r_{lk}^{\mu\nu}(x, -i\nabla)$. It follows from property (2.41) of $r_{lk}^{\mu\nu}$, the definition of q^M and (2.42) that for $|\mu| = |\nu| = n+1$, and suitable constants C, C_n and C' that

$$\begin{aligned}\|M_\mu \partial^\nu v\|_M &\leq C \sum_{l,k} \Big(\sum_{X\in\Pi} \|T_X^{1M} r_{lk}^{\mu\nu} v\|_M^2 + \|v\|_M^2 \Big)^{\frac{1}{2}} \\ &\leq C' \sum_{l,k} \Big(\sum_{X\in\Pi} (\|r_{lk}^{\mu\nu} T_X^{1M} v\|_M + \|[T_X^{1M}, r_{lk}^{\mu\nu}]v\|_M) + \|v\|_M \Big) \\ &\leq C_n \sum_{X\in\Pi} q_n(T_X^{1M} v) + C' \sum_{l,k} \|[T_X^{1M}, r_{lk}^{\mu\nu}]v\|_M,\end{aligned}$$

where $r_{lk}^{\mu\nu} = r_{lk}^{\mu\nu}(x, -i\nabla)$.

According to the induction hypothesis this gives with new constants

$$\begin{aligned}\|M_\mu \partial^\nu v\|_M &\leq C_n' \Big(\sum_{X\in\Pi} \Big(\|T_X^{1M} v\|_{M_n} + C' \sum_{l,k} \|[T_X^{1M}, r_{lk}^{\mu\nu}]v\|_M \Big)\Big) \\ &\leq C_n \|v\|_{M_{n+1}} + C \sum_{X\in\Pi} \sum_{l,k} \|[T_X^{1M}, r_{lk}^{\mu\nu}]v\|_M,\end{aligned} \tag{2.43}$$

$|\mu| = |\nu| = n+1$, $r^{\mu\nu}_{lk} = r^{\mu\nu}_{lk}(x, -i\nabla)$. Let $v = (f, \dot f) \in M_\infty$ and let r be one of the partial differential operators $r^{\mu\nu}_{lk}$, $|\mu| = |\nu| = n+1$. Definition (1.5) of T^1 restricted to M_∞ give for $v = (f, \dot f)$

$$\begin{aligned}
&\sum_{X\in\Pi} \|[T^{1M}_X, r]v\|_M \\
&\quad \le C\Big(\sum_i \|[\partial_i, r]v\|_M + \sum_{i<j}\|[x_i\partial_j - x_j\partial_i, r]v\|_M + \||\nabla|^{\rho-1}[\Delta, r]f\|_{L^2} \\
&\qquad + \sum_i \||\nabla|^{\rho}[x_i, r]\dot f\|_{L^2} + \sum_i \||\nabla|^{\rho-1}[\sum_l \partial_l x_i \partial_l, r]f\|_{L^2}\Big), \quad r = r(x, -i\nabla).
\end{aligned} \tag{2.44}$$

By the same argument which led to (2.37) it follows that there exist polynomials $p(X, x, \xi)$, for $X = P_\mu$ and for $X = M_{ij}$, $1 \le i < j \le 3$, and polynomials $p^{(i)}(X, x, \xi)$, for $X = M_{0j}$, $i = 1, 2$, and $1 \le j \le 3$, such that

$$\begin{aligned}
[\partial_i, r(x, -i\nabla)] &= p(P_i, x, -i\nabla), \\
[\Delta, r(x, -i\nabla)] &= p(P_0, x, -i\nabla), \\
[x_i\partial_j - x_j\partial_i, r(x, -i\nabla)] &= p(M_{ij}, x, -i\nabla), \quad 1 \le i < j \le 3, \\
[x_i, r(x, -i\nabla)] &= p^{(1)}(M_{0i}, x, -i\nabla), \\
[\sum_l \partial_l x_i \partial_l, r(x, -i\nabla)] &= p^{(2)}(M_{0i}, x, -i\nabla).
\end{aligned} \tag{2.45}$$

Since, according to (2.41) r satisfies $r(a^{-1}x, a\xi) = r(x, \xi)$ and $\deg r \le 2n$, it follows from (2.45) that

$$p(P_i, a^{-1}x, a\xi) = ap(P_i, x, \xi), \qquad \deg p(P_i) \le 2n-1, \tag{2.46a}$$
$$p(P_0, a^{-1}x, a\xi) = a^2 p(P_0, x, \xi), \qquad \deg p(P_0) \le 2n, \tag{2.46b}$$
$$p(M_{ij}, a^{-1}x, a\xi) = p(M_{ij}, x, \xi), \qquad \deg p(M_{ij}) \le 2n, \tag{2.46c}$$
$$p^{(1)}(M_{0i}, a^{-1}x, a\xi) = a^{-1}p^{(1)}(M_{0i}, x, \xi), \qquad \deg p^{(1)}(M_{0i}) \le 2n-1, \tag{2.46d}$$
$$p^{(2)}(M_{0i}, a^{-1}x, a\xi) = ap^{(2)}(M_{0i}, x, \xi), \qquad \deg p^{(2)}(M_{0i}) \le 2n+1. \tag{2.46e}$$

It follows from (2.45), (2.46a) and (2.46c) that

$$\sum_i \|[\partial_i, r]v\|_M + \sum_{i<j}\|[x_i\partial_j - x_j\partial_i, r]v\| \le C_n q^M_n(v). \tag{2.47}$$

Due to (2.46b) and (2.46e) we can write:

$$\begin{aligned}
&\text{a) } p(P_0, x, -i\nabla) = \sum_j \partial_j p_j(P_0, x, -i\nabla), \quad \deg p_j(P_0) \le 2n-1, \\
&\quad p_j(P_0, a^{-1}x, a\xi) = ap_j(P_0, x, \xi),
\end{aligned} \tag{2.48}$$

$$\begin{aligned}
&\text{b) } p^{(2)}(M_{0j}, x, -i\nabla) = \sum_l \partial_l p^{(2)}_l(M_{0j}, x, -i\nabla), \quad \deg p^{(2)}_l(M_{0j}) \le 2n, \\
&\quad p^{(2)}_l(M_{0j}, a^{-1}x, a\xi) = p^{(2)}_l(M_{0j}, x, \xi).
\end{aligned} \tag{2.49}$$

It follows from (2.45), (2.48) and (2.49) that, (leaving out the arguments $x, -i\nabla$)

$$\begin{aligned} \| |\nabla|^{\rho-1}[\Delta, r]f\|_{L^2} &+ \sum_j \| |\nabla|^{\rho-1}[\sum_k \partial_k x_j \partial_k, r]f\|_{L^2} \\ &\leq \sum_j \| |\nabla|^{\rho-1}\partial_j p_j(P_0)f\|_{L^2} + \sum_{j,l} \| |\nabla|^{\rho-1}\partial_l p_l^{(2)}(M_{0j})f\|_{L^2} \\ &\leq \sum_j \| |\nabla|^{\rho} p_j(P_0)f\|_{L^2} + \sum_{j,l} \| |\nabla|^{\rho} p_l^{(2)}(M_{0j})f\|_{L^2} \\ &\leq C_n q_n^M((f,0)), \end{aligned} \tag{2.50}$$

for some constant C_n. Due to (2.46d) we can write

$$\partial_k[x_l, r] = \partial_k p^{(1)}(M_{0l}, x, -i\nabla) = p_k^{(1)}(M_{0l}, x, -i\nabla), \tag{2.51}$$

$$\deg p_k^{(1)}(M_{0l}) \leq 2n, \quad p_k^{(1)}(M_{0l}, a^{-1}x, a\xi) = p_k^{(1)}(M_{0l}, x, \xi).$$

It follows from (2.51) that

$$\begin{aligned} \sum_i \| |\nabla|^{\rho}[X_i, r]\dot{f}\|_{L^2} &\leq C \sum_{i,j} \| |\nabla|^{\rho-1}\partial_j[X_i, r]\dot{f}\|_{L^2} \\ &= C \sum_{i,j} \| |\nabla|^{\rho-1} p_j^{(1)}(M_{0i})\dot{f}\|_{L^2} \\ &\leq C_n q_n^M((0, \dot{f})). \end{aligned} \tag{2.52}$$

We obtain from inequalities (2.44), (2.47), (2.50) and (2.52) that

$$\sum_{X \in \Pi} \|[T_X^{1M}, r]v\|_M \leq C_n q_n^M(v), \tag{2.53}$$

for some constant C_n.

It follows from (2.43) and (2.53) that (with new C_n)

$$\|M_\mu \partial^\nu v\|_M \leq C_n(\|v\|_{M_{n+1}} + q_n^M(v)), \quad |\mu| = |\nu| = n+1. \tag{2.54}$$

It now follows from inequalities (2.39), (2.40) and (2.54) that

$$q_{n+1}^M(v) \leq C'_n\big(q_n^M(v) + \|v\|_{M_{n+1}}\big), \tag{2.55}$$

and then by the induction hypothesis that

$$q_{n+1}^M(v) \leq C_n \|v\|_{M_{n+1}}, \tag{2.56}$$

for some constant C_n.

The inequality $q^D_{n+1}(\alpha) \leq C_n \|\alpha\|_{D_{n+1}}$ is proved in a similar way, but one has only to consider the degree of the corresponding polynomials $r(x,\xi)$ and not their homogeneity properties. Since the proof of this part is very similar, we omit it. Together with (2.56) we then have

$$q_{n+1}(u) \leq C_n \|u\|_{E_{n+1}},$$

which according to the induction hypothesis proves the second inequality of the theorem.

To prove that E_c is dense in E_∞ we first observe that $C_0^\infty(\mathbb{R}^3,\mathbb{C})$ is dense in $S(\mathbb{R}^3,\mathbb{C})$ which after inverse Fourier transform shows that D_c is dense in D_∞. We therefore only have to prove that M_c is dense in M_∞. Let $\varphi \in C_0^\infty(\mathbb{R}^3)$, $0 \leq \varphi(k) \leq 1$ for $k \in \mathbb{R}^3$, $\varphi(k) = 1$ for $|k| \leq 1$, $\varphi(k) = 0$ for $|k| \geq 2$. Let $\varphi_n(k) = \varphi(n^{-1}k)(1-\varphi(nk))$ for $k \in \mathbb{R}^3$, $n \in \mathbb{N}$, $n \geq 2$, and let $\psi_n = 1 - \varphi_n$. Then $0 \leq \psi_n(k) \leq 1$ for $k \in \mathbb{R}^3$ and $\psi_n(k) = 0$ for $2/n \leq |k| \leq n$. Moreover if $|\alpha| \geq 1$ then $|k|^{|\alpha|}|\partial^\alpha \psi_n(k)| \leq 2^{|\alpha|}C_{|\alpha|}$ for $|k| \leq 2/n$ and $|\partial^\alpha \psi_n(k)| \leq n^{-|\alpha|}C_{|\alpha|}$ for $|k| \geq n$, where the C_l are constants independent of n and k. For $(f,\dot f) \in M_\infty$ we define $\hat f^{(n)}(k) = \varphi_n(k)\hat f(k)$ and $\hat{\dot f}^{(n)}(k) = \varphi_n(k)\hat{\dot f}(k)$. Then $(f^{(n)}, \dot f^{(n)}) \in M_c$ and $\hat f^{(n)} - \hat f = \psi_n \hat f$, $\hat{\dot f}^{(n)} - \hat{\dot f} = \psi_n \hat{\dot f}$. The first inequality of the theorem and the Plancherel theorem now show that

$$\begin{aligned}
&\|(\hat f^{(n)} - \hat f, \hat{\dot f}^{(n)} - \hat{\dot f})\|^2_{M_N} \\
&\quad \leq C''_N \sum_{|\mu|\leq|\nu|\leq N} \int_{\mathbb{R}^3 - B_n} \Big(|k|^{2\rho}|\partial^\mu(k^\nu \psi_n(k)\hat f(k))|^2 + |k|^{2\rho-2}|\partial^\mu(k^\nu \psi_n(k)\hat{\dot f}(k))|^2\Big)\, dk,
\end{aligned}$$

$n \geq 2$, where $B_n = \{k \in \mathbb{R}^3 \big| 2/n \leq |k| \leq n\}$. The estimates of ψ_n and commutation of ∂_μ and k^ν give that

$$\begin{aligned}
&\|(\hat f^{(n)} - \hat f, \hat{\dot f}^{(n)} - \hat{\dot f})\|^2_{M_N} \\
&\quad \leq C''_N \sum_{|\mu|\leq|\nu|\leq N} \int_{\mathbb{R}^3 - B_n} \Big(|k|^{2\rho}|\partial^\mu(k^\nu \hat f(k))|^2 + |k|^{2\rho-2}|\partial^\mu(k^\nu \hat{\dot f}(k))|^2\Big)\, dk.
\end{aligned}$$

The last inequality converges to zero when $n \to \infty$ since $q_N((f,\dot f))$ is finite. This proves the theorem.

The elements of the space have important asymptotic decrease properties:

Lemma 2.10. *Let $1 \leq p < \infty$ and let $f \in L^p(\mathbb{R}^3)$ be such that $M_\alpha \partial^\beta f \in L^p(\mathbb{R}^3)$, $M_\alpha(x) = x^\alpha$, for $0 \leq |\alpha| \leq |\beta| \leq n$. If ν is a multi-index and $(n - |\nu|)p > 3$, then (after a change on a set of measure zero)*

$$(1+|x|)^{3/p+|\nu|}|\partial^\nu f(x)| \leq C_{\nu,p} \sum_{|\alpha|\leq|\beta|\leq n} \|M_\alpha \partial^\beta f\|_{L^p}. \tag{2.57}$$

Proof. A Sobolev embedding gives at once with $lp > 3$ that

$$\|\partial^\nu f\|_{L^\infty} \leq C \sum_{|\mu|\leq l} \|\partial^{\mu+\nu} f\|_{L^p} \leq C \sum_{|\beta|\leq n} \|\partial^\beta f\|_{L^p}, \tag{2.58}$$

and that $\partial^\nu f$ is a continuous function vanishing at ∞.

Let $\varphi \in C_0^\infty(\mathbb{R}^3)$, $\varphi(x) = 1$ for $1/2 \leq |x| \leq 2$, $\varphi(x) = 0$ for $|x| \leq 1/3$, $\varphi(x) = 0$ for $|x| > 3$ and $\varphi(x) \geq 0$, $x \in \mathbb{R}^3$. We define $g_a(x) = f(ax)$. Since $a^{|\nu|}(\partial^\nu f)(ax) = \partial^\nu g_a(x)$ and $\partial^\nu g_a(x) = \partial^\nu(g_a\varphi)(x)$ for $1/2 \leq |x| \leq 2$, we obtain from (2.58)

$$\begin{aligned}\sup_{1/2\leq|x|\leq 2} |a^{|\nu|}(\partial^\nu f)(ax)| &\leq \|\partial^\nu g_a\varphi\|_{L^\infty} \\ &\leq \sum_{|\beta|\leq n} \|\partial^\beta(g_a\varphi)\|_{L^p} \\ &\leq C_n \sum_{|\beta_1|+|\beta_2|\leq n} \|(\partial^{\beta_1}g_a)(\partial^{\beta_2}\varphi)\|_{L^p},\end{aligned} \tag{2.59}$$

where $(n - |\nu|)p > 3$ and where we have used the Leibniz rule in the last inequalities.

A change of coordinates gives

$$\|(\partial^{\beta_1}g_a)(\partial^{\beta_2}\varphi)\|_{L^p} = a^{|\beta_1|-3/p}\Big(\int |(\partial^{\beta_1}f)(x)(\partial^{\beta_2}\varphi)(x/a)|^p dx\Big)^{1/p}.$$

Since the integrand vanishes outside the set $\{x \big| a/3 \leq |x| \leq 3a\}$, it follows that

$$\begin{aligned}a^{3/p}\|(\partial^{\beta_1}g_a)(\partial^{\beta_2}\varphi)\|_{L^p} &\leq C_{n,p}\|\partial^{\beta_2}\varphi\|_{L^\infty}\||x|^{|\beta_1|}\partial^{\beta_1}f\|_{L^p} \\ &\leq C'_{n,p}\sum_{|\alpha|\leq|\beta_1|}\|M_\alpha\partial^{\beta_1}f\|_{L^p} \\ &\leq C'_{n,p}\sum_{|\alpha|\leq|\beta|\leq n}\|M_\alpha\partial^\beta f\|_{L^p}.\end{aligned}$$

This inequality and (2.59) show that for $1/2 \leq |y| \leq 2$

$$a^{3/p+|\nu|}|(\partial^\nu f)(ay)| \leq C_{\nu,p}\sum_{|\alpha|\leq|\beta|\leq n}\|M_\alpha\partial^\beta f\|_{L^p},$$

which with $a = |x| > 0, y = x/|x|$, together with (2.58) gives estimate (2.57). This proves the lemma.

Remark 2.11. Lemma 2.10 is still true if $p = \infty$, $n = |\nu|$. In this case the proof is trivial.

Theorem 2.12. *If* $(f, 0) \in M_1^\rho, 1/2 < \rho < 1,$ *then*

$$(1 + |x|)^{3/2-\rho}|f(x)| \leq C\|(f,0)\|_{M_1^\rho} \tag{2.60a}$$

and if $(f, \dot f) \in M^\rho_{|\nu|+2}, 1/2 < \rho < 1,$ *then*

$$(1 + |x|)^{5/2+|\nu|-\rho}\big(|\partial^\nu\partial_i f(x)| + |\partial^\nu \dot f(x)|\big) \leq C_{|\nu|}\|(f,\dot f)\|_{M_{|\nu|+2}}. \tag{2.60b}$$

Proof. Let $p = 6(3-2\rho)^{-1}$. Then $3 < p < 6$ for $1/2 < \rho < 1$. Suppose for the moment that $f \in C_0^\infty(\mathbb{R}^3)$. The inequality (cf. Theorem 4.5.3. of [11])

$$\|f\|_{L^p(\mathbb{R}^3)} \leq C_p \||\nabla|^\rho f\|_{L^2(\mathbb{R}^3)}, \quad p = 6(3-2\rho)^{-1}, \tag{2.61}$$

where the constant C_p is independent of the support of f, and inequality (2.57) with $n - |\mu| = 1 > 3/p$ of lemma 2.10 then give

$$\begin{aligned}(1+|x|)^{3/2-\rho+|\mu|}|\partial^\mu f(x)| &\leq C_{\mu,\rho} \sum_{|\alpha|\leq|\beta|\leq|\mu|+1} \||\nabla|^\rho M_\alpha \partial^\beta f\|_{L^2} \\ &\leq C'_{\mu,\rho} q^M_{|\mu|+1}(f,0),\end{aligned} \tag{2.62a}$$

where q^M was defined in (2.32a). It follows from this inequality and Theorem 2.9 that

$$(1+|x|)^{3/2-\rho+|\mu|}|\partial^\mu f(x)| \leq C_{\mu,\rho}\|(f,0)\|_{M_{1+|\mu|}}, \tag{2.62b}$$

for $(f,0) \in M_{1+|\mu|}$. As a matter of fact, $C_0^\infty(\mathbb{R}^3,\mathbb{R}^4) \oplus C_0^\infty(\mathbb{R}^3,\mathbb{R}^4)$ is dense in $S(\mathbb{R}^3,\mathbb{R}^4) \oplus S(\mathbb{R}^3,\mathbb{R}^4)$, which by definition is dense in M_n^ρ. In particular, with $\mu = 0$, this proves (2.60a).

According to (2.62a) we have

$$(1+|x|)^{3/2-\rho+|\mu|}|\partial^\mu x_i \dot{f}(x)| \leq C_{\mu,\rho} q^M_{|\mu|+1}(Q_i\dot{f},0), \tag{2.63}$$

where $(Q_i\dot{f}) = x_i\dot{f}(x)$. By the definition of q^M

$$\begin{aligned}q^M_{|\mu|+1}(Q_i\dot{f},0)^2 &= \sum_{|\alpha|\leq|\beta|\leq|\mu|+1} \||\nabla|^\rho M_\alpha \partial^\beta Q_i \dot{f}\|^2_{L^2} \\ &= \sum_{\substack{|\alpha|\leq|\beta|\leq|\mu|+1 \\ 1\leq j\leq 3}} \||\nabla|^{\rho-1} \partial_j M_\alpha \partial^\beta Q_j \dot{f}\|^2_{L^2}.\end{aligned} \tag{2.64}$$

But $\partial_j M_\alpha \partial^\beta Q_l = Q^\beta_{j\alpha l}(x, -i\nabla)$, where $Q^\beta_{j\alpha l}(x,\xi)$ is a polynomial of degree $|\alpha| + |\beta| + 2 \leq 2|\mu| + 4$ and $Q^\beta_{j\alpha l}(a^{-1}x, a\xi) = a^{|\beta|-|\alpha|} Q^\beta_{j\alpha l}(x,\xi)$. Therefore

$$q^M_{|\mu|+1}(Q_i\dot{f},0) \leq C_{|\mu|} q^M_{|\mu|+2}(0,\dot{f}),$$

which together with (2.63) and (2.64) shows that

$$(1+|x|)^{3/2-\rho+|\mu|}|\partial^\mu x_i \dot{f}| \leq C_{\rho,\mu} q^M_{|\mu|+2}(0,\dot{f}) \leq C'_{\rho,\mu}\|(0,\dot{f})\|_{M^\rho_{|\mu|+2}}, \tag{2.65}$$

where the last inequality follows from Theorem 2.9.

In a similar way it follows from (2.62a) that

$$|\partial^\mu \dot{f}(x)| \leq C_{\mu,\rho}\|(0,\dot{f})\|_{M^\rho_{|\mu|+2}}. \tag{2.66}$$

Inequalities (2.65) and (2.66) with $\mu = 0$ prove (2.60b) with $\nu = 0$ and $f = 0$. Since $[\partial^\mu, x_i]$ is a monomial of degree $|\mu| - 1$ in ∇, it follows from (2.65) and (2.66) by induction that (2.60b) is true for $|\nu| \geq 0$, with $f = 0$.

Let $\partial^\mu = \partial^\nu \partial_i$ in (2.62b). Then (2.60b), with $\dot{f} = 0$, follows from (2.62b). Since (2.60b) is true for the particular cases $(f, 0) \in M^\rho_{|\nu|+2}$ and $(0, \dot{f}) \in M^\rho_{|\nu|+2}$, it is also true for $(f, \dot{f}) \in M^\rho_{|\nu|+2}$, because $\|(f, 0)\|_{M_n} + \|(0, \dot{f})\|_{M_n} \leq \sqrt{2}\|(f, \dot{f}\|_{M_n}$. This proves the theorem.

We shall prove that M^ρ *contains the long range potentials* which it is expected to contain. It follows from the next theorem that $(f, \dot{f}) \in E^\rho_\infty$ if $(1 + |x|)^{a+|\nu|}|\partial^\nu f(x)|$ and $(1 + |x|)^{a+1+|\nu|}|\partial^\nu \dot{f}(x)|$ are uniformly bounded in x for some $a > 3/2 - \rho$.

Theorem 2.13. *Let $1/2 < \rho < 1, p = 6(5 - 2\rho)^{-1}, q = 6(3 - 2\rho)^{-1}$, $f \in L^q$ and $\dot{f} \in L^p$. If $M_\alpha \partial^\beta \partial_i f \in L^p(\mathbb{R}^3, \mathbb{R}^4)$ and $M_\alpha \partial^\beta \dot{f} \in L^p(\mathbb{R}^3, \mathbb{R}^4)$ for $0 \leq |\alpha| \leq |\beta|, 1 \leq i \leq 3$, then $(f, \dot{f}) \in M_n$ and*

$$\|(f, \dot{f})\|_{M_n} \leq C_n \Big(\sum_{\substack{0 \leq |\alpha| \leq |\beta| \leq n \\ 1 \leq i \leq 3}} \|M_\alpha \partial^\beta \partial_i f\|_{L^p} + \sum_{0 \leq |\alpha| \leq |\beta| \leq n} \|M_\alpha \partial^\beta \dot{f}\|_{L^p} \Big), \quad n \geq 0.$$

Proof. Suppose for the moment that $\dot{f} \in C_0^\infty(\mathbb{R}^3)$. The inequality (cf. Theorem 4.5.3. of [11])

$$\||\nabla|^{\rho-1} \dot{f}\|_{L^2(\mathbb{R}^3)} \leq C_p \|\dot{f}\|_{L^p(\mathbb{R}^3)}, \quad p = 6(5 - 2\rho)^{-1}, \tag{2.67}$$

where C_p is independent of the support of $\dot{f}$, gives $\|(0, \dot{f})\|_{M^\rho} \leq C_p \|f\|_{L^p}, f \in C_0^\infty$. Since C_0^∞ is dense L^p, for our given p, and dense in $S(\mathbb{R}^3)$, it follows by continuity and by the definition of the space M^ρ that

$$\|(0, \dot{f})\|_{M^\rho} \leq C_p \|\dot{f}\|_{L^p}, \quad p = 6(5 - 2\rho)^{-1}, \dot{f} \in L^p, \tag{2.68}$$

and that $(0, \dot{f}) \in M^\rho$ for $\dot{f} \in L^p$.

According to inequality (2.68), Theorem 2.9 and definition (2.32a) of q^M_n we have

$$\|(0, \dot{f})\|_{M^\rho_n} \leq C_{\rho,n} q^M_n(0, \dot{f}) \leq C'_{\rho,n} \sum_{0 \leq |\alpha| \leq |\beta| \leq n} \|M_\alpha \partial^\beta \dot{f}\|_{L^p}, \tag{2.69}$$

$p = 6(5 - 2\rho)^{-1}$, $n \geq 0$, if $M_\alpha \partial^\beta \dot{f} \in L^p$ for $0 \leq |\alpha| \leq |\beta| \leq n$. Since $\|(f, 0)\|_{M^\rho_n} \leq C_{\rho,n} q^M_n(f, 0)$ according to Theorem 2.9 and $q^M_n(f, 0) \leq C_n \sum_i q^M_n(0, \partial_i f)$ using definition (2.32a) of q^M_n it follows from (2.69) and the triangle inequality that the inequality of the theorem is true.

Corollary 2.14. *Let $n \geq 0$, $1/2 < \rho < 1$, $f \in C^{n+1}(\mathbb{R}^3, \mathbb{R}^4)$, $\dot{f} \in C^n(\mathbb{R}^3, \mathbb{R}^4)$ and let*

$$\Gamma_{n,a}(f, \dot{f}) = \sum_{|\nu| \leq n+1} \sup_x \big((1 + |x|)^{a+|\nu|}|\partial^\nu f(x)|\big) + \sum_{|\nu| \leq n} \sup_x \big((1 + |x|)^{a+1+|\nu|}|\partial^\nu \dot{f}(x)|\big).$$

If $\Gamma_{n,a}(f, \dot{f}) < \infty$ for some $a > 3/2 - \rho$, then $(f, \dot{f}) \in M^\rho_n$ and $\|(f, \dot{f})\|_{M^\rho_n} \leq C_n \Gamma_{n,a}(f, \dot{f})$.

Proof. Let a be such that $\Gamma_{n,a}(f,\dot f)$ is finite. Then the hypotheses of Theorem 2.13 are satisfied and the right-hand side of the inequality in that theorem is bounded by $C_n\Gamma_{n,a}$, after redefining the constants. This proves the corollary.

Later we shall need estimates of weighted supremum norms of solution of the homogeneous wave equation. These estimates follow directly from Kirchhoff's formula and Theorem 2.12.

Proposition 2.15. *If $n \geq 1$, $(f,\dot f) \in M^\rho_{n+2}$, $1/2 < \rho < 1$, then the solution u of the wave equation $\square u = 0$, with initial conditions $u(0,x) = f(x)$, $\frac{\partial}{\partial t}u(t,x)|_{t=0} = \dot f(x)$, satisfies*

$$\begin{aligned}(1+|x|+|t|)^{3/2-\rho}|u(t,x)| + (1+|x|+|t|)& \\ \sum_{1\leq|\nu|+l\leq n}(1+\big||t|-|x|\big|)^{1/2-\rho+|\nu|+l}|\partial^\nu\big(\frac{\partial}{\partial t}\big)^l u(t,x)|& \\ \leq C_{n,\rho}\|(f,\dot f)\|_{M^\rho_{n+2}}&\end{aligned}$$

Proof. Give first $f,\dot f \in S(\mathbb{R}^3,\mathbb{R}^4)$. Then $\partial^\alpha(\partial/\partial t)^m u(t,x) = u_{\alpha,m}(t,x), \alpha = (\alpha_1,\alpha_2,\alpha_3)$, is the solution of the wave equation with initial conditions $f_{\alpha,m},\dot f_{\alpha,m}$, where $(f_{\alpha,m},\dot f_{\alpha,m}) = T^{M1}_{P^\beta}(f,\dot f)$, where $\beta = (m,\alpha_1,\alpha_2,\alpha_3)$, $P^\beta = P_0^{\beta_0}P_1^{\beta_1}P_2^{\beta_2}P_3^{\beta_3}$ is an element in the enveloping algebra $U(\mathfrak{p})$ and where T^{M1}_X denotes the restriction to the space M^ρ of the linear Lie algebra representation T^1_X as defined by (1.5). As the initial data are in $S(\mathbb{R}^3,\mathbb{R}^4)$ it is sure that the solution is given by Kirchhoff's formula (cf. [11]):

$$u_\beta(t,x) = (4\pi)^{-1}\int_{|\omega|=1}\Big(f_\beta(x+t\omega) + t\sum_i \omega^i\partial_i f_\beta(x+t\omega) + t\dot f_\beta(x+t\omega)\Big)d\omega. \tag{2.70}$$

Let $\Gamma_{n,a}$ be as in Corollary 2.14 and let

$$j_a(t,x) = (4\pi)^{-1}\int_{|\omega|=1}(1+|x+\omega t|^2)^{-a/2}d\omega, \quad a\in\mathbb{R}. \tag{2.71}$$

It follows from (2.70) and (2.71) that

$$|u_\beta(t,x)| \leq \Gamma_{0,a}(f_\beta,\dot f_\beta)\big(j_a(t,x)+|t|j_{a+1}(t,x)\big). \tag{2.72}$$

The easy explicit evaluation of the integral in (2.71) in polar coordinates leads to

$$j_a(t,x) \leq C_a(1+|x|+|t|)^{-a} \quad \text{for } 0<a<2, \tag{2.73a}$$
$$j_a(t,x) \leq C_a(1+|x|+|t|)^{-2}(1+\big||t|-|x|\big|)^{2-a} \quad \text{for } a>2. \tag{2.73b}$$

We choose $a = 3/2-\rho+|\beta|$ in (2.72), where $1/2<\rho<1$. Then $1/2+|\beta| < a < 1+|\beta|$ and it follows from (2.73) that

$$j_a(t,x)+|t|j_{a+1}(t,x) \leq C_{\rho,|\beta|}(1+|x|+|t|)^{-(3/2-\rho)}, \quad |\beta|=0, \tag{2.74a}$$
$$j_a(t,x)+|t|j_{a+1}(t,x) \leq C_{\rho,|\beta|}(1+|x|+|t|)^{-1}(1+\big||t|-|x|\big|)^{-(1/2-\rho+|\beta|)}, \quad |\beta|\geq 1, \tag{2.74b}$$

where $a = 3/2 - \rho + |\beta|$.

Estimates (2.72) and (2.74) show that, if

$$Q_n(u) = \sup_{t,x} \big((1+|x|+|t|)^{3/2-\rho}|u(t,x)|\big) + \sum_{1 \le |\beta| \le n} \sup_{t,x} \big((1+|x|+|t|)(1+\big||t|-|x|\big|)^{1/2-\rho+|\beta|}|u_\beta(t,x)|\big),$$

then

$$Q_n(u) \le \sum_{0 \le |\beta| \le n} C_{\rho,|\beta|}\Gamma_{0,3/2-\rho+|\beta|}(f_\beta, \dot{f}_\beta). \tag{2.75}$$

It follows from the definition of $\Gamma_{n,a}$ and from $T^1_{P^\beta} = \partial^\alpha \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}^t$, $\beta = (t, \alpha)$, that

$$\Gamma_{0,3/2-\rho+|\beta|}(f_\beta, \dot{f}_\beta) \le \Gamma_{|\beta|,3/2-\rho}(f, \dot{f}),$$

which inserted into (2.75) gives (with new constant)

$$Q_n(u) \le C_{\rho,n}\Gamma_{n,3/2-\rho}(f, \dot{f}). \tag{2.76}$$

Inserting the result $\Gamma_{n,3/2-\rho}(f, \dot{f}) \le C_{n,\rho}\|(f, \dot{f})\|_{M_{n+2}}$ from Theorem 2.12 into inequality (2.76) we obtain for some constant $C_{\rho,n}$

$$Q_n(u) \le C_{\rho,n}\|(f, \dot{f})\|_{M_{n+2}}. \tag{2.77}$$

Since $S(\mathbb{R}^3, \mathbb{R}^4) \oplus S(\mathbb{R}^3, \mathbb{R}^4)$ is dense in the space M_{n+2} by construction, it follows that (2.77) is true for $(f, \dot{f}) \in M_{n+2}$. This proves the proposition.

It follows directly from definitions (1.5), (1.7) and (1.8) of $T_X, X \in \mathfrak{p}$, that $[T_X, T_Y] \equiv DT_X.T_Y - DT_Y.T_X = T_{[X,Y]}$ on the space of C^∞ functions. The next lemmas and corollary prove in particular that T_X, $X \in \mathfrak{p}$, is a continuous polynomial from E_∞ to E_∞, which assures that $X \mapsto T_X$ is a *nonlinear representation* of $\mathfrak{p}$ in E_∞.

Lemma 2.16. *Let $N \ge 0$, $u_i = (f_i, \dot{f}_i, \alpha_i) \in E_\infty$, $i = 1, 2$, and let $p = 6(5-2\rho)^{-1}$. Then*

$$\|T^2_{P_0}(u_1 \otimes u_2)\|_{E_N} \le C_N\Big(\sum_{|\mu| \le |\nu_1|+|\nu_2| \le N} \|M_\mu|\partial^{\nu_1}\alpha_1||\partial^{\nu_2}\alpha_2|\|_{L^p} + \sum_{\substack{|\mu| \le N \\ |\nu_1|+|\nu_2| \le N}} \big(\|M_\mu|\partial^{\nu_1}f_1||\partial^{\nu_2}\alpha_2|\|_{L^2} + \|M_\mu|\partial^{\nu_1}f_2||\partial^{\nu_2}\alpha_1|\|_{L^2}\big)\Big)$$

and

$$\|T^2_{M_{0j}}(u_1 \otimes u_2)\|_{E_N} \le C_N\Big(\sum_{|\mu| \le |\nu_1|+|\nu_2|+1 \le N+1} \|M_\mu|\partial^{\nu_1}\alpha_1||\partial^{\nu_2}\alpha_2|\|_{L^p} + \sum_{\substack{|\mu| \le N+1 \\ |\nu_1|+|\nu_2| \le N}} \big(\|M_\mu|\partial^{\nu_1}f_1||\partial^{\nu_2}\alpha_2|\|_{L^2} + \|M_\mu|\partial^{\nu_1}f_2||\partial^{\nu_2}\alpha_1|\|_{L^2}\big)\Big).$$

Proof. It follows from (1.8a) that

$$\|T^2_{P_0}(u_1 \otimes u_2)\|^2_{E_N} \tag{2.78}$$
$$= \|(0, \frac{1}{2}(\overline{\alpha}_1 \gamma \alpha_2 + \overline{\alpha}_2 \gamma \alpha_1))\|^2_{M_n} + \|\frac{1}{2}\Big(\sum_{\mu=0}^{3}(f_{1\mu}\gamma^0\gamma^\mu\alpha_2 + f_{2\mu}\gamma^0\gamma^\mu\alpha_1)\Big)\|^2_{D_n}.$$

It follows from Theorem 2.13 and Liebniz rule that

$$\|(0, \overline{\alpha}_i\gamma\alpha_j)\|_{M_n} \le C_n \sum_{|\beta| \le |\nu_1|+|\nu_2| \le n} \|M_\beta|\partial^{\nu_1}\alpha_i||\partial^{\nu_2}\alpha_j|\|_{L^p}, \tag{2.79}$$

where $p = 6(5-2\rho)^{-1}$. By definition (2.32b) of q_n^D and by the first part of Theorem 2.9:

$$\sum_{\mu=0}^{3} \|f_{i\mu}\gamma^0\gamma^\mu\alpha_j\|_{D_n} \le C'_n \sum_{\mu=0}^{3} q_n^D(f_{i\mu}\gamma^0\gamma^\mu\alpha_j) \tag{2.80}$$
$$\le C''_n \sum_{\substack{|\beta|\le n \\ |\nu_1|+|\nu_2|\le n}} \|M_\beta|\partial^{\nu_1}f_i||\partial^{\nu_2}\alpha_j|\|_{L^2}.$$

The triangle inequality, inequalities (2.78), (2.79) and (2.80) prove the first inequality in the lemma. The proof of the second inequality is so similar that we omit it.

Lemma 2.17. *If $u_i \in E_\infty$, $i = 1,2$, and $X \in \Pi$, then*

i) $\|T^2_X(u_1 \otimes u_2)\|_{E_N} \le C_N\big(\|u_1\|_{E_{N+1}}\|u_2\|_{E_0} + \|u_1\|_{E_0}\|u_2\|_{E_{N+1}}\big), \quad N \ge 0,$

ii) $\|T^2_X(u_1 \otimes u_2)\|_{E_N} \le C_N\big(\|u_1\|_{E_N}\|u_2\|_{E_1} + \|u_1\|_{E_1}\|u_2\|_{E_N}\big)^{3/2-\rho}$
$\big(\|u_1\|_{E_{N+1}}\|u_2\|_{E_0} + \|u_1\|_{E_0}\|u_2\|_{E_{N+1}}\big)^{\rho-1/2}, \quad N \ge 0,$

iii) $\|T^2_X(u_1 \otimes u_2)\|_{E} \le C\min\big(\|u_1\|_{E_1}\|u_2\|^{\rho-1/2}_{E_1}\|u_2\|^{3/2-\rho}_{E_0}, \|u_1\|^{\rho-1/2}_{E_1}\|u_1\|^{3/2-\rho}_{E_0}\|u_2\|_{E_1}\big).$

Proof. Since $T^2_X = 0$ if $X \in \Pi$ and $X \ne P_0$, $X \ne M_{0j}$, $j = 1,2,3$, we have according to Lemma 2.16 for $X \in \Pi$, $u_i = (f_i, \dot{f}_i, \alpha_i) \in E_\infty$, $i = 1,2$,

$$\|T^2_X(u_1 \otimes u_2)\|_E \le C\Big(\sum_{|\mu|\le 1} \|M_\mu|\alpha_1||\alpha_2|\|_{L^p} \tag{2.81}$$
$$+ \sum_{|\mu|\le 1}\big(\|M_\mu|f_1||\alpha_2|\|_{L^2} + \|M_\mu|f_2||\alpha_1|\|_{L^2}\big)\Big), \quad p = 6(5-2\rho)^{-1}.$$

Since $\|M_\mu|\alpha_1||\alpha_2|\|_{L^p} \le \|\alpha_1\|_{L^2}\|M_\mu\alpha_2\|_{L^q}$, $p = 6(5-2\rho)^{-1}$, $q = 3(1-\rho)^{-1}$ and $\|M_\mu\alpha_2\|_{L^q} \le C\sum_{|\nu|\le 1}\|\partial^\nu M_\mu\alpha_2\|_{L^2}$, we obtain

$$\sum_{|\mu|\le 1} \|M_\mu|\alpha_1||\alpha_2|\|_{L^p} \le C\|\alpha_1\|_{L^2} \sum_{\substack{|\mu|\le 1 \\ |\nu|\le 1}} \|M_\mu\partial^\nu\alpha_2\|_{L^2} \tag{2.82}$$
$$\le C'\|\alpha_1\|_D\|\alpha_2\|_{D_1}, \quad p = 6(5-2\rho)^{-1}, 1/2 < \rho < 1.$$

The inequalities $\|M_\mu|f_1||\alpha_2|\|_{L^2} \le \|f_1\|_{L^q}\|M_\mu\alpha_2\|_{L^{3/\rho}}$, $q = 6(3-2\rho)^{-1}$, $\|f_1\|_{L^q} \le C \|(f_1,0)\|_M$ (cf. (2.61)) and $\|M_\mu\alpha_2\|_{L^{3/\rho}} \le C\sum_{|\nu|\le 1}\|\partial^\nu M_\mu\alpha_2\|_{L^2}$ give

$$\sum_{|\mu|\le 1}\|M_\mu|f_1||\alpha_2|\|_{L^2} \le C\|(f_1,0)\|_M\|\alpha_2\|_{D_1}. \tag{2.83}$$

Since $\|(f_i,0)\|_{M_n} \le \|u_i\|_{E_n}, \|\alpha_i\|_{D_n} \le \|u_i\|_{E_n}$ it follows from (2.81), (2.82) and (2.83) that

$$T_X^2(u_1\otimes u_2)\|_E \le C\big(\|u_1\|_E\|u_2\|_{E_1} + \|u_1\|_{E_1}\|u_2\|_E\big),$$

which proves the first statement of the lemma in the case where $N=0$.

Let $u = (f,\dot f,\alpha) \in E_\infty$ and $|\mu|\le 1$. It follows from (2.60a) of Theorem 2.12 and from the inequality

$$\|(1+|x|)^{\rho-1/2}\alpha\|_{L^2} \le \|(1+|x|)\alpha\|_{L^2}^{\rho-1/2}\|\alpha\|_{L^2}^{3/2-\rho} \le \|\alpha\|_{D_1}^{\rho-1/2}\|\alpha\|_D^{3/2-\rho},$$

that

$$\begin{aligned}\|M_\mu|f||\alpha|\|_{L^2} &\le \sup_x\big((1+|x|)^{3/2-\rho}|f(x)|\big)\|(1+|x|)^{\rho-1/2}\alpha\|_{L^2} \\ &\le C\|(f,0)\|_{M_1}\|\alpha\|_{D_1}^{\rho-1/2}\|\alpha\|_D^{3/2-\rho}, |\mu|\le 1.\end{aligned} \tag{2.84}$$

It follows from Lemma 2.16 that, for $X\in\Pi$, $\|T_X^2(u_1\otimes u_2)\|_{E_N}$ is bounded by a sum of terms of the form

$$C_N\|M_\mu|M_{\mu_1}\partial^{\nu_1}\alpha_1||M_{\mu_2}\partial^{\nu_2}\alpha_2|\|_{L^p} = C_N I(\mu,\mu_1,\mu_2,\nu_1,\nu_2), \quad p = 6(5-2\rho)^{-1}, \tag{2.85}$$

where $|\mu|\le 1, |\nu_1|+|\nu_2|\le N, |\mu_1|\le|\nu_1|, |\mu_2|\le|\nu_2|$ and of terms of the form

$$\begin{aligned}&C_N J(\mu,\mu_1,\mu_2,\nu_1,\nu_2) \\ &\quad = C_N(\|M_\mu|M_{\mu_1}\partial^{\nu_1}f_1||M_{\mu_2}\partial^{\nu_2}\alpha_2|\|_{L^2} + \|M_\mu|M_{\mu_1}\partial^{\nu_1}f_2||M_{\mu_2}\partial^{\nu_2}\alpha_1|\|_{L^2}),\end{aligned} \tag{2.86}$$

where $|\mu|\le 1, |\nu_1|+|\nu_1|\le N, |\mu_1|+|\mu_2|\le N$.

Let first $|\nu_1|\ge|\nu_2|$ in (2.85). Then (2.82) gives

$$\begin{aligned}\|M_\mu|M_{\mu_1}\partial^{\nu_1}\alpha_1||M_{\mu_2}\partial^{\nu_2}\alpha_2|\|_{L^p} &\le C_{|\nu_1|}\|M_{\mu_1}\partial^{\nu_1}\alpha_1\|_D\|M_{\mu_2}\partial^{\nu_2}\alpha_2\|_{D_1} \\ &\le C_{|\nu_1|}\|\alpha_1\|_{D_{|\nu_1|}}\|\alpha_2\|_{D_{|\nu_2|+1}} \\ &\le C_{|\nu_1|}\|u_1\|_{E_{|\nu_1|}}\|u_2\|_{E_{|\nu_2|+1}} \\ &= I'(|\nu_1|,|\nu_2|),\end{aligned} \tag{2.87}$$

where we have used Theorem 2.9. Since $|\nu_1|+|\nu_2|+1 \le N+1$, $|\nu_1|\le N$, $|\nu_2|+1\le [\frac{N}{2}]+1\le N$ for $N\ge 1$, Corollary 2.6 gives with $N_0 = 1$

$$I'(|\nu_1|,|\nu_2|) \le C'_N\big(\|u_1\|_{E_1}\|u_2\|_{E_N} + \|u_1\|_{E_N}\|u_2\|_{E_1}\big). \tag{2.88}$$

If we now take $|\nu_1| < |\nu_2|$, we obtain the same estimate (2.88), so (2.88) is true for $|\nu_1| + |\nu_2| \leq N, N \geq 1$. From (2.85) and (2.88) it follows that

$$I(\mu, \mu_1, \mu_2, \nu_1, \nu_2) \leq C'_N \big(\|u_1\|_{E_1} \|u_2\|_{E_N} + \|u_1\|_{E_N} \|u_2\|_{E_1}\big), \tag{2.89}$$

with the range of multi-indices defined in (2.85).

Let $|\nu_1| > |\nu_2|$ in (2.86). We can choose μ_1 and μ_2, without changing the value of $J(\mu, \mu_1, \mu_2, \nu_1, \nu_2)$, such that $|\mu_1| \leq |\nu_1|, |\mu_2| \leq N - |\nu_1|$. Inequality (2.83) then gives

$$\begin{aligned} \|M_\mu |M_{\mu_1} \partial^{\nu_1} f_1| |M_{\mu_2} \partial^{\nu_2} \alpha_2|\|_{L^2} &\leq C_{|M_1|} \|(M_{\mu_1} \partial^{\nu_1} f_1, 0)\|_M \|M_{\mu_2} \partial^{\nu_2} \alpha_2\|_{D_1} \\ &\leq C'_{|\nu_1|} \|(f_1, 0)\|_{M_{|\nu_1|}} \|\alpha_2\|_{D_{1+N-|\nu_1|}} \\ &\leq C'_{|\nu_1|} \|u_1\|_{E_{|\nu_1|}} \|u_2\|_{E_{1+N-|\nu_1|}}, \end{aligned} \tag{2.90}$$

where we have used Theorem 2.9. An estimate for the second term in (2.86) is obtained by permuting u_1 and u_2 in (2.90). This gives

$$J(\mu, \mu_1, \mu_2, \nu_1, \nu_2) \leq C_N \big(\|u_1\|_{E_{|\nu_1|}} \|u_2\|_{E_{N+1-|\nu_1|}} + \|u_1\|_{E_{N+1-|\nu_1|}} \|u_2\|_{E_{|\nu_1|}}\big), \quad |\nu_1| > |\nu_2|.$$

Since $|\nu_1| + |\nu_2| \leq N$ and $N + 1 - |\nu_1| \leq N$ in this inequality, Corollary 2.6 gives

$$J(\mu, \mu_1, \mu_2, \nu_1, \nu_2) \leq C_N \big(\|u_1\|_{E_1} \|u_2\|_{E_N} + \|u_1\|_{E_N} \|u_2\|_{E_1}\big), \quad |\nu_1| > |\nu_2|. \tag{2.91}$$

Let $|\nu_1| \leq |\nu_2|$ in (2.86). As above we can choose $|\mu_1| \leq |\nu_1|$ and $|\mu_2| \leq N - |\nu_1|$. Application of the inequality (2.84) to the two terms in (2.86) gives

$$\begin{aligned} J(\mu, \mu_1, \mu_2, \nu_1, \nu_2) \leq C_N \Big(&\|(f_1, 0)\|_{M_{|\nu_1|}} \|\alpha_2\|^{\rho-1/2}_{D_{1+N-|\nu_1|}} \|\alpha_2\|^{3/2-\rho}_{D_{N-|\nu_1|}} \\ &+ \|(f_2, 0)\|_{M_{|\nu_1|}} \|\alpha_1\|^{\rho-1/2}_{D_{1+N-|\nu_1|}} \|\alpha_1\|^{3/2-\rho}_{D_{N-|\nu_1|}} \Big), \quad |\nu_1| \leq |\nu_2|. \end{aligned}$$

It now follows from the definition of the norms that

$$\begin{aligned} J(\mu, \mu_1, \mu_2, \nu_1, \nu_2) \leq C_n \Big(&\|u_1\|_{E_{|\nu_1|}} \|u_2\|^{\rho-1/2}_{E_{N+1-|\nu_1|}} \|u_2\|^{3/2-\rho}_{E_{N-|\nu_1|}} \\ &+ \|u_2\|_{E_{|\nu_1|}} \|u_1\|^{\rho-1/2}_{E_{N+1-|\nu_1|}} \|u_1\|^{3/2-\rho}_{E_{N-|\nu_1|}} \Big), \quad |\nu_1| \leq |\nu_2|. \end{aligned} \tag{2.92}$$

Application of Corollary 2.6 to the terms $(\|u_1\|_{E_{|\nu_1|}} \|u_2\|_{E_{N+1-|\nu_1|}})^{\rho-1/2}$ and $(\|u_1\|_{E_{|\nu_1|}} \|u_2\|_{E_{N-|\nu_2|}})^{3/2-\rho}$ and the corresponding terms with u_1 and u_2 permuted in (2.92), gives

$$\begin{aligned} J(\mu, \mu_1, \mu_2, \nu_1, \nu_2) \leq C_N &\big(\|u_1\|_E \|u_2\|_{E_{N+1}} + \|u_1\|_{E_{N+1}} \|u_2\|_E\big)^{\rho-1/2} \\ &\big(\|u_1\|_{E_1} \|u_2\|_{E_N} + \|u_1\|_{E_N} \|u_2\|_{E_1}\big)^{3/2-\rho}, \quad |\nu_1| \leq |\nu_2|. \end{aligned} \tag{2.93}$$

Factorization of the right-hand side of (2.91) into a factor with exponent $\rho - 1/2$ and a factor with exponent $3/2 - \rho$ and the application of Corollary 2.6 to the terms in the first factor show that (2.93) is also true for $|\nu_1| > |\nu_2|$. This proves the second statement of the lemma for $N > 1$. The case $N = 0$ is the same as in the first statement of the lemma.

The first statement of the lemma for $N \geq 1$ follows from the second by application of Corollary 2.6 to the factor with exponent $3/2 - \rho$.

Finally statement iii) follows from (2.82), application of estimate (2.83) to one of the terms $\|M_\mu |f_1||\alpha_2|\|_{L^2}$ or $\|M_\mu |f_2||\alpha_1|\|_{L^2}$, and application to the other terms of estimate (2.84). This proves the lemma.

Corollary 2.18. *If $u \in E_\infty$ and $X \in \Pi$, then*

$$\|T_X^2(u)\|_{E_N} \leq C_N(\|u\|_{E_1}\|u\|_{E_N})^{3/2-\rho}(\|u\|_{E_0}\|u\|_{E_{N+1}})^{\rho-1/2}, \quad N \geq 1,$$

and

$$\|T_X^2(u)\|_{E_N} \leq C\|u\|_{E_0}\|u\|_{E_{N+1}}, \quad N \geq 0.$$

We shall need an analogy of Lemma 2.17 and Corollary 2.18 for $T_Y = T_Y^1 + \tilde{T}_Y$, where $Y \in \Pi'$, the basis for the enveloping algebra $U(\mathfrak{p})$.

Lemma 2.19. *If $u_1, \dots, u_n \in E_\infty$ and $Y \in \Pi'$, then*

i) $\|T_Y^n(u_1 \otimes \cdots \otimes u_n)\|_{E_N} \leq C\sum_i \prod_{1 \leq l \leq n-1} \|u_{i_l}\|_E \|u_{i_n}\|_{E_{N+|Y|}}$,

for $n \geq 1$, $N \geq 0$ and

ii) $\|T_Y^n(u_1 \otimes \cdots \otimes u_n)\|_{E_N}$

$$\leq C\Big(\sum_i \|u_{i_1}\|_{E_{N+|Y|-1}}\|u_{i_2}\|_{E_1}\prod_{l=3}^n \|u_{i_l}\|_E\Big)^{3/2-\rho}\Big(\sum_i \|u_{i_1}\|_{E_{N+|Y|}}\|u_{i_2}\|_{E_1}\prod_{l=3}^n \|u_{i_l}\|_E\Big)^{\rho-1/2},$$

for $n \geq 2$ and $|Y| + N \geq 1$. Here the summation is over all permutation i of $(1, \dots, n)$ and the constant C depends on $|Y|, n, N, \rho$.

Proof. We prove the first statement by induction. It is true for $Y = \mathbb{I}$, because $T_{\mathbb{I}}(u) = u$. It follows from Lemma 2.17 that it is also true for $Y = X \in \Pi$. Suppose it is true for $|Y| \leq L$. If $Y' = YX, |Y| \leq L, X \in \Pi$, then it follows from definition (1.9) of T_{YX} that with $I_q = \otimes^q I$, $I =$ identity in E,

$$T_{YX}^n = \sum_{0 \leq q \leq n-1} T_Y^n(I_q \otimes T_X^1 \otimes I_{n-q-1})\tau_n + \sum_{0 \leq q \leq n-2} T_Y^{n-1}(I_q \otimes T_X^2 \otimes I_{n-q-2})\tau_n, \quad (2.94)$$

where τ_n is the normalized symmetrization operator on $\hat{\otimes}^n E$ $(= E\hat{\otimes}E\hat{\otimes}\cdots\hat{\otimes}E$, n times). By the induction hypothesis we have, after reindexation for $n \geq 2$,

$$\|T_Y^{n-p+1}(I_q \otimes T_X^p \otimes I_{n-q-p})(\tau_n \otimes_{j=1}^n u_j)\|_{E_N}$$
$$\leq C\sum_i \|u_{i_1}\|_E \cdots \|u_{i_{n-2}}\|_E$$
$$\big(\|u_{i_{n-1}}\|_E\|T_X^1 u_{i_n}\|_{E_{|Y|+N}} + \|T_X^1 u_{i_{n-1}}\|_E\|u_{i_n}\|_{E_{|Y|+N}}\big), \quad p = 1, 2,\ n - p \geq 0.$$

Since, according to the definition of $\|\cdot\|_{E_l}$ and Corollary 2.6

$$\begin{aligned}&\|u_{i_{n-1}}\|_E\|T^1_X u_{i_n}\|_{E_{|Y|+N}}+\|T^1_X u_{i_{n-1}}\|_E\|u_{i_n}\|_{E_{|Y|+N}}\\&\quad\le C'\big(\|u_{i_{n-1}}\|_E\|u_{i_n}\|_{E_{|Y|+1+N}}+\|u_{i_{n-1}}\|_{E_{|Y|+1+N}}\|u_{i_n}\|_E\big),\end{aligned}$$

and since the case $n=1$ is trivial, we obtain

$$\begin{aligned}&\|T^{n-p+1}_Y(I_q\otimes T^p_X\otimes I_{n-q-p})(\tau_n\otimes^n_{j=1}u_j)\|_{E_N}\\&\quad\le C''\sum_i\prod_{j=1}^{n-1}\|u_{i_j}\|_E\|u_{i_n}\|_{E_{N+1+|Y|}},\quad n\ge 1, p=1,2, n-p\ge 0.\end{aligned}\tag{2.95}$$

Formula (2.94) and inequality (2.95) prove, after changing the constant C, that the first statement of the lemma is true for $|Y'|=L+1$. So, by induction it is true for all $|Y|\ge 0$.

According to Theorem 2.4 of [20] we have, for $X\in\Pi$ and $Z\in\Pi'$,

$$T^n_{XZ}=\sum_{n_1+n_2=n}\sum_{Z,2}{}' T^2_X(T^{n_1}_{Z_1}\otimes T^{n_2}_{Z_2})\tau_n+T^1_XT^n_Z,\quad n\ge 2,\tag{2.96}$$

where $\sum'_{Z,2}$ is a sum over a subset of couples (Z_1,Z_2) with $Z_i\in\Pi'$ and $0\le|Z_i|\le|Z|$, $|Z_1|+|Z_2|=|Z|$ and where τ_n is the normalized symmetrization operator on $\hat{\otimes}^n E$. Let $Y=XZ, X\in\Pi, Z\in\Pi'$, let $f_l\in E_\infty, 1\le l\le n_1, g_j\in E_\infty, 1\le j\le n_2$, and let $f=f_1\otimes\cdots\otimes f_{n_1}, g=g_1\otimes\cdots\otimes g_{n_2}$. According to statement ii) of Lemma 2.17, we obtain

$$\begin{aligned}&\|T^2_X(T^{n_1}_{Z_1}(f)\otimes T^{n_2}_{Z_2}(g))\|_{E_N}\\&\quad\le C_N\big(\|T^{n_1}_{Z_1}(f)\|_{E_N}\|T^{n_2}_{Z_2}(g)\|_{E_1}+\|T^{n_1}_{Z_1}(f)\|_{E_1}\|T^{n_2}_{Z_2}(g)\|_{E_N}\big)^{3/2-\rho}\\&\qquad\big(\|T^{n_1}_{Z_1}(f)\|_{E_{N+1}}\|T^{n_2}_{Z_2}(g)\|_E+\|T^{n_1}_{Z_1}(f)\|_E\|T^{n_2}_{Z_2}(g)\|_{E_{N+1}}\big)^{\rho-1/2},\quad N\ge 0.\end{aligned}\tag{2.97}$$

Let $N\ge 1$. Then it follows from the first inequality of Lemma 2.19 which is already proved and from the second inequality of Corollary 2.6, that

$$\begin{aligned}&\|T^{n_1}_{Z_1}(f)\|_{E_N}\|T^{n_2}_{Z_2}(g)\|_{E_1}+\|T^{n_1}_{Z_1}(f)\|_{E_1}\|T^{n_2}_{Z_2}(g)\|_{E_N}\\&\quad\le C_{N,n,|Z|}\sum_{l,j}\|f_{l_2}\|_E\cdots\|f_{l_{n_1}}\|_E\|g_{j_2}\|_E\cdots\|g_{j_{n_2}}\|_E\\&\qquad\big(\|f_{l_1}\|_{E_{N+|Z_1|}}\|g_{j_1}\|_{E_{1+|Z_2|}}+\|f_{l_1}\|_{E_{1+|Z_1|}}\|g_{j_1}\|_{E_{N+|Z_2|}}\big)\\&\quad\le C'_{N,n,|Z|}\sum_{l,j}\|f_{l_2}\|_E\cdots\|f_{l_{n_1}}\|_E\|g_{j_2}\|_E\cdots\|g_{j_{n_2}}\|_E\\&\qquad\big(\|f_{l_1}\|_{E_{N+|Z|}}\|g_{j_1}\|_{E_1}+\|f_{l_1}\|_{E_1}\|g_{j_1}\|_{E_{N+|Z|}}\big),\quad N\ge 1.\end{aligned}\tag{2.98}$$

For the last inequality we have also used that $|Z_1| + |Z_2| = |Z|$. We obtain in the same way for $N \geq 0$,

$$\|T^{n_1}_{Z_1}(f)\|_{E_{N+1}}\|T^{n_2}_{Z_2}(g)\|_E + \|T^{n_1}_{Z_1}(f)\|_E\|T^{n_2}_{Z_2}(g)\|_{E_{N+1}} \tag{2.99}$$
$$\leq C_{N,n,|Z|}\sum_{l,j}\|f_{l_2}\|_E\cdots\|f_{l_{n_1}}\|_E\|g_{j_2}\|_E\cdots\|g_{n_2}\|_E$$
$$\big(\|f_{l_1}\|_{E_{N+|Z|+1}}\|g_{j_1}\|_E + \|f_{l_1}\|_E\|g_{j_1}\|_{E_{N+|Z|+1}}\big), \quad N \geq 0.$$

Let $u_1, \ldots, u_n \in E_\infty$. Then it follows from inequalities (2.97), (2.98) and (2.99) that

$$\|T^2_X(T^{n_1}_{Z_1} \otimes T^{n_2}_{Z_2})\tau_n(\otimes^n_{l=1}u_l)\|_{E_N} \tag{2.100}$$
$$\leq C_{N,n,|Z|}\Big(\sum_i \|u_{i_1}\|_{E_{N+|Z|}}\|u_{i_2}\|_{E_1}\|u_{i_3}\|_E\cdots\|u_{i_n}\|_E\Big)^{3/2-\rho}$$
$$\Big(\sum_i \|u_{i_1}\|_{E_{N+|Z|+1}}\|u_{i_2}\|_E\cdots\|u_{i_n}\|_E\Big)^{\rho-1/2}$$
$$\leq C_{N,n,|Z|}\Big(\sum_i \|u_{i_1}\|_{E_{N+|Z|}}\|u_{i_2}\|_{E_1}\|u_{i_3}\|_E\cdots\|u_{i_n}\|_E\Big)^{3/2-\rho}$$
$$\Big(\sum_i \|u_{i_1}\|_{E_{N+|Z|+1}}\|u_{i_2}\|_{E_1}\|u_{i_3}\|_E\cdots\|u_{i_n}\|_E\Big)^{\rho-1/2}, \quad N \geq 1.$$

Let $f_1, \ldots, f_{n_1},\ g_1, \ldots, g_{n_2},\ f, g$ be defined as previously. Statement iii) of Lemma 2.17 gives

$$\|T^2_X(T^{n_1}_{Z_1}(f) \otimes T^{n_2}_{Z_2}(g))\|_E \leq C\min\Big(\|T^{n_1}_{Z_1}(f)\|_{E_1}\|T^{n_2}_{Z_2}(g)\|^{\rho-1/2}_{E_1}\|T^{n_2}_{Z_2}(g)\|^{3/2-\rho}_E \tag{2.101}$$
$$\|T^{n_1}_{Z_1}(f)\|^{\rho-1/2}_{E_1}\|T^{n_1}_{Z_1}(f)\|^{3/2-\rho}_E\|T^{n_2}_{Z_2}(g)\|_{E_1}\Big).$$

It follows from the first inequality of Lemma 2.19 that

$$\|T^{n_1}_{Z_1}(f)\|_{E_1}\|T^{n_2}_{Z_2}(g)\|^{\rho-1/2}_{E_1}\|T^{n_2}_{Z_2}(g)\|^{3/2-\rho}_{E_0} \tag{2.102}$$
$$\leq C_{|Z|,n}\Big(\sum_{l,j}\|f_{l_1}\|_{E_{1+|Z_1|}}\|f_{l_2}\|_E\cdots\|f_{l_{n_1}}\|_E\|g_{j_1}\|_{E_{1+|Z_2|}}\|g_{j_2}\|_E\cdots\|g_{j_{n_2}}\|_E\Big)^{\rho-1/2}$$
$$\Big(\sum_{l,j}\|f_{l_1}\|_{E_{1+|Z_1|}}\|f_{l_2}\|_E\cdots\|f_{l_{n_1}}\|_E\|g_{j_1}\|_{E_{|Z_2|}}\|g_{j_2}\|_E\cdots\|g_{j_{n_2}}\|_E\Big)^{3/2-\rho}.$$

Inequalities (2.101) and (2.102) give

$$\|T^2_X(T^{n_1}_{Z_1} \otimes T^{n_2}_{Z_2})\tau_n(\otimes^n_{j=1}u_j)\|_E \leq C_{n,|Z|}\min(Q(|Z_1|,|Z_2|), Q(|Z_2|,|Z_1|)), \quad n \geq 2, \tag{2.103}$$

where

$$Q(|Z_1|,|Z_2|)=\Big(\sum_i \|u_{i_1}\|_{E_{1+|Z_1|}}\|u_{i_2}\|_{E_{1+|Z_2|}}\|u_{i_3}\|_E\cdots\|u_{i_n}\|_E\Big)^{\rho-1/2}$$
$$\Big(\sum_i \|u_{i_1}\|_{E_{1+|Z_1|}}\|u_{i_2}\|_{E_{|Z_2|}}\|u_{i_3}\|_E\cdots\|u_{i_n}\|_E\Big)^{3/2-\rho}. \tag{2.104}$$

If $|Z_1|=|Z_2|=0$, then

$$Q(0,0)\leq\Big(\sum_i \|u_{i_1}\|_{E_1}\|u_{i_2}\|_{E_1}\|u_{i_3}\|_E\cdots\|u_{i_n}\|_E\Big)^{\rho-1/2}$$
$$\Big(\sum_i \|u_{i_1}\|_{E}\|u_{i_2}\|_{E_1}\|u_{i_3}\|_E\cdots\|u_{i_n}\|_E\Big)^{3/2-\rho}. \tag{2.105a}$$

If $|Z_1|=|Z_2|\geq 1$, then $1+|Z_1|\leq|Z|$, so it follows from the second inequality of Corollary 2.6 that

$$Q(|Z_1|,|Z_2|)\leq C\Big(\sum_i \|u_{i_1}\|_{E_{1+|Z|}}\|u_{i_2}\|_{E_1}\|u_{i_3}\|_E\cdots\|u_{i_n}\|_E\Big)^{\rho-1/2}$$
$$\Big(\sum_i \|u_{i_1}\|_{E_{|Z|}}\|u_{i_2}\|_{E_1}\|u_{i_3}\|_E\cdots\|u_{i_n}\|_E\Big)^{3/2-\rho},\quad |Z_1|=|Z_2|\geq 1. \tag{2.105b}$$

If $|Z_1|<|Z_2|$, then $|Z_1|+1\leq|Z_2|$, so it follows from Corollary 2.6 that

$$Q(|Z_1|,|Z_2|)\leq C\Big(\sum_i \|u_{i_1}\|_{1+|Z|}\|u_{i_2}\|_{E_1}\|u_{i_3}\|_E\cdots\|u_{i_n}\|_E\Big)^{\rho-1/2}$$
$$\Big(\sum_i \|u_{i_1}\|_{E_{|Z|}}\|u_{i_2}\|_{E_1}\|u_{i_3}\|_E\cdots\|u_{i_n}\|_E\Big)^{3/2-\rho},\quad |Z_1|<|Z_2|. \tag{2.105c}$$

It follows from inequalities (2.103) and (2.105) that

$$\|T^2_X(T^{n_1}_{Z_1}\otimes T^{n_2}_{Z_2})\tau_n(\otimes^n_{j=1}u_j)\|_E$$
$$\leq C_{n,|Z|}\Big(\sum_i \|u_{i_1}\|_{E_{1+|Z|}}\|u_{i_2}\|_{E_1}\|u_{i_3}\|_E\cdots\|u_{i_n}\|_E\Big)^{\rho-1/2}$$
$$\Big(\sum_i \|u_{i_1}\|_{E_{|Z|}}\|u_{i_2}\|_{E_1}\|u_{i_3}\|_E\cdots\|u_{i_n}\|_E\Big)^{3/2-\rho}, \tag{2.106}$$

where $|Z_1|+|Z_2|=|Z|$. Inequalities (2.100) and (2.106) show that

$$\|T^2_X(T^{n_1}_{Z_1}\otimes T^{n_2}_{Z_2})\tau_n(\otimes^n_{j=1}u_j)\|_{E_N}$$
$$\leq C_{N,n,|Z|}\Big(\sum_i \|u_{i_1}\|_{E_{N+|Z|}}\|u_{i_2}\|_{E_1}\|u_{i_3}\|_E\cdots\|u_{i_n}\|_E\Big)^{3/2-\rho}$$
$$\Big(\sum_i \|u_{i_1}\|_{E_{N+|Z|+1}}\|u_{i_2}\|_{E_1}\|u_{i_3}\|_E\cdots\|u_{i_n}\|_E\Big)^{\rho-1/2},\quad N\geq 0, \tag{2.107}$$

where $|Z_1|+|Z_2|=|Z|, |Z_j|\geq 0, n\geq 2$.

We now prove the second statement of the lemma by induction in $|Y|$, $Y=XZ$, using formula (2.96). If $|Y|=0$ then the statement is true since $T^n_{\mathbf{1}}=0$ for $n\geq 2$. Let $|Y|=L+1$. Then, supposing inequality ii) of the lemma true for all $|Y|\leq L$, we get

$$\begin{aligned} \|T^1_X T^n_Z(\otimes^n_{j=1}u_j)\|_{E_N} &\leq \|T^n_Z(\otimes^n_{j=1}u_j)\|_{E_{N+1}} \\ &\leq C_{|Z|,N,n}\Big(\sum_i \|u_i\|_{E_{N+|Z|}}\|u_{i_2}\|_{E_1}\prod_{l=3}^n\|u_{i_l}\|_E\Big)^{3/2-\rho} \\ &\qquad \Big(\sum_i \|u_{i_1}\|_{E_{N+1+|Z|}}\|u_{i_2}\|_{E_1}\prod_{l=3}^n\|u_{i_l}\|_E\Big)^{\rho-1/2}. \end{aligned} \tag{2.108}$$

It now follows from equality (2.96) and inequalities (2.107) and (2.108) that inequality ii) of the lemma is true. This proves the lemma.

According to Lemma 2.19 $T^n_Y, Y\in\Pi'$, has continuous extensions to spaces larger than E_∞. These extensions are denoted by the same symbol T^n_Y.

Remark 2.20. Let $Y\in\Pi'$ and $N\geq 0$. Then T^n_Y is a continuous linear map from $\hat{\otimes}^n E_{N+|Y|}$ to E_N.

Lemma 2.19 immediately gives estimates for the polynomial T_Y.

Corollary 2.21. *$T_Y, Y\in\Pi'$ is a continuous polynomial from $E_{N+|Y|}$ to E_N satisfying:*

i) $\|T_Y(u)\|_{E_N}\leq C_{N,|Y|}(\|u\|_E)\|u\|_{E_{N+|Y|}}$,

ii) $\|\tilde{T}_Y(u)\|_{E_N}\leq C_{N,|Y|}(\|u\|_E)\|u\|_E\|u\|_{E_{N+|Y|}}$,

iii) $\|\tilde{T}_Y(u)\|_{E_N}\leq C_{N,|Y|}(\|u\|_E)\|u\|_{E_1}(\|u\|_{E_{N+|Y|-1}})^{3/2-\rho}\|u\|^{\rho-1/2}_{E_{N+|Y|}}$,

where $C_{N,|Y|}$ is an increasing continuous function and $\tilde{T}_Y=T_Y-T^1_Y$.

Let $Y\mapsto a_Y$ be a linear function from $U(\mathfrak{p})$ to a Banach space B. We introduce (cf. (2.38) of [20])

$$\wp^B_N(a)=\Big(\sum_{\substack{Y\in\Pi'\\|Y|\leq N}}\|a_Y\|^2_B\Big)^{1/2},\quad N\geq 0. \tag{2.109}$$

When $B=E$ we write $\wp_N$ instead of $\wp^E_N$. According to definition (1.6a) of $\|\cdot\|_{E_N}$ we have

$$\wp_N(T^1(u))=\|u\|_{E_N},\quad N\geq 0. \tag{2.110}$$

We can now prove that the linear representation T^1 of $U(\mathfrak{p})$ is bounded by the nonlinear representation T, and vice-versa, on a E-neighbourhood of zero in E_∞.

Theorem 2.22. *There is a neighbourhood $\mathcal{O}$ of zero in E such that for $u \in E_N \cap \mathcal{O}$:*

i) $\wp_N(T(u)) \leq C_N \|u\|_{E_N}, \quad N \geq 0,$

ii) $\|u\|_{E_N} \leq C_N \wp_N(T(u)), \quad 0 \leq N \leq 1,$

iii) $\|u\|_{E_N} \leq F_N(\wp_1(T(u)))\wp_N(T(u)), \quad N \geq 0.$

Here C_N is a constant and F_N an increasing continuous function, only depending on the neighbourhood $\mathcal{O}$.

Proof. For $K > 0$ we define $\mathcal{O} = \{u \in E \big| \|u\|_E < K\}$. Let $Y \in \Pi'$, $|Y| \leq N$ and $N \geq 0$. It follows from i) of Corollary 2.21 that for $u \in E_N \cap \mathcal{O}$,

$$\|T_Y(u)\|_E \leq C_{0,|Y|}(\|u\|_E)\|u\|_{E_{|Y|}} \leq C_{0,|Y|}(K)\|u\|_{E_N}.$$

Summation over $Y \in \Pi'$, $|Y| \leq N$ now gives inequality i) of the theorem for $N \geq 0$.

To prove the second inequality we note that it is true for $N = 0$ and that, according to ii) of Corollary 2.21,

$$\begin{aligned}\wp_1(\tilde{T}(u)) &\leq \Big(\sum_{\substack{Y \in \Pi' \\ |Y| \leq 1}} (C_{0,|Y|}\big(\|u\|_E)\|u\|_E \|u\|_{E_{|Y|}}\big)^2 \Big)^{1/2} \\ &\leq C(K)K\|u\|_{E_1}, \quad u \in E_1 \cap \mathcal{O},\end{aligned}$$

where $C(K)$ is continuous and increasing in $K \geq 0$. For a given χ, $0 < \chi < 1$, we choose K such that $C(K)K \leq \chi$. Then

$$\wp_1(\tilde{T}(u)) \leq \chi \|u\|_{E_1}, \quad u \in E_1 \cap \mathcal{O}.$$

Since $\|u\|_{E_1} = \wp_1(T^1 u) \leq \wp_1(T(u)) + \wp_1(\tilde{T}(u))$ we obtain $\|u\|_{E_1} \leq \chi\|u\|_{E_1} + \wp_1(T(u))$, which gives

$$\|u\|_{E_1} \leq (1-\chi)^{-1}\wp_1(T(u)), \quad u \in E_1 \cap \mathcal{O}, \tag{2.111}$$

where $0 < \chi < 1$. This proves inequality ii).

We prove the third inequality by induction. It follows from inequality ii) that it is true for $N = 0$ and $N = 1$. Suppose it is true up to $N-1, N \geq 1$. Then inequality iii) of Corollary 2.21 gives for $|Y| \leq N, Y \in \Pi'$,

$$\begin{aligned}\|\tilde{T}_Y(u)\|_E &\leq C_{0,|Y|}(K)\|u\|_{E_1}\|u\|_{E_{N-1}}^{3/2-\rho}\|u\|_{E_N}^{\rho-1/2} \\ &\leq C_{0,|Y|}(K)C_1\wp_1(T(u))\big(F_{N-1}(\wp_1(T(u)))\wp_{N-1}(T(u))\big)^{3/2-\rho}\|u\|_{E_N}^{\rho-1/2},\end{aligned}$$

where inequality ii) and the induction hypothesis were used for the second inequality. Since $\|u\|_{E_N} \leq \wp_N(T(u)) + \wp_N(\tilde{T}(u))$ we get after summation over Y

$$\|u\|_{E_N} \leq \wp_N(T(u)) + H_N\big(\wp_1(T(u))\big)\wp_{N-1}(T(u))^{3/2-\rho}\|u\|_{E_N}^{\rho-1/2}, \tag{2.112}$$

where H_N is an increasing continuous function depending only on K. Since $\wp_1(T(u)) = \wp_{N-1}(T(u)) = 0$ if $\wp_N(T(u)) = 0, N \geq 1$, it follows from (2.112) that inequality iii) of the theorem is true for $\wp_N(T(u)) = 0$. Let $\wp_N(T(u)) > 0$ and let $x = \|u\|_{E_N} / \wp_N(T(u))$. Since $\wp_{N-1}(T(u)) \leq \wp_N(T(u))$, inequality (2.112) gives

$$x \leq 1 + A_N x^{\rho - 1/2}, \quad x \geq 0, \tag{2.113}$$

where $A_N = H_N(\wp_1(T(u)))$. If $x \leq 1$ it then follows from the definition of x that inequality iii) of the theorem is true. If $x > 1$, it then follows from (2.113) that $x \leq 1 + A_N x^{1/2}$, since $0 < \rho < 1/2$, which shows that $x^{1/2} \leq A_N/2 + ((A_N/2)^2 + 1)^{1/2} \leq 2((A_N/2)^2 + 1)^{1/2}$. So $x \leq A_N^2 + 4$ and $\|u\|_{E_N} \leq (A_N^2 + 4)\wp_N(T(u))$, which proves that inequality iii) of the theorem is also true for $x > 1$. This proves the theorem.

3. The asymptotic nonlinear representation.

If we suppose that the limit (1.17c) exists and that $t \mapsto u(t) = \Omega_+(U^{(+)}_{\exp(tP_0)}(v))$ is a solution of the evolution equation (1.2), then we can easily find the explicit expressions (1.17a) and (1.17b) of $U^{(+)}_g$, $g \in \mathcal{P}_0$, since the Maxwell-Dirac equations are manifestly covariant. Let $T^{(+)}$ be the Lie algebra representation of $\mathfrak{p}$ which is the differential of $U^{(+)}$. In this chapter we shall prove that $U^{(+)}_g$ maps $E^{0\rho}_N$ into $E^{0\rho}_N$, for N sufficiently large, that $T^{(+)}_X$ maps E_∞ into E_∞ and deduce properties of the C^n-vectors of $U^{(+)}$. We give a direct proof (theorem 3.14) of the fact that $X \mapsto T^{(+)}_X$ is a representation of $\mathfrak{p}$. We return to the question of the existence of the limit (1.17c) in chapter 6.

To begin with we study *the phase function* in (1.23a) of the definition of $U^{(+)}$.

Lemma 3.1. *Let f be a continuous function on $D_0 = \{(t,x) \in \mathbb{R}\times\mathbb{R}^3 \big| t > |x| \text{ and } t > 0\}$, which is absolutely Lebesgue integrable on every half-line in D_0 starting at the origin. f is defined to be zero outside D_0. Let*

$$\hat{g}(k) = \int_0^\infty f(\tau\omega(k)/m, \tau k/m)d\tau, \quad k \in \mathbb{R}^3, w(k) = (m^2+k^2)^{1/2}.$$

i) *if $f(t,x) = 0$ for $0 \le t \le 1$ and $f \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3))$, then,*
$\|(\omega(-i\partial))^{-2} g\|_{L^2} \le C \sup_{t \ge 1} \|f(t)\|_{L^2}$,

ii) *if the function $(t,x) \mapsto (1+t)(1+t-|x|)^\varepsilon f(t,x)$ belongs to $L^\infty(D_0)$, where $\omega(k) = (m^2+k^2)^{1/2}$ and $\omega(-i\partial) = (m^2+|-i\partial|^2)^{1/2} = (m^2-\Delta)^{1/2}$, then,*
$|\hat{g}(k)| \le \dfrac{m}{\omega(k)}(\ln(1+2(\omega(k)/m)^2)+\varepsilon^{-1}) \sup_{(t,x)\in D_0}((1+t)(1+t-|x|)^\varepsilon |f(t,x)|)$, $\varepsilon > 0$.

Proof. To prove the first statement of the lemma, let $D_1 = \{(t,x) \in \mathbb{R}\times\mathbb{R}^3 \big| t \ge (1+|x|^2)^{1/2}\}$ and let

$$\hat{g}_0(k) = \int_0^1 f(\tau\omega(k)/m, \tau k/m)d\tau, \quad \hat{g}_1(k) = \int_1^\infty f(\tau\omega(k)/m, \tau k/m)d\tau. \tag{3.1}$$

For $\hat{g}_1$, Schwarz inequality gives

$$\begin{aligned} |\hat{g}_1(k)| &\le \Big(\int_1^\infty \tau^{-2\delta} d\tau\Big)^{1/2} \Big(\int_1^\infty \tau^{2\delta} |f(\tau\omega(k)/m, \tau k/m)|^2 d\tau\Big)^{1/2} \\ &\le \Big(C_\delta \int_1^\infty \tau^{2\delta} |f(\tau\omega(k)/m, \tau k/m)|^2 d\tau\Big)^{1/2}, \quad \delta > 1/2. \end{aligned}$$

Plancherel theorem now gives

$$\begin{aligned} \|\omega(-i\partial)^{-2} g_1\|^2_{L^2} &= \|\omega^{-2}\hat{g}\|^2_{L^2} \\ &\le C_\delta \int_{\mathbb{R}^3} dk \int_1^\infty d\tau \tau^{2\delta} |f(\tau\omega(k)/m, \tau k/m)|^2 \omega(k)^{-4}. \end{aligned}$$

Making the variable transformation $t = \tau\omega(k)/m$, $x = \tau k/m$, with Jacobian equal to $m^2\tau^3/\omega(k)$, in the right-hand side of the last inequality and using the fact that $\omega(k) = mt/\tau$, we obtain

$$\|\omega(-i\partial)^{-2}g_1\|_{L^2}^2 \leq C_\delta \int_{D_1} |f(t,x)|^2(1-(x/t)^2)^\delta t^{2\delta-3}m^{-5}dtdx.$$

Since $t \geq 1$ in D_1 we obtain, choosing $1/2 < \delta < 1$,

$$\|\omega(-i\partial)^{-2}g_1\|_{L^2}^2 \leq C'_\delta m^{-5} \sup_{t\geq 1} \|f(t)\|_{L^2}^2. \tag{3.2}$$

For $\hat{g}_0$, we have by Schwarz inequality

$$|\hat{g}_0(k)|^2 \leq \int_0^1 |f(\tau\omega(k)/m, \tau k/m)|^2 d\tau.$$

Multiplication by $\omega(k)^{-4}$, integration on k and changing variables as in the case for $\hat{g}_1$ give, since $D_1 \subset D_0$,

$$\|\omega(-i\partial)^{-2}g_0\|_{L^2}^2 \leq \int_{D_0-D_1} |f(t,x)|^2 t^{-3}m^{-5}dtdx.$$

Since $t \geq 1$ in the support of f, we obtain

$$\|\omega(-i\partial)^{-2}g_0\|_{L^2}^2 \leq Cm^{-5} \sup_{t\geq 1} \|f(t)\|_{L^2}^2. \tag{3.3}$$

Definition (3.1), inequalities (3.2) and (3.3) prove statement i) of the lemma.

To prove the second statement, let

$$C = \sup_{(t,x)\in D_0} ((1+t)(1+t-|x|)^\varepsilon |f(t,x)|).$$

It follows from the definition of $\hat{g}$ that

$$|\hat{g}(k)| \leq C \int_0^\infty (1+\tau\omega(k)/m)^{-1}(1+\tau\omega(k)/m - \tau|k|/m)^{-\varepsilon} d\tau.$$

Since $\omega(k) - |k| \geq m^2/(2\omega(k))$ we obtain, with $t = \tau\omega(k)/m$,

$$\begin{aligned}
|\hat{g}(k)| &\leq C\frac{m}{\omega(k)} \int_0^\infty (1+t)^{-1}(1+tm^2/(2\omega(k)^2))^{-\varepsilon}dt \\
&\leq C\frac{m}{\omega(k)}\Big(\int_0^{2(\omega(k)/m)^2} (1+t)^{-1}dt + \int_{2(\omega(k)/m)^2}^\infty t^{-(1+\varepsilon)}dt\ (2(\omega(k)/m)^2)^\varepsilon\Big) \\
&= C\frac{m}{\omega(k)}(\ln(1+2(\omega(k)/m)^2) + \varepsilon^{-1}),
\end{aligned}$$

which proves the last statement of the lemma.

In the case where f in Lemma 3.1 is a solution of the wave equation and if the spatial Fourier transform of f vanishes in a neighbourhood of zero, then $\hat{g}$ exists and its L^∞-norm can be estimated directly in terms of weighted L^2-norms of the Fourier transform of the initial data. We recall that M^ρ_c, $\rho \in]-1/2, \infty[$, was defined in Theorem 2.9, and since there is no possibility of confusion we keep the previous notation $\rho(t,x) = (t^2 - |x|^2)^{1/2}$ for $|t| \geq |x|$.

Lemma 3.2. *Let* $(h, \dot{h}) \in M^\rho_c$, *let*

$$f(t,x) = \chi_0(\rho(t,x))\big(\cos((-\Delta)^{1/2}t)h + (-\Delta)^{-1/2}\sin((-\Delta)^{1/2}t)\dot{h}\big)(x),$$

where $\chi_0 \in C^\infty([0,\infty[)$, *and let* $\hat{g}$ *be defined as in Lemma 3.1.*

i) *If* $-1/2 < \rho$, $a, b \in \mathbb{R}$, $a - \rho > -1/2$, $a - \rho - b < -1/2$ *and if* $0 \leq \chi_0(\tau) \leq 1$ *for* $\tau \in [0, \infty[$, $\chi_0(\tau) = 1$ *for* $\tau \geq 2$, *then*

$$\|\hat{g}\|_{L^\infty} \leq C_{\rho,a,b} \||\nabla|^{-a}(1+|\nabla|)^b (h,\dot{h})\|_{M^\rho}.$$

ii) *If* $1/2 < \rho < 1$, *and if* $\chi_0(\tau) = 0$ *for* $\tau \geq 2$, *then*

$$\|\omega^{3/2-\rho}\hat{g}\|_{L^\infty} \leq C_\rho \|(h,\dot{h})\|_{M^\rho}.$$

Before proving the lemma we remark that the definition of $\hat{g}$ makes sense. In fact if $(h,\dot{h})$ is as in the lemma, then f is absolutely integrable on half-lines starting at the origin according to Proposition 2.15. Since M^ρ_c is dense in M^ρ_∞ according to Theorem 2.9, it follows from the inequality in Lemma 3.2 that the linear map $(h,\dot{h}) \mapsto \hat{g}$ has a unique continuous extension to the Hilbert space $|\nabla|^a(1+|\nabla|)^{-b}M^\rho$.

Proof. To prove statement i), let $0 < r < R$ be such that the supports of $\hat{h}$ and $\hat{\dot{h}}$ are contained in $\{p \in \mathbb{R} | r \leq |p| \leq R\}$. We extend the domain of definition of χ_0 to $\mathbb{R}$ by defining $\chi_0(\tau) = 0$ for $\tau < 0$. Let $\varphi \in S(\mathbb{R})$, $0 \leq \varphi(\tau) \leq 1$, $\varphi(\tau) = 1$ for $|\tau| \leq 1$ and $\varphi(\tau) = 0$ for $|\tau| \geq 2$. Let $\chi_1 = (1-\varphi)\chi_0$ and let $\theta \in S'(\mathbb{R})$ (resp. $\theta_1 \in S'(\mathbb{R})$) be the inverse Fourier transform of $\sqrt{2\pi}\chi_0$ (resp. $\sqrt{2\pi}\chi_1$). Since the derivative $\chi_1' \in C^\infty_0$ it follows that $s \mapsto s\theta_1(s)$ is a function in $S(\mathbb{R})$, so θ_1 restricted to $\mathbb{R} - \{0\}$ is a C^∞ function satisfying $\theta_1(s) \leq A_n(1+|s|)^{-n}|s|^{-1}$, $n \geq 0$, $s \neq 0$ for some constants A_n. Since $\chi_0 - \chi_1 \in C^\infty_0(\mathbb{R}-\{0\})$ and since $\chi_0 - \chi_1$ has bounded left and right derivatives of all orders at zero, it follows that $\theta - \theta_1$ is an entire function satisfying $|\theta(s) - \theta_1(s)| \leq A(1+|s|)^{-1}$, $s \in \mathbb{R}$, for some $A > 0$. Therefore

$$|\theta(s)| \leq A|s|^{-1}, \quad s \neq 0, \tag{3.4a}$$

for some constant A.

Since $h \in S(\mathbb{R}^3)$, we get by Fourier transformation

$$\begin{aligned}&(e^{i\varepsilon\tau\omega(k)|\nabla|/m}h)(\tau k/m)\\ &\quad = (2\pi)^{-3/2}\int e^{i\tau(\omega(k)|p|\varepsilon + k\cdot p)/m}\hat{h}(p)dp, \quad \tau \in \mathbb{R}, k \in \mathbb{R}^3.\end{aligned}$$

This gives

$$\int_{-\infty}^{\infty} (e^{i\varepsilon\tau\omega(k)|\nabla|/m} h)(\tau k/m)\chi_0(\tau)d\tau$$
$$= (2\pi)^{-3/2} \int \chi_0(\tau) e^{i\tau(\omega(k)|p|\varepsilon + k\cdot p)/m} \hat{h}(p) dp d\tau.$$

Since $\frac{1}{m}(\omega(k)|p| + \varepsilon k \cdot p) \geq (2\omega(k))^{-1} m|p| \geq (2\omega(k))^{-1} mr$ in the support of $\hat{h}$, we obtain

$$\int_{-\infty}^{\infty} (e^{i\varepsilon\tau\omega(k)|\nabla|/m} h)(\tau k/m)\chi_0(\tau)d\tau \tag{3.4b}$$
$$= (2\pi)^{-2} \int \theta(m^{-1}(\varepsilon\omega(k)|p| + k.p))\hat{h}(p)dp,$$

where according to (3.4a)

$$|\theta(m^{-1}(\varepsilon\omega(k)|p| + k\cdot p))| \leq Am|\omega(k)|p| + \varepsilon k\cdot p|^{-1} \leq 2A\omega(k)/(m|p|), \quad |p| > 0.$$

Let $d, b \in \mathbb{R}$, $d > -1/2$ and $d - b < -1/2$. Schwarz inequality and Plancherel theorem give

$$|\int_{-\infty}^{\infty} (e^{i\varepsilon\tau\omega(k)|\nabla|/m} h)(\tau k/m)\chi_0(\tau)d\tau| \tag{3.5}$$
$$\leq \Big(\int_{r\leq|p|\leq R} |\theta(m^{-1}(\varepsilon\omega(k)|p| + k\cdot p))|^2 |p|^{2d}(1+|p|)^{-2b} dp\Big)^{1/2} \||\nabla|^{-d}(1+|\nabla|)^b h\|_{L^2}.$$

Let $F_\varepsilon(s) = \varepsilon \int_s^{\varepsilon\infty} |\theta(\xi)|^2 d\xi$. Then $0 \leq F_\varepsilon(s) \leq B|s|^{-1}$, for $\varepsilon s > 0, \varepsilon = \pm$ and some $B > 0$. We have

$$\begin{aligned} I(k) &= \int_{r\leq|p|\leq R} |\theta(m^{-1}(\varepsilon\omega(k)|p| + k\cdot p))|^2 |p|^{2d}(1+|p|)^{-2b} dp \\ &= 2\pi \int_r^R d|p||p|^{2+2d}(1+|p|)^{-2b} \int_0^\pi d\nu \sin\nu \big|\theta(m^{-1}(\varepsilon\omega(k)|p| + \cos\nu|k||p|))\big|^2 \\ &= 2\pi \int_{|p|\geq r} d|p||p|^{1+2d}(1+|p|)^{-2b} m|k|^{-1}\big(F_\varepsilon(m^{-1}\varepsilon(\omega(k) - |k|)|p|) \\ &\quad - F_\varepsilon(m^{-1}\varepsilon(\omega(k) + |k|)|p|)\big). \end{aligned}$$

This gives $I(k) \leq C_{d,b}\,\omega(k)$ for $|k| \geq m$. It follows directly from the definition of $I(k)$ and from inequality $|\theta(m^{-1}(\varepsilon\omega(k)|p| + k\cdot p))| \leq C|p|^{-1}$ for $|k| \leq m$, $|p| > 0$, that $I(k) \leq C'_{d,b}$ for $|k| \leq m$. Inequality (3.5) now gives

$$|\int_{-\infty}^{\infty} (e^{i\varepsilon\tau\omega(k)|\nabla|/m} h)(\tau k/m)\chi_0(\tau)d\tau| \tag{3.6}$$
$$\leq C_{d,b}\||\nabla|^{-d}(1+|\nabla|)^b h\|_{L^2}, \quad d > -1/2, d - b < -1/2.$$

Since

$$\hat{g}(k) = \int_{-\infty}^{\infty} f(\tau\omega(k)/m, \tau k/m) d\tau$$

and

$$f(\tau\omega(k)/m, \tau k/m) = \frac{\chi_0(\tau)}{2} \sum_{\varepsilon=\pm} (e^{\varepsilon i\tau\omega(k)|\nabla|/m} h - ie^{\varepsilon i\tau\omega(k)|\nabla|/m} |\nabla|^{-1}\dot{h}),$$

the inequality in the lemma follows from (3.6) with h and (3.6) with $|\nabla|^{-1}\dot{h}$ instead of h and by defining $a = d + \rho$. This proves statement i) of the lemma.

To prove statement ii) of the lemma, we observe that in this case, instead of inequality (3.4a), we have

$$|\theta(s)| \leq A(1 + |s|)^{-1}, \quad s \in \mathbb{R}.$$

In the same way as we obtained (3.5), it now follows that

$$\begin{aligned} &\Big| \int_{-\infty}^{\infty} (e^{i\varepsilon\tau\omega(k)|\nabla|/m} h)(\tau k/m) \chi_0(\tau) d\tau \Big| \\ &\qquad \leq \Big(\int |\theta(m^{-1}(\varepsilon\omega(k)|p| + k \cdot p))|^2 |p|^{-2\rho} dp \Big)^{1/2} \| |\nabla|^\rho h \|_{L^2}, \quad 1/2 < \rho < 1. \end{aligned}$$

The last two inequalities give that

$$\begin{aligned} &\Big| \int_{-\infty}^{\infty} (e^{i\varepsilon\tau\omega(k)|\nabla|/m} h)(\tau k/m) \chi_0(\tau) d\tau \Big| \\ &\qquad \leq C_\rho (\omega(k))^{-3/2+\rho} \| |\nabla|^\rho h \|_{L^2}, \quad 1/2 < \rho < 1. \end{aligned}$$

The inequality of statement ii) now follows as in the proof of statement i). This proves the lemma.

We are now ready to study *the phase function* in the definitions (1.17a) and (1.17b) of $U^{(+)}$. For $(f, \dot{f}) \in M^\rho_\infty$, $1/2 < \rho < 1$, we introduce, if $X = a^\nu P_\nu$,

$$(\Phi_{\varepsilon,X}(f, \dot{f}))(k) = \int_0^\infty a^\nu l^\mu_\varepsilon \chi_0(m\tau) B^{(+)1}_{\nu\mu}(\tau l_\varepsilon) d\tau, \quad 0 \leq \nu \leq 3, \varepsilon = \pm, \tag{3.7}$$

where $l^0_\varepsilon = \omega(k)$, $l^j_\varepsilon = -\varepsilon k_j$, $1 \leq j \leq 3$,

$$B^{(+)1}_\mu(t) = \cos((-\Delta)^{1/2} t) f_\mu + (-\Delta)^{-1/2} \sin((-\Delta)^{1/2} t) \dot{f}_\mu, \tag{3.8a}$$

$$B^{(+)1}_{0\mu}(t) = \frac{d}{dt} B^{(+)1}_\mu(t), \tag{3.8b}$$

$$B^{(+)1}_{j\mu}(t) = \partial_j B^{(+)1}_\mu(t), \quad 1 \leq j \leq 3,$$

and where $\chi_0 \in C^\infty([0, \infty[)$, $\chi_0(s) = 1$ for $s \geq 2$ and $0 \leq \chi_0(s) \leq 1$. We use Einstein's summation convention over μ and ν in (3.7). It follows directly from Proposition 2.15 and statement ii) of Lemma 3.1 that $k \mapsto (\Phi_{\varepsilon,X}(f, \dot{f}))(k)$ is a C^∞ function for $(f, \dot{f}) \in M^\rho_\infty$.

We denote by $\Phi_\epsilon(f,\dot f)$ the linear map $X \mapsto \Phi_{\epsilon,X}(f,\dot f)$ from $\mathbb{R}^4$ to $C^\infty(\mathbb{R}^3)$.

To state the next proposition we note that

$$\eta^{(\varepsilon)}_{M_{ij}} = -k_i \frac{\partial}{\partial k_j} + k_j \frac{\partial}{\partial k_i}, \quad 1 \le i < j \le 3, \tag{3.9a}$$

$$\eta^{(\varepsilon)}_{M_{0i}} = -\varepsilon \frac{\partial}{\partial k_j} \omega(k), \quad 1 \le i \le 3, \varepsilon = \pm, \tag{3.9b}$$

are the generators for a representation $\eta^{(\varepsilon)}$ of $so(3,1)$ on $C^\infty(\mathbb{R}^3,\mathbb{C})$ and that the matrices $n_{\mu\nu}$, $0 \le \mu < \nu \le 3$, in (1.5) define a representation which we denote $X \mapsto n_X$ of $so(3,1)$ in $\mathbb{C}^4$.

Proposition 3.3. *Let $N \ge 0$ be an integer and Π'' be the restriction of the standard basis Π' of $U(\mathfrak{p})$ to the enveloping algebra $U(so(3,1))$. Then*

i) $$\sum_{\substack{Y \in \Pi'' \\ |Y| \le N}} \|\omega^{-5/2-a} \eta^{(\varepsilon)}_Y \Phi_\varepsilon(f,\dot f)\|_{L^2} \le C_{\rho,N,a,b} \|(1-\Delta)^{b/2}(f,\dot f)\|_{M^\rho_N},$$

if $(1-\Delta)^{b/2}(f,\dot f) \in M^\rho_N$, $-1/2 < \rho \le 3/2$, $a > 0$, $b > 3/2 - \rho$,

ii) $$\sum_{\substack{Y \in \Pi'' \\ |Y| \le N}} \|\omega^{-1} \eta^{(\varepsilon)}_Y \Phi_\varepsilon(f,\dot f)\|_{L^\infty} \le C_{\rho,N} \|(1-\Delta)^{b/2}(f,\dot f)\|_{M^\rho_N},$$

if $(1-\Delta)^{b/2}(f,\dot f) \in M^\rho_N$, $-1/2 < \rho < 3/2$, $b > 3/2 - \rho$,

iii) $$\sum_{\substack{Y \in \Pi'' \\ |Y| \le N}} |(\eta^{(\varepsilon)}_Y \Phi_\varepsilon(f,\dot f)(k)| \le C_{\rho,N}(1+\ln(1+\omega(k)/m)) \|(f,\dot f)\|_{M^\rho_{N+3}},$$

if $(f,\dot f) \in M^\rho_{N+3}$, $1/2 < \rho < 1$.

Proof. Let $(f,\dot f) \in M_c$, where M_c is defined in Theorem 2.9. The initial conditions $f^{(\nu)}_\mu, \dot f^{(\nu)}_\mu$ of the solution $B_{\nu\mu}$, given by (3.8b), of the wave equation satisfy

$$\||\nabla|^{-1}(1-\Delta)^{b/2}(f^{(\nu)},\dot f^{(\nu)})\|_{M^\rho_0} \le \|(1-\Delta)^{b/2}(f,\dot f)\|_{M^\rho_0}, \quad b \in \mathbb{R}.$$

It follows from definition (3.7) of $\Phi_{\varepsilon,X}$, $X \in \mathfrak{p}$, and from Lemma 3.2 (with $a = 1$) that

$$\|\omega^{-1}\Phi_{\varepsilon,P_\nu}(f,\dot f)\|_{L^\infty} \le C_{\rho,b} \|(1-\Delta)^{b/2}(f,\dot f)\|_{M^\rho_0}, \tag{3.10}$$

where $\varepsilon = \pm$, $0 \le \nu \le 3$, $-1/2 < \rho < 3/2$, $b > 3/2 - \rho$. We define a group representation $V^{(\varepsilon)}$, $\varepsilon = \pm$, of $SL(2,\mathbb{C})$ on the space of distributions $r \in S'(\mathbb{R}^3,\mathbb{C}^4)$ by

$$(V^{(\varepsilon)}_A r)(k) = \Lambda(A) F^{(\varepsilon)}(\Lambda(A)^{-1}(\omega(k), -\varepsilon k)),$$

where $A \mapsto \Lambda(A)$ is the canonical projection of $SL(2,\mathbb{C})$ onto $SO(3,1)$ and where the function $F^{(\varepsilon)}\colon \{p \in \mathbb{R}^4 | p_0 > 0, p^\mu p_\mu = m^2\} \to \mathbb{C}^4$ is given by $F^{(\varepsilon)}(\omega(k), -\varepsilon k) = r(k)$.

Let $r_\nu^{(\varepsilon)}(f,\dot f)=\Phi_{\varepsilon,P_\nu}(f,\dot f)$. It follows from definition (3.7) of $\Phi_{\varepsilon,X}$ that $V_A^{(\varepsilon)}r^{(\varepsilon)}(f,\dot f)=r^{(\varepsilon)}(U^{M1}_{(0,A)}(f,\dot f))$. Since the differential of $V^{(\varepsilon)}$ is the Lie algebra representation $\eta^{(\varepsilon)}+n$ and since $(f,\dot f)\in M_c\subset M^\rho_\infty$, it follows that

$$\eta_X^{(\varepsilon)}\Phi_{\varepsilon,P_\nu}(f,\dot f)+(n_X)^\mu{}_\nu\Phi_{\varepsilon,P_\mu}(f,\dot f)=\Phi_{\varepsilon,P_\nu}(T_X^{M1}(f,\dot f)),\quad X\in\mathfrak{p}, \tag{3.11}$$

so

$$\sum_\nu|n_X^{(\varepsilon)}\Phi_{\varepsilon,P_\nu}(f,\dot f))\le C\sum_\nu(|\Phi_{\varepsilon,P_\nu}(T_X^{M1}(f,\dot f))|+|\Phi_{\varepsilon,P_\nu}(f,\dot f)|),\quad X\in\Pi\cap so(3,1),$$

which extends to the enveloping algebra:

$$\sum_\nu|\eta_Y^{(\varepsilon)}\Phi_{\varepsilon,P_\nu}(f,\dot f)|\le C_{|Y|}\sum_\nu\sum_{\substack{Z\in\Pi''\\|Z|\le|Y|}}|\Phi_{\varepsilon,P_\nu}(T_Z^{M1}(f,\dot f))|,\quad Y\in\Pi''. \tag{3.12}$$

The inequality in statement ii) of the proposition, for $(f,\dot f)\in M_c$, now follows from inequality (3.10) and inequality (3.11), since

$$\|(1-\Delta)^{b/2}T_Y^{M1}(f,\dot f)\|_{M^\rho}\le C_{|Y|,b}\|(1-\Delta)^{b/2}(f,\dot f)\|_{M_N^\rho},\quad Y\in\Pi'',|Y|\le N.$$

We note that M_c is dense in M_∞ according to Theorem 2.19. So M_c is dense in M_N^ρ which proves statement ii). Statement i) follows trivially from statement ii).

To prove statement iii) we suppose that $(f,\dot f)\in M_c$. It follows from statement ii) of Lemma 3.1 that

$$\begin{aligned}|\Phi_\varepsilon(f,\dot f)|(k)\le{}& C\big(\ln(1+2(\omega(k)/m)^2)+(3/2-\rho)^{-1}\big)\\ &\sum_{\mu,\nu}\sup_{\substack{t\ge0\\t\ge|x|}}((1+t)(1+t-|x|)^{3/2-\rho}|B^{(+)1}_{\nu,\mu}(t,x)|),\quad\rho<3/2.\end{aligned} \tag{3.13}$$

According to definition (3.8) of $B^{(+)1}_{\nu\mu}$, we obtain using Proposition 2.15 that statement iii) is true for $N=0$ and $(f,\dot f)\in M_c$. Inequality (3.12) then proves the inequality in statement iii) for $(f,\dot f)\in M_c$. Since M_c is dense in M^ρ_{3+N}, this proves the proposition.

We now define rigourously $T^{(+)}$ by $T^{(+)}=T^1+T^{(+)2}$, where $T^{(+)2}_{M_{\mu\nu}}=0$ for $0\le\mu<\nu\le3$ and, if $u=(f,\dot f,\alpha)\in E_\infty, 0\le\mu\le3$,

$$(T^{(+)2}_{P_\mu}(u))^\wedge(k)=i(0,0,\sum_{\varepsilon=\pm}(\Phi_{\varepsilon,P_\mu}(f,\dot f))(k)P_\varepsilon(k)\hat\alpha(k)), \tag{3.14}$$

Proposition 3.3 and the algebraic properties of $T^{(+)}$ will permit us to prove that $T^{(+)}$ is a nonlinear representation of $\mathfrak{p}$ on E_∞.

Theorem 3.4. *$T_X^{(+)}$, $X \in \mathfrak{p}$ is a continuous polynomial of degree two from E_∞^ρ to E_∞^ρ and $T^{(+)}$ is a nonlinear representation of $\mathfrak{p}$ on E_∞^ρ, $1/2 < \rho < 1$.*

Proof. T_X^1, $X \in \mathfrak{p}$ is a continuous linear map from E_∞ to E_∞ by the definition of E_∞. It follows from definition (3.9b) of $\eta_{M_{0i}}^{(\varepsilon)}$, $1 \leq i \leq 3$, and statement iii) of Proposition 3.3 that Φ_{ε,P_ν} is a continuous linear map from M_∞^ρ into the space of C^∞ functions on $\mathbb{R}^3$ uniformly bounded together with their derivatives by a constant times the function $k \mapsto (\omega(k))$. This proves that the bilinear map $M_\infty^\rho \times D_\infty \to S(\mathbb{R}^3, \mathbb{C}^4) = D_\infty$ defined by $((f,\dot{f}),\alpha) \mapsto \Phi_{\varepsilon,P_\nu}(f,\dot{f})P_\varepsilon\hat{\alpha}$ is continuous, so $T_X^{(+)}\colon E_\infty^\rho \to E_\infty^\rho$ is continuous.

To prove that $T^{(+)}$ is a representation we have to show that

$$T_{XY}^{(+)} - T_{YX}^{(+)} = T_{[X,Y]}^{(+)}, \quad X, Y \in \mathfrak{p}. \tag{3.15}$$

Since $T_X^{(+)} = T_X^1$ for $X \in sl(2,\mathbb{C})$, (3.15) is true for $X, Y \in sl(2,\mathbb{C})$. It follows from definition (1.5) of T^1 and definition (3.9) of $\eta^{(\varepsilon)}$ that

$$(T_X^{D1}\alpha)^\wedge = \sum_{\varepsilon=\pm}(\eta_X^{(\varepsilon)} + \sigma_X)P_\varepsilon\hat{\alpha}, \quad X \in \mathfrak{sl}(2,\mathbb{C}), \tag{3.16}$$

where $X \mapsto \sigma_X$ is the representation of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}^4$ defined by the matrices $\sigma_{\mu\nu}$, $0 \leq \mu < \nu \leq 3$, and where P_ε is given by (1.16). A direct calculation shows that

$$(-n_X\Phi_\varepsilon)_\mu = \Phi_{\varepsilon,[X,P_\mu]}, \quad X \in \mathfrak{sl}(2,\mathbb{C}), 0 \leq \mu \leq 3. \tag{3.17}$$

Formulas (3.11), (3.16) and (3.17) give (since $\eta_X^{(\varepsilon)}$ is a derivation and since T_X^{D1} commutes with the projector $P_\varepsilon(-i\partial)$) with $X \in \mathfrak{sl}(2,\mathbb{C})$, $Y \in \mathbb{R}^4$,

$$\begin{aligned}(T_{XY}^{D2}(f,\dot{f},\alpha))^\wedge &= (T_X^{D1}T_Y^{D2}(f,\dot{f},\alpha))^\wedge \\ &= i\sum_{\varepsilon=\pm}(\eta_X^{(\varepsilon)} + \sigma_X)\Phi_{\varepsilon,Y}(f,\dot{f})(P_\varepsilon(-i\partial)\alpha)^\wedge \\ &= i\sum_{\varepsilon=\pm}((\eta_X^{(\varepsilon)}\Phi_{\varepsilon,Y}(f,\dot{f}))P_\varepsilon\hat{\alpha} + \Phi_{\varepsilon,Y}(f,\dot{f})(T_X^{D1}P_\varepsilon(-i\partial))\alpha)^\wedge \\ &= i\sum_{\varepsilon=\pm}(\Phi_{\varepsilon,Y}(T_X^{M1}(f,\dot{f})) + \Phi_{\varepsilon,[X,Y]}(f,\dot{f}))(P_\varepsilon(-i\partial)\alpha)^\wedge \\ &\qquad + i\sum_{\varepsilon=\pm}\Phi_{\varepsilon,Y}(f,\dot{f}))(P_\varepsilon(-i\partial)T_X^{D1}\alpha)^\wedge.\end{aligned} \tag{3.18}$$

On the other hand we have

$$\begin{aligned}(T_{YX}^{D2}(f,\dot{f},\alpha))^\wedge &= ((DT_Y^{D2}.T_X^1)(f,\dot{f},\alpha))^\wedge \\ &= i\sum_{\varepsilon=\pm}\Phi_{\varepsilon,Y}(T_X^{M1}(f,\dot{f}))(P_\varepsilon(-i\partial)\alpha)^\wedge + i\sum_{\varepsilon=\pm}\Phi_{\varepsilon,Y}(f,\dot{f}))(P_\varepsilon(-i\partial)T_X^{D1}\alpha)^\wedge.\end{aligned} \tag{3.19}$$

Equalities (3.18) and (3.19) give

$$((T_{XY}^{(+)D2} - T_{YX}^{(+)D2})(f, \dot{f}, \alpha))^{\wedge} = i \sum_{\varepsilon=\pm} \Phi_{\varepsilon,[X,Y]}(f, \dot{f})(P_\varepsilon(-i\partial)\alpha)^{\wedge} \tag{3.20}$$
$$= (T_{[X,Y]}^{(+)D2}(f, \dot{f}, \alpha))^{\wedge}, \quad X \in \mathfrak{sl}(2, \mathbb{C}), Y \in \mathbb{R}^4,$$

where the last step follows from definition (3.14) and from the fact that $[X, Y]$ is in the subalgebra $\mathbb{R}^4$ of $\mathfrak{p}$. This proves equality (3.15) for $X \in sl(2, \mathbb{C})$, $Y \in \mathbb{R}^4$, since $[T_X^1, T_Y^1] = T_{[X,Y]}^1$. Now let $X, Y \in \mathbb{R}^4$; then for $u = (f, \dot{f}, \alpha) \in E_\infty^\rho$

$$\begin{aligned}(T_{XY}^{(+)D}(u))^{\wedge} &= ((DT_X^{(+)D}.T_Y^{(+)})(u))^{\wedge} \\
&= (T_X^{(+)D1}T_Y^{(+)D}(u))^{\wedge} + ((DT_X^{(+)D2}.T_Y^{(+)})(u)^{\wedge} \\
&= (T_X^{(+)D1}T_Y^{(+)D}(u))^{\wedge} + i\sum_{\varepsilon=\pm} \Phi_{\varepsilon,X}(T_Y^{M1}(f, \dot{f}))(P_\varepsilon(-i\partial)\alpha)^{\wedge} \\
&\quad + i\sum_{\varepsilon=\pm} \Phi_{\varepsilon,X}(f, \dot{f})(P_\varepsilon(-i\partial)(T_Y^{(+)D}(u))^{\wedge} \\
&= (T_{XY}^{D1}\alpha)^{\wedge} + i\sum_{\varepsilon=\pm} \Phi_{\varepsilon,Y}(f, \dot{f})(P_\varepsilon(-i\partial)T_X^{D1}\alpha)^{\wedge} \\
&\quad + i\sum_{\varepsilon,X} \Phi_{\varepsilon,X}(T_Y^{M1}(f, \dot{f}))(P_\varepsilon(-i\partial)\alpha)^{\wedge} \\
&\quad + i\sum_{\varepsilon=\pm} \Phi_{\varepsilon,X}(f, \dot{f})(P_\varepsilon(-i\partial)T_Y^{D1}\alpha)^{\wedge} \\
&\quad - \sum_{\varepsilon,\varepsilon'=\pm} \Phi_{\varepsilon,X}(f, \dot{f})P_\varepsilon\Phi_{\varepsilon',Y}(f, \dot{f})P_{\varepsilon'}\hat{\alpha}.\end{aligned} \tag{3.21}$$

It follows from (3.21) that for $X, Y \in \mathbb{R}^4$

$$\begin{aligned}&((T_{XY}^{(+)D} - T_{YX}^{(+)D})(u))^{\wedge} \\
&\quad = i\sum_{\varepsilon}(\Phi_{\varepsilon,X}(T_Y^{M1}(f, \dot{f})) - \Phi_{\varepsilon,Y}(T_X^{M1}(f, \dot{f})))(P_\varepsilon(-i\partial)\alpha)^{\wedge}.\end{aligned} \tag{3.22}$$

It follows from definitions (3.7) of Φ and (3.8) of B that

$$\Phi_{\varepsilon,X}(T_Y^{M1}(f, \dot{f})) - \Phi_{\varepsilon,Y}(T_X^{M1}(f, \dot{f})) = 0, \quad X, Y \in \mathbb{R}^4, \tag{3.23}$$

which according to (3.22) shows that (3.15) is true for $X, Y \in \mathbb{R}^4$. This proves the theorem.

Lemma 3.5. *If* $(f, \dot{f}, \alpha) \in E_\infty^\rho$, $\frac{1}{2} < \rho < 1$, $0 \leq \mu \leq 3$, $\varepsilon = \pm$, $0 < a \leq 1$ *and* $\frac{3}{2} - \rho < b \leq 1$, *then*

$$\begin{aligned}\text{i) } \|\Phi_{\varepsilon,P_\mu}(f, \dot{f})\hat{\alpha}\|_{L^2} &\leq C_{\rho,a}\|(f, \dot{f}\|_{M_3^\rho}\|\omega(-i\partial)^a\alpha\|_{D_0} \\
&\leq C'_{\rho,a}\|(f, \dot{f})\|_{M_3^\rho}\|\alpha\|_{D_0}^{1-a}\|\omega(-i\partial)\alpha\|_{D_0}^a, \\
\text{ii) } \|\Phi_{\varepsilon,P_\mu}(f, \dot{f})\hat{\alpha}\|_{L^2} &\leq C_{\rho,b}\|(1 - \Delta)^{b/2}(f, \dot{f})\|_{M_0^\rho}\|\omega(-i\partial)\alpha\|_{D_0} \\
&\leq C'_{\rho,b}\|(f, \dot{f})\|_{M_0^\rho}^{1-b}\|(1 - \Delta)^{1/2}(f, \dot{f})\|_{M_0^\rho}^b\|\omega(-i\partial)\alpha\|_{D_0}.\end{aligned}$$

Proof. It follows from statement iii) of Proposition 3.3 that

$$\begin{aligned}\|\Phi_{\varepsilon,P_\nu}(f,\dot f)\hat\alpha\|_{L^2} &\le \|\omega^{-a}\Phi_{\varepsilon,P_\nu}(f,\dot f)\|_{L^\infty}\|\omega^a\hat\alpha\|_{L^2}\\ &\le C_{\rho,a}\|(f,\dot f)\|_{M_3}\|\omega^a\hat\alpha\|_{L^2}, \quad a>0.\end{aligned}$$

Since $\|\omega^a\hat\alpha\|_{L^2} \le \|\omega(-i\partial)\alpha\|_{L^2}^a\|\alpha\|_{L^2}^{1-a}$, $0<a\le 1$, we obtain the inequality in statement i).

We get using statement ii) of Proposition 3.3

$$\begin{aligned}\|\Phi_{\varepsilon,P_\nu}(f,\dot f)\hat\alpha\|_{L^2} &\le \|\omega^{-1}\Phi_{\varepsilon,P_\nu}(f,\dot f)\|_{L^\infty}\|\omega\hat\alpha\|_{L^2}\\ &\le C_{\rho,b}\|(1-\Delta)^{b/2}(f,\dot f)\|_{M_0^\rho}\|\omega(-i\partial)\alpha\|_{L^2}, \quad b>3/2-\rho.\end{aligned}$$

Since $\|\omega^{5/2-\rho}\hat\alpha\|_{L^2} \le \|\omega(-i\partial)^3\alpha\|_{L^2} \le C\|\alpha\|_{D_3}$ and by Sobolev embedding $\|\omega^3\hat\alpha\|_{L^\infty} \le C\|(1-\Delta)\omega^3\hat\alpha\|_{L^2} \le C'\|\alpha\|_{D_3}$ and $\|(f^{(i)},\dot f^{(i)})\|_{M^\rho} \le C\|(f,\dot f)\|_{M^\rho}$. This proves the first inequality in statement ii) of the lemma. The second follows from the first, since

$$\|(1-\Delta)^{b/2}(f,\dot f)\|_{M_0^\rho} \le \|(f,\dot f)\|_{M_0^\rho}^{1-b}\|(1-\Delta)^{1/2}(f,\dot f)\|_{M_0^\rho}^b \quad \text{for } b\le 1.$$

This proves the lemma.

Lemma 3.6. *If $u_1,u_2\in E_\infty$, $X\in\Pi$, $N\ge 0$ and $\frac32-\rho<a\le 1$, then*

i) $\|T_X^{(+)2}(u_1\otimes u_2)\|_{E_N} \le C_N(\|u_1\|_{E_1}\|u_2\|_{E_{N+1}} + \|u_1\|_{E_{N+1}}\|u_2\|_{E_1})$,

ii) $\|T_X^{(+)2}(u_1\otimes u_2)\|_{E_N} \le C_{N,a}(\|u_1\|_{E_3}\|u_2\|_{E_N} + \|u_1\|_{E_N}\|u_2\|_{E_3})^{1-a}$
$(\|u_1\|_{E_3}\|u_2\|_{E_{N+1}} + \|u_1\|_{E_{N+1}}\|u_2\|_{E_3})^a$,

iii) $\|T_X^{(+)2}(u_1\otimes u_2)\|_E \le C\min\big(\|u_1\|_{E_3}\|u_2\|_{E_0}^{1-a}\|u_2\|_{E_1}^a, \|u_2\|_{E_3}\|u_1\|_{E_0}^{1-a}\|u_1\|_{E_1}^a\big)$.

Proof. Let $\mathcal O_i$, $i=1,2$, be two orderings on the basis Π of $\mathfrak p$ and let Π_i' be the canonical basis of $U(\mathfrak p)$ corresponding to the ordering $\mathcal O_i$. The norms on the space of C^N-vectors given by (1.6a) corresponding to $\mathcal O_1$ and $\mathcal O_2$ respectively, are equivalent. Let $\mathcal O_1$ be an ordering such that $P_\alpha < M_{\mu\nu}$ and $\mathcal O_2$ an ordering such that $M_{\mu\nu}<P_\alpha$ for $0\le\mu<\nu\le 3$ and $0\le\alpha\le 3$.

If $Y\in U(\mathfrak{sl}(2,\mathbb C))$, then $T_Y^{(+)} = T_Y^1$ so it follows from (1.9) that $T_Y^1T_X^{(+)2} = T_{YX}^{(+)2}$ for $X\in\mathfrak p$. If moreover $X\in\Pi\cap\mathbb R^4$ and $Y\in\Pi_2'$, then $YX\in\Pi_2'$. Since YX can be written, as it is seen by commutation, as a linear combination of elements $Z'Z\in\Pi_1'$, where $Z'\in\mathbb R^4$, $Z\in\Pi_1'\cap U(\mathfrak{sl}(2,\mathbb C))$ and $|Z|\le|Y|$, we get, remembering that $T^{(+)2}$ vanishes on $\mathfrak{sl}(2,\mathbb C)$

$$\sum_{|Y|\le N}\sum_{X\in\Pi}\|T_Y^1T_X^{(+)2}(u_1\otimes u_2)\|_E^2 \le C_N\sum_{|Z|\le N}\sum_{X\in\Pi}\|T_{XZ}^{(+)2}(u_1\otimes u_2)\|_E^2, \tag{3.24a}$$

where the summation is taken over

$$Y\in U(\mathfrak{sl}(2,\mathbb C))\cap\Pi_2', \quad Z\in U(\mathfrak{sl}(2,\mathbb C))\cap\Pi_1', \tag{3.24b}$$

and $u_1, u_2 \in E_\infty$.

It follows by definition (3.14) of $T^{(+)2}$ that

$$T^1_Y T^{(+)2}_X (u_1 \otimes u_2) = T^{(+)2}_X \big((f_1, \dot f_1, T^{D1}_Y \alpha_1) \otimes (f_2, \dot f_2, T^{D1}_Y \alpha_2)\big), \quad Y \in U(\mathbb{R}^4), \tag{3.25}$$

where $(f_i, \dot f_i, \alpha_i) = u_i \in E_\infty$ for $i = 1, 2$. Inequality (3.24a) with domain of summation (3.24b) and equality (3.25) give, using that different orderings on Π define equivalent norms,

$$\sum_{X \in \Pi} \|T^{(+)2}_X (u_1 \otimes u_2)\|^2_{E_N} \leq C_N \sum_{|Y|+|Z| \leq N} \sum_{X \in \Pi} \|T^{(+)2}_{XZ} \big((f_1, \dot f_1, T^{D1}_Y \alpha_1) \otimes (f_2, \dot f_2, T^{D1}_Y \alpha_2)\big)\|^2_E, \tag{3.26a}$$

with summation over

$$Y \in U(\mathbb{R}^4) \cap \Pi'_2, \quad Z \in U(\mathfrak{sl}(2, \mathbb{C})) \cap \Pi'_2. \tag{3.26b}$$

According to Theorem 2.4 of [20] (cf. (2.96)) and since $T^{(+)}_Z = T^1_Z$ for $Z \in U(\mathfrak{sl}(2, \mathbb{C}))$, it is enough to estimate

$$I_X(Z_1, Z_2, Y) = \|T^{(+)2}_X (T^1_{Z_1} u^Y_1 \otimes T^1_{Z_2} u^Y_2\|_E, \tag{3.27}$$

$X \in \Pi$, $Z_1, Z_2 \in U(\mathfrak{sl}(2, \mathbb{C})) \cap \Pi'_2$, $Y \in U(\mathbb{R}^4) \cap \Pi'_2$, $|Z_1| + |Z_2| + |Y| \leq N$, $u^Y_i = (f_i, \dot f_i, T^{D1}_Y \alpha_i)$, in order to establish a bound on the right-hand side of (3.26a). It follows from definition (3.14) of $T^{(+)}$ and Lemma 3.5 that, for $0 < a \leq 1$ and $3/2 - \rho < b \leq 1$,

$$\begin{aligned} \|T^{(+)2}_X (u_1 \otimes u_2)\|_E \leq C_a \Big(&\min \big(\|(f_1, \dot f_1)\|_{M_3} \|\alpha_2\|^{1-a}_{D_0} \|\omega(-i\partial)\alpha_2\|^a_{D_0}, \\ &\|(f_1, \dot f_1)\|^{1-b}_{M_0} \|(1-\Delta)^{1/2}(f_1, \dot f_1)\|^b_{M_0} \|\omega(-i\partial)\alpha_2\|_{D_0}\big) \\ &+ \min \big(\|(f_2, \dot f_2)\|_{M_3} \|\alpha_1\|^{1-a}_{D_0} \|\omega(-i\partial)\alpha_1\|^a_{D_0}, \\ &\|(f_2, \dot f_2)\|^{1-b}_{M_0} \|(1-\Delta)^{1/2}(f_2, \dot f_2)\|^b_{M_0} \|\omega(-i\partial)\alpha_1\|_{D_0}\big)\Big), \end{aligned} \tag{3.28}$$

where $u_i = (f_i, \dot f_i, \alpha_i) \in E_\infty$. Definition (3.27) and inequality (3.28) prove that

$$\begin{aligned} I_X(Z_1, Z_2, Y) \leq C_a \Big(&\min \big(\|(f_1, \dot f_1)\|_{M_{|Z_1|+3}} \|\alpha_2\|^{1-a}_{D_{|Z_2|+|Y|}} \|\alpha_2\|^a_{D_{|Z_2|+|Y|+1}}, \\ &\|(f_1, \dot f_1)\|^{1-b}_{M_{|Z_1|}} \|(f_1, \dot f_1)\|^b_{M_{|Z_1|+1}} \|\alpha_2\|_{D_{|Z_2|+|Y|+1}}\big) \\ &+ \min \big(\|(f_1, \dot f_2)\|_{M_{|Z_2|+3}} \|\alpha_1\|^{1-a}_{D_{|Z_1|+|Y|}} \|\alpha_1|^a_{D_{|Z_1|+|Y|+1}}, \\ &\|(f_2, \dot f_2)\|^{1-b}_{M_{|Z_2|}} \|\alpha_1\| (f_2, \dot f_2)\|^b_{M_{|Z_1+1}} \|\alpha_1\|_{D_{|Z_1|+|Y|+1}}\big)\Big). \end{aligned} \tag{3.29}$$

It follows from (3.26a), (3.27) and (3.29) that for $3/2-\rho<a\le 1$ and $a=b$

$$
\begin{aligned}
&\Big(\sum_{X\in\Pi}\|T_X^{(+)2}(u_1\otimes u_2)\|^2_{E_N}\Big)^{1/2} \qquad (3.30)\\
&\le C_{N,a}\sum_{N_1+N_2+n\le N}\Big(\min\big(\|u_1\|_{E_{N_1+3}}\|u_2\|^{1-a}_{E_{N_2+n}}\|u_2\|^{a}_{E_{N_2+n+1}},\\
&\qquad \|u_1\|^{1-a}_{E_{N_1}}\|u_1\|^{a}_{E_{N_1+1}}\|u_2\|_{E_{N_2+n+1}}\big)\\
&\qquad +\min\big(\|u_2\|_{E_{N_2+3}}\|u_1\|^{1-a}_{E_{N_1+n}}\|u_1\|^{a}_{E_{N_1+n+1}},\\
&\qquad \|u_2\|_{E_{N_2}}\|u_2\|^{a}_{E_{N_2+1}}\|u_1\|_{E_{N_1+n+1}}\big)\Big)\\
&\le C'_{N,a}\sum_{N_1+N_2\le N}\Big(\min\big(\|u_1\|_{E_{N_1+3}}\|u_2\|^{1-a}_{E_{N_2}}\|u_2\|^{a}_{E_{N_2+1}},\\
&\qquad \|u_1\|^{1-a}_{E_{N_1}}\|u_1\|^{a}_{E_{N_1+1}}\|u_2\|_{E_{N_2+1}}\big)\\
&\qquad +\min\big(\|u_2\|_{E_{N_2+3}}\|u_1\|^{1-a}_{E_{N_1}}\|u_1\|^{a}_{E_{N_1+1}},\\
&\qquad \|u_2\|_{E_{N_2}}\|u_2\|^{a}_{E_{N_2+1}}\|u_1\|_{E_{N_1+1}}\big)\Big).
\end{aligned}
$$

To prove statement iii) we observe that for $N=2$, the right hand side of (3.30) is smaller than

$$
\begin{aligned}
&C'_{0,a}\Big(\min\big(\|u_1\|_{E_3}\|u_2\|^{1-a}_{E_0}\|u_2\|^{a}_{E_1},\|u_1\|^{1-a}_{E_0}\|u_1\|^{a}_{E_1}\|u_2\|_{E_1}\big)\\
&\qquad +\min\big(\|u_2\|_{E_3}\|u_1\|^{1-a}_{E_0}\|u_1\|^{a}_{E_1},\|u_2\|^{1-a}_{E_0}\|u_2\|^{a}_{E_1}\|u_1\|_{E_1}\big)\Big)\\
&\le 2C'_{0,a}\min\big(\|u_1\|_{E_3}\|u_2\|^{1-a}_{E_0}\|u_2\|^{a}_{E_1},\|u_2\|_{E_3}\|u_1\|^{1-a}_{E_0}\|u_1\|^{a}_{E_1}\big).
\end{aligned}
$$

To prove statement i) we choose on the right-hand side of (3.30) the second term in both minima. Since $\|u_i\|^{1-a}_{E_{N_i}}\|u_i\|^{a}_{E_{N_i+1}}\le\|u_i\|_{E_{N_i+1}}$ we obtain the majorization

$$2C'_{N,a}\sum_{N_1+N_2\le N}\|u_1\|_{E_{N_1+1}}\|u_2\|_{E_{N_2+1}}.$$

In this expression no seminorm has a higher index than $N+1$. Corollary 2.6 now gives, since $N_1+N_2\le N$, that the last expression is smaller than

$$C''_{N,a}(\|u_1\|_{E_1}\|u_2\|_{E_{N+1}}+\|u_1\|_{E_{N+1}}\|u_2\|_{E_1}),\quad N\ge 0,$$

which proves statement i).

To prove statement ii) we choose on the right-hand side of (3.30) the second term in the first minimum for $N_2\le N-1$ and the first term for $N_2=N$. A similar choice for the second minimum gives the following majorization of the last member of inequality (3.30):

$$C'_{N,a}\Big(\sum_{\substack{N_1+N_2\le N\\ N_2\le N-1}} \|u_1\|^{1-a}_{E_{N_1}}\|u_2\|^{a}_{E_{N_1+1}}\|u_2\|_{E_{N_2+1}} + \|u_1\|_{E_3}\|u_2\|^{1-a}_{E_N}\|u_2\|^{a}_{E_{N+1}}$$
$$+\sum_{\substack{N_1+N_2\le N\\ N_1\le N-1}} \|u_2\|^{1-a}_{E_{N_2}}\|u_2\|^{a}_{E_{N_2+1}}\|u_1\|_{E_{N_1+1}} + \|u_2\|_{E_3}\|u_1\|^{1-a}_{E_N}\|u_1\|^{a}_{E_{N+1}}\Big).$$

It follows, using the second inequality of Corollary 3.6, that this expression is bounded by

$$C''_{N,a}(\|u_1\|_{E_3}\|u_2\|_{E_N}+\|u_1\|_{E_N}\|u_2\|_{E_3})^{1-a}$$
$$(\|u_1\|_{E_3}\|u_2\|_{E_{N+1}}+\|u_1\|_{E_{N+1}}\|u_2\|_{E_3})^{a},\quad 3/2-\rho<a\le 1, N\ge 0,$$

which proves statement ii). This proves the lemma.

Corollary 3.7. *If $N \ge 0$, $u \in E_\infty$ and $X \in \Pi$, then*

i) $\|T^{(+)2}_X(u)\|_{E_N} \le C_{N,a}\|u\|_{E_3}\|u\|^{1-a}_{E_N}\|u\|^{a}_{E_{N+1}},\quad 3/2-\rho<a\le 1,$

ii) $\|T^{(+)2}_X(u)\|_{E_N} \le C_N\|u\|_{E_1}\|u\|_{E_{N+1}}.$

The properties in Corollary 2.6 of the spaces E_N and the estimates in Lemma 3.6 of $T^{(+)2}_X$, $X\in\mathfrak{p}$, lead to estimates of $T^{(+)}_Y$, $Y\in\Pi'$. The proof of this is so similar to that of Lemma 2.19 that we only state the result.

Lemma 3.8. *If $u_1,\dots,u_n \in E_\infty$ and $Y\in\Pi'$, then*

i) $\|T^{(+)n}_Y(u_1\otimes\cdots\otimes u_n)\|_{E_N} \le C\sum_i \prod_{1\le l\le n-1}\|u_{i_l}\|_{E_3}\|u_{i_n}\|_{E_{N+|Y|}}$, *for $n\ge 1$, $N\ge 0$,*

ii) $$\|T^{(+)n}_Y(u_1\otimes\cdots\otimes u_n)\|_{E_N} \le C_a\Big(\sum_i \|u_{i_1}\|_{E_{N+|Y|-1}}\prod_{l=2}^n\|u_{i_l}\|_{E_3}\Big)^{1-a}$$
$$\Big(\sum_i \|u_{i_1}\|_{E_{N+|Y|}}\prod_{l=2}^n\|u_{i_l}\|_{E_3}\Big)^{a},\quad 3/2-\rho<a\le 1,$$

for $n\ge 2$ and $|Y|+N\ge 1$. Here the summation is taken over all permutations i of $(1,\dots,n)$ and the constants C, C_a depend on $|Y|$, n, N, ρ.

To prove that *the nonlinear representation $T^{(+)}$ majorizes the linear representation* T^1 in E_N we first need to prove this for E_3. This can be done easily using the fact that $\alpha\mapsto T^{(+)D}_Y((f,\dot f,\alpha))$ is a linear function for $Y\in U(\mathbb{R}^4)$.

Lemma 3.9. *Let $u=(f,\dot f,\alpha)\in E_\infty$, $g=(f,\dot f,0)$ and $v=(0,0,\alpha)$. If $Y\in U(\mathbb{R}^4)$ then*

$$T^{(+)Dn}_Y(u\otimes\cdots\otimes u) = nT^{(+)Dn}_Y(g\otimes\cdots\otimes g\otimes v),\quad n\ge 1.$$

Proof. The statement is trivial for $Y = \mathbb{I}$. For $Y \in \Pi \cap \mathbb{R}^4$ it follows from definition (3.14) of $T^{(+)}$. Suppose it is true for $|Y| = L$, $Y \in \Pi' \cap U(\mathbb{R}^4)$ and let $X \in \Pi \cap \mathbb{R}^4$. Then it follows from definition (1.9) and the induction hypothesis that:

$$T^{(+)Dn}_{YX}(u \otimes \cdots \otimes u) = nT^{(+)Dn}_{Y}\Big(\sum_{0 \le q \le n-1} I_q \otimes T^1_X \otimes I_{n-q-1}(g \otimes \cdots \otimes g \otimes v)\Big)$$
$$+ n\frac{2(n-1)}{n}T^{(+)D(n-1)}_{Y}(g \otimes \cdots \otimes g \otimes T^{(+)2}_{X}(g \otimes v)) = nT^{(+)Dn}_{YX}(g \otimes \cdots \otimes g \otimes v),$$

where we have used that $T^{(+)n}$ is symmetric. This proves the lemma.

Lemma 3.8 and Lemma 3.9 have the following immediate corollary which we state without proof:

Corollary 3.10. *$T^{(+)}_Y$, $Y \in \Pi'$, is a C^∞ polynomial from $E_{N+|Y|}$ to E_N, for $N + |Y| \ge 3$. If $u = (f, \dot{f}, \alpha) \in E_{N+|Y|} \cap E_3$, $g = (f, \dot{f})$, then*

i) $\|T^{(+)}_Y(u)\|_{E_N} \le \|u\|_{E_{N+|Y|}} + \|\tilde{T}^{(+)}_Y(u)\|_{E_N}$,
where $N + |Y| \ge 0$,
ii) $\|\tilde{T}^{(+)}_Y(u)\|_{E_N} \le C_{N,|Y|}(\|g\|_{M_3})(\|g\|_{M_3}\|\alpha\|_{D_{N+|Y|}} + \|g\|_{M_{N+|Y|}}\|\alpha\|_{D_3})$,
where $N + |Y| \ge 0$, $C_{0,0} = 0$,
iii) $\|\tilde{T}^{(+)}_Y(u)\|_{E_N} \le C_{N,|Y|,a}(\|g\|_{M_3})(\|g\|_{M_3}\|\alpha\|_{D_{N+|Y|-1}} + \|g\|_{M_{N+|Y|-1}}\|\alpha\|_{D_3})^{1-a}$
$$(\|g\|_{M_3}\|\alpha\|_{D_{N+|Y|}} + \|g\|_{M_{N+|Y|}}\|\alpha\|_{D_3})^a,$$

where $3/2 - \rho < a \le 1$, $N + |Y| \ge 1$. *$C_{N,|Y|}$ and $C_{N,|Y|,a}$ are increasing continuous functions from $\mathbb{R}^+$ to $\mathbb{R}^+$, $\tilde{T}^{(+)} = T^{(+)} - T^1$ and $\tilde{T}^{(+)}_{\mathbb{I}} = 0$.*

We can now prove that the linear representation T^1 is bounded by the nonlinear representation $T^{(+)}$.

Theorem 3.11.

i) $\wp_N(T^{(+)}(u)) \le C_N(\|u\|_{E_3})\|u\|_{E_N}, \quad N \ge 0$,
ii) *If $K > 0$ is sufficiently small, $u = (f, \dot{f}, \alpha) \in E$, $g = (f, \dot{f}) \in M_3$ and $\|g\|_{M_3} \le K$ then*
$$\|u\|_{E_N} \le F_N(\wp_3(T^{(+)}(u)))\wp_N(T^{(+)}(u)), \quad N \ge 3.$$
iii) *If $u \in E_3$ then*
$$\|u\|_{E_N} \le C_N(\|u\|_{E_3})(\wp_N(T^{(+)}(u)) + \|u\|_{E_3}), \quad N \ge 3.$$
C_N and F_N are increasing continuous functions from $\mathbb{R}^+$ to $\mathbb{R}^+$.

Proof. $u \mapsto T^{(+)}_Y(u)$, $Y \in \Pi'$, is a polynomial. It follows from statements i) and ii) of Corollary 3.10 that

$$\|T^{(+)}_Y(u)\|_E \le \|u\|_{E_Y} + C_{|Y|}(\|u\|_{E_3})\|u\|_{E_3}\|u\|_{E_{|Y|}}$$

for $Y \in \Pi'$ and $|Y| \geq 0$, where $C_{|Y|}$ is a continuous increasing function. According to definition (2.109) of $\wp_N$ we then have

$$\wp_N(T^{(+)}(u)) \leq C'_N(\|u\|_{E_3})\|u\|_{E_N}, \quad N \geq 0,$$

which proves statement i) of the theorem.

Statement ii) of Corollary 3.10 shows that, if $u = (f, \dot{f}, \alpha) \in E$, $g = (f, \dot{f})$, then

$$\begin{aligned}\|T^1_Y u\|_E &\leq \|T^{(+)}_Y(u)\|_E + \|\tilde{T}^{(+)}_Y(u)\|_E \\ &\leq \|T^{(+)}_Y(u)\|_E + C_{|Y|}(\|g\|_{M_3})(\|g\|_{M_3}\|\alpha\|_{D_{|Y|}} + \|g\|_{M_{|Y|}}\|\alpha\|_{D_3}),\end{aligned}$$

for $Y \in \Pi'$ and $|Y| \geq 0$. This gives

$$\|u\|_{E_N} \leq \wp_N(T^{(+)}(u)) + C'_N(\|g\|_{M_3})(\|g\|_{M_3}\|\alpha\|_{D_N} + \|g\|_{M_N}\|\alpha\|_{D_3}), \quad N \geq 0, \tag{3.31}$$

where C'_N is a continuous increasing function.

Inequality (3.31) and $\|\alpha\|_{D_3} \leq \|u\|_{E_3}$ give

$$\|u\|_{E_3} \leq \wp_3(T^{(+)}(u)) + 2C'_3(\|g\|_{M_3})\|g\|_{M_3}\|u\|_{E_3}.$$

Since C'_3 is continuous we can choose $K > 0$ such that $2C'_3(K)K < 1/2$, which gives

$$\|u\|_{E_3} \leq 2\wp_3(T^{(+)}(u)). \tag{3.32a}$$

This proves statement ii) for $N = 3$. Suppose for the moment that statement iii) of the theorem is true. It then follows from inequality (3.32) and by defining $F_N = 3C_N$, $N \geq 4$, where C_N is given by statement iii), that statement ii) of the theorem is true.

We need to prove statement iii) of the theorem. It follows from statement iii) of Corollary 3.10 that

$$\|u\|_{E_N} \leq \wp_N(T^{(+)}(u)) + C_{N,a}(\|g\|_{M_3})\|u\|_{E_3}\|u\|^{1-a}_{E_{N-1}}\|u\|^a_{E_N}, \tag{3.32b}$$

$3/2 - \rho < a \leq 1$, $N \geq 1$. Let $u \neq 0$, since the case $u = 0$ is trivial. Then $\wp_N(T^{(+)}(u)) > 0$. Let

$$x_N = \|u\|_{E_N}/\wp_N(T^{(+)}(u)).$$

Inequality (3.32b), with $a < 1$, shows that

$$x_N \leq 1 + A_N x^{1-a}_{N-1} x^a_N, \quad 3/2 - \rho < a \leq 1, N \geq 4,$$

where

$$A_N = C_{N,a}(\|g\|_{M_3})\|u\|_{E_3}\big(\wp_{N-1}(T^{(+)}(u))/\wp_N(T^{(+)}(u))\big)^{1-a}.$$

As we shall show at the end of this proof, there exists $q_a \in \mathbb{R}^+$, independent of N, A_N, x_{N-1}, x_N, such that $x_N \leq q_a(1 + A_N^{1/(1-a)} x_{N-1})$. This gives that

$$\|u\|_{E_N} \leq q_a\big(\wp_N(T^{(+)}(u)) + (C_{N,a}(\|g\|_{M_3})\|u\|_{E_3})^{1/(1-a)}\|u\|_{E_{N-1}}\big), \quad N \geq 4.$$

Induction in N now gives the inequality of statement iii).

Let $x, y, c \in [0, \infty[$, let $b \in]0,1[$ and let $x \leq 1 + cy^{1-b}x^b$. If $x \geq 1$, then it follows that $x \leq x^b + cy^{1-b}x^b$, which gives that $x \leq (1 + cy^{1-b})^{1/(1-b)}$. Since the function $z \mapsto z^{1/(1-b)}$, $z \geq 0$, is convex, it follows that $x \leq 2^{b/(1-b)}(1 + c^{1/(1-b)}y)$. The right-hand side of this inequality is larger than 1, so it follows that $x \leq 2^{b/(1-b)}(1 + c^{1/(1-b)}y)$ for $x \geq 0$. This completes the proof.

We next prove that $U^{(+)}$ extended to E, defining $\varphi_g, g \in \mathcal{P}_0$, by (1.23a) for $u \in E$, is a *continuous representation* in E_N, $N \geq 0$, and that it has the same C^∞-vectors as U^1 for $N \geq 3$ (i.e. the map $g \mapsto U_g^{(+)}(u)$ from $\mathcal{P}_0$ t_0 E_N is C^∞ if and only if $u \in E_\infty$). We also prove *analyticity* properties of $u \mapsto U_g^{(+)}(u)$.

Theorem 3.12.

i) *Let $N \geq 0$. Then $(g,u) \mapsto U_g^{(+)}(u)$ is a continuous function from $\mathcal{P}_0 \times E_N$ to E_N and $E_N^{\circ\rho}$ is invariant under $U^{(+)}$.*

ii) *If $N \geq 3$, then E_∞ is the set of C^∞ vectors for $U^{(+)}$ in E_N, and $U^{(+)}\colon \mathcal{P}_0 \times E_\infty \to E_\infty$ is a C^∞ function and $T_X^{(+)}(u) = (d/dt)\, U^+_{exp(tX)}(u) \,|_{t=0}, u \in E_\infty$.*

iii) *If $b > 0$, $N \geq 3$ and if $\mathcal{J}u = (f, \dot{f}, (1-\Delta)^{-1/2}\alpha)$, $u = (f, \dot{f}, \alpha)$, then $u \mapsto \mathcal{J}^b U_g^{(+)}(u)$ is a real analytic map from E_N to E_N for $g \in \mathcal{P}_0$. Moreover if $\sum_{n\geq 0} K^n_{g,g_0}(u_0; u-u_0)$ is the Taylor development of $U^1_{g^{-1}}U_g^{(+)}(u) - U^1_{g_0^{-1}}U_{g_0}^{(+)}(u_0) - (u-u_0)$ at u_0, then there exists $R > 0$ such that*

$$\lim_{g\to g_0} \sup_{\|u-u_0\|_{E_N}\leq R} \sum_{n\geq 1} \|K^n_{g,g_0}(u_0; u-u_0)\|_{E_N} = 0.$$

Proof. Let $u = (f, \dot{f}, \alpha) \in E_c$, where E_c was defined in Theorem 2.9, $g = (f, \dot{f})$ and let $h(a) = \exp(a^\nu P_\nu)$, a_0, a_1, a_2, $a_3 \in \mathbb{R}$. It follows from definitions (1.17a) and (1.17b) of $U^{(+)}$ and from definition (1.23a) of the phase function φ, that

$$(U_{h(a)}^{(+)D}(u))^\wedge = e_{h(a)}(g)(U^{1D}_{h(a)}\alpha)^\wedge, \tag{3.33a}$$

where

$$((e_{h(a)}(g))\alpha)^\wedge(k) = \sum_{\varepsilon=\pm} \exp(i(q_\varepsilon(g,a))(k))P_\varepsilon(k)\hat{\alpha}(k), \tag{3.33b}$$

$$(q_\varepsilon(g,a))(k) = \varphi_{\exp(a^\nu P_\nu)}(u, -\varepsilon k). \tag{3.33c}$$

Let $F\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the solution of the wave equation with initial data $(U^{1M}_{h(a)} - I)g$. According to (1.23a) and (3.33c),

$$(q_\varepsilon(g,a))(k) = \int_0^\infty \chi_0(\tau m) l^\mu_\varepsilon(k) F_\mu(\tau l_\varepsilon(k)) d\tau, \tag{3.34}$$

where $l^0_\varepsilon(k) = \omega(k)$ and $l^j_\varepsilon(k) = -\varepsilon k_j$ for $1 \le j \le 3$. Since $(U^{1M}_{h(a)} - I)g \in M_c$, it follows from Lemma 3.2 that

$$|q_\varepsilon(g,a)|(k) \le C_{\rho,b}\ \omega(k) \||\nabla|^{-1/2}(1+|\nabla|)^b (U^{1M}_{h(a)} - I)g\|_{M^\rho_0}, \tag{3.35a}$$

where $1/2 < \rho < 1$ and $1-\rho < b \le 1/2$. Since ϑ^∞, defined in (1.23b), satisfies $\vartheta^\infty(H,\Lambda y) = \vartheta^\infty(\Lambda^{-1}H, y)$ for $\Lambda \in SO(3,1)$, it follows by differentiation with respect to Λ in this expression and from (3.34) and (3.35a), that $k \mapsto (q_\varepsilon(g,a))(k)$ is a C^∞ function and that

$$|\eta^{(\varepsilon)}_Z q_\varepsilon(g,a)|(k) \le C_{\rho,b}\ \omega(k) \||\nabla|^{-1/2}(1+|\nabla|)^b T^{1M}_Z (U^{1M}_{h(a)} - I)g\|_{M^\rho_0}, \tag{3.35b}$$

for $Z \in U(\mathfrak{sl}(2,\mathbb{C}))$, where the representation $\eta^{(\varepsilon)}$ is defined in (3.9b). Using that

$$T^{1M}_X U^{1M}_{h(a)} = U^{1M}_{h(a)} T^{1M}_{X-[a^\mu P_\mu, X]}$$

for $X \in \mathfrak{sl}(2,\mathbb{C})$, that $[P_\mu, X] \in \mathbb{R}^4$ and that U^{1M} is strongly continuous on M^ρ_0, it follows from (3.35b) that

$$\begin{aligned} &\sum_{\substack{Z\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))\\ |Z|\le N}} |\eta^{(\varepsilon)}_Z q_\varepsilon(g,a)|(k) \\ &\quad\le C_{\rho,N,b}\ \omega(k)\Big(\sum_{\substack{Z\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))\\ |Z|\le N}} \||\nabla|^{-1/2}(1+|\nabla|)^b (U^{1M}_{h(a)} - I) T^{1M}_Z g\|_{M^\rho_0} \\ &\qquad + \sum_{\substack{Y\in\Pi'\\ |Y|\le N-1}} \||\nabla|^{1/2}(1+|\nabla|)^b T^{1M}_Y g\|_{M^\rho_0}\Big), \end{aligned} \tag{3.35c}$$

for $N \ge 0$, $|a| = \sum_\nu |a^\nu| \le 1$. Using elementary properties of the exponential function, it follows that

$$\||\nabla|^{-1/2}(1+|\nabla|)^b (U^{1M}_{h(a)} - I) T^{1M}_Z g\|_{M^\rho_0} \le C_b |a|^{1/2-b} \|T^{1M}_Z g\|_{M^\rho_0}, \quad |a| \le 1.$$

The norm $\||\nabla|^{1/2}(1+|\nabla|)^b \cdot\|_{M^\rho_0}$ is weaker than the norm $\||\nabla|^{1/2}\cdot\|_{M^\rho_0}$. This shows, together with inequality (3.35c), that $k \mapsto (q_\varepsilon(g,a))(k)$ is a C^N function and that

$$\sum_{\substack{Z\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))\\ |Z|\le N}} |\eta^{(\varepsilon)}_Z q_\varepsilon(g,a)|(k) \le |a|^{1/2-b} C_{\rho,N,b}\ \omega(k) \|g\|_{M^\rho_N}, \tag{3.36}$$

where $N \ge 0$, $1/2 < \rho < 1$, $1-\rho < b \le 1/2$, $|a| \le 1$ and where $g \in M^\rho_N$, since M^ρ_∞ is dense in M^ρ_N and M_c is dense in M^ρ_∞, according to Theorem 2.9.

Definition (3.34) of $q_\varepsilon(g,a)$ shows that $\frac{\partial}{\partial a^\mu} q_\varepsilon(g,a) = \Phi_{\varepsilon,P_\mu}(U^{1M}_{h(a)} g)$. This gives that

$$q_\varepsilon(g,a) = \int_0^1 a^\mu \Phi_{\varepsilon,P_\mu}(U^{1M}_{h(as)} g) ds. \tag{3.37}$$

This equality, statement iii) of Proposition 3.3 and the strong continuity of U^{1M} on M^ρ_N show that

$$\sum_{\substack{Z\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))\\ |Z|\le N}} |\eta^{(\varepsilon)}_Z q_\varepsilon(g,a)|(k) \le C_{\rho,N}|a|(1+\ln(1+\omega(k)/m))\|g\|_{M^\rho_{N+3}}, \tag{3.38}$$

where $N\ge 0$, $1/2<\rho<1$, $|a|\le 1$ and $g\in M^\rho_{N+3}$.

Let $N\ge 0$, $g\in M^\rho_N$, $a\in\mathbb{R}^4$, $|a|\le 1$, $\alpha\in D_N$, and let $Y\in\Pi'\cap U(\mathbb{R}^4)$ and $Z\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))$, $|Y|+|Z|\le N$. We obtain using definition (3.9) of $\eta^{(\varepsilon)}$

$$\begin{aligned}&(T^{D1}_{YZ}(e_{h(a)}(g)-I)P_\varepsilon(-i\partial)\alpha)^\wedge \\ &\quad=\sum_{Z',Z''\in D(Z)} C(Z,Z',Z'')(\eta^{(\varepsilon)}_{Z'}(e^{iq_\varepsilon(g,a)}-1))P_\varepsilon(-i\partial)T^{D1}_{YZ''}\alpha)^\wedge,\end{aligned} \tag{3.39}$$

where $C(Z,Z',Z'')\in\mathbb{R}$ and $D(Z)$ is the set of $Z',Z''\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))$ such that $|Z|=|Z'|+|Z''|$. We have used here that $T^{D1}_X e_{h(a)}(g)\alpha=e_{h(a)}(g)T^{D1}_X\alpha$, $X\in\mathbb{R}^4$, according to definition (3.33b) of $e_{h(a)}(g)$. It follows from equation (3.39) that

$$\begin{aligned}&\|(e_{h(a)}(g)-I)\alpha\|_{D_N} \\ &\quad\le C_N \sum_{\substack{Y\in\Pi',Z\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))\\ |Z|+|Y|\le N,\varepsilon=\pm}} \|(\eta^{(\varepsilon)}_Z(e^{iq_\varepsilon(g,a)}-1))P_\varepsilon(-i\partial)T^{D1}_Y\alpha)^\wedge\|_{L^2}.\end{aligned} \tag{3.40}$$

If $Z_1,\dots,Z_n\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))$, $n\ge 1$, $L=|Z_1|+\dots+|Z_n|$, then it follows from inequality (3.36) and Corollary 2.6 that

$$|(\eta^{(\varepsilon)}_{Z_1}q_\varepsilon(g,a))(k)\cdots(\eta^{(\varepsilon)}_{Z_n}q_\varepsilon(g,a))(k)|\le |a|^{n(1/2-b)}\omega(k)^n C_{\rho,b,N,L}\|g\|^{n-1}_{M^\rho_0}\|g\|_{M^\rho_L}, \tag{3.41}$$

for $|a|\le 1$, $1/2<\rho<1$, $1-\rho<b\le 1/2$. Inequality (3.40), Leibniz rule and inequality (3.41) with $L=|Z|$ and $|Z_i|\ge 1$ for $1\le i\le n$, give that

$$\begin{aligned}&\|(e_{h(a)}(g)-I)\alpha\|_{D_N} \\ &\quad\le C_N\sum_{\substack{Y\in\Pi'\\ |Y|=N}}\|(e_{h(a)}(g)-I)T^{D1}_Y\alpha\|_{D_0}+C_N|a|^{1/2-b}\sum_{\substack{N_1+N_2\le N\\ N_1\ge 1}}\|g\|_{M^\rho_{N_1}}\|\alpha\|_{D_{N_1+N_2}},\end{aligned} \tag{3.42}$$

where C_N depends only on ρ, b and $\|g\|_{M^\rho_0}$. Given a bounded subset $K\subset\mathbb{R}^3$ and a bounded subset $B\subset M^\rho_0$, according to inequality (3.36) for $|a|\le 1$, $\exp(iq_\varepsilon(g,a))(k))-1$ converges to zero, uniformly for $(k,g)\in K\times B$, (resp. $(k,a)\in K\times\{a\in\mathbb{R}^4\big||a|\le 1\}$) when $a\to 0$ (resp. $g\to 0$ in M^ρ_0).

Moreover, since $|\exp(iq_\varepsilon(g,a))(k))-1|\le 2$, it follows from inequality (3.40), with $N=0$, and from the Lebesgue dominated convergence theorem, that for $Y\in\Pi'$, $|Y|\le N$, $N\ge 0$, $\alpha\in D_N$,

$$\|(e_{h(a)}(g)-I)T^{D1}_Y\alpha\|_{D_0}\to 0$$

uniformly on $\{a||a| \leq 1\}$ (resp. bounded subsets of M_0^ρ) when $g \to 0$ in M_0^ρ (resp. $a \to 0$). Inequality (3.42) now shows that, if B is a bounded subset of M_N^ρ and $\alpha \in D_N$, then $(e_{h(a)}(g) - I)\alpha \to 0$ in D_N uniformly on $\{|a| \leq 1\}$ (resp. B) when $g \to 0$ in M_N^ρ (resp. $a \to 0$). Moreover it follows from inequality (3.42) that $\|e_{h(a)}(g)\alpha\|_{D_N} \leq C_B \|\alpha\|_{D_N}$ for $|a| \leq 1$, $g \in B$. Since

$$e_{h(a)}(g_1)\alpha_1 - \alpha = e_{h(a)}(g_1)(\alpha_1 - \alpha) + (e_{h(a)}(g_1) - I)\alpha,$$

it then follows that $\|e_{h(a)}(g_1)\alpha_1 - \alpha\|_{D_N} \to 0$, when $(a, g_1, \alpha_1) \to (0, (g, \alpha))$ in $\mathbb{R}^4 \times E_N^\rho$. Equality (3.33a) and the fact that U^{1D} is a strongly continuous linear representation of $\mathcal{P}_0$ in D_N give that the function $(a, u) \mapsto U_{h(a)}^{(+)D}(u)$ from $\{|a| \leq 1\} \times E_N^\rho$ to D_N is continuous at $(0, u)$ and therefore that the function $(a, u) \mapsto U_{h(a)}^{(+)}(u)$ from $\{|a| \leq 1\} \times E_N^\rho$ to E_N^ρ is continuous at $(0, u)$, since $U^{(+)} = U^{1M} \oplus U^{(+)D}$ and U^{1M} is a strongly continuous linear representation of $\mathcal{P}_0$ in E_N^ρ. The fact that the set of elements $U_{h(a)}^{(+)D}$, $a \in \mathbb{R}^4$, is a connected group now shows that $(a, u) \mapsto U_{h(a)}^{(+)}(u)$ is a continuous map from $\mathbb{R}^4 \times E_N^\rho$ to E_N^ρ. Finally, since $U_A^{(+)} = U_A^1$ for $A \in SL(2, \mathbb{C})$ and since U^1 is a strongly continuous linear representation, it follows that $(j, u) \mapsto U_j^{(+)}(u)$ is a continuous map from $\mathcal{P}_0 \times E_N^\rho$ to E_N^ρ, $N \geq 0$. $M_N^{\circ\rho}$ is invariant under U^{1M}, so by the definition of $U^{(+)}$, the space $M_N^{\circ\rho}$ is invariant under $U^{(+)}$.

To prove statement iii), let $e_{h(a)}^{n-1}(g)$, $n \geq 1$, $g \in M_\infty^\rho$, $a \in \mathbb{R}^4$, be defined by

$$(e_{h(a)}^{n-1}(g)\alpha)^\wedge(k) = \sum_{\varepsilon = \pm} \frac{1}{(n-1)!} (i(q_\varepsilon(g, a))(k))^{n-1} P_\varepsilon(k)\hat{\alpha}(k), \tag{3.43}$$

where $\alpha \in D_\infty$ and $k \in \mathbb{R}^3$. Similarly as we obtained inequality (3.40), it follows from (3.43) that

$$\begin{aligned} &\|e_{h(a)}^{n-1}(g)\alpha\|_{D_N} \\ &\quad \leq C_N((n-1)!)^{-1} \sum_{\substack{Y \in \Pi', Z \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C})) \\ |Z| + |Y| \leq N, \varepsilon = \pm}} \|(\eta_Z^{(\varepsilon)}(q_\varepsilon(g, a)^{n-1}))(P_\varepsilon(-i\partial)T_Y^{D1}\alpha)^\wedge\|_{L^2}, \end{aligned} \tag{3.44}$$

where $N \geq 0$, $(g, \alpha) \in E_\infty^\rho$, $a \in \mathbb{R}^4$ and $n \geq 1$.

Let $N \geq 3$, $n \geq 2$, Z, $Z_1, \ldots, Z_{n-1} \in \Pi' \cap U(\mathfrak{sl}(2, \mathbb{C}))$, $|Z_1| + \ldots + |Z_{n-1}| = |Z|$ and let $0 \leq |Z| \leq N$. If L is the number of elements Z_i such that $|Z_i| + 3 > N$, then $L(N-2) \leq |Z|$. This gives that $L = 0$ for $|Z| \leq N - 3$, $L \leq 1$ for $|Z| = N - 2$, $L \leq 2$ for $|Z| = N - 1$ and $L \leq 2$ for $|Z| = N$. Therefore if $L \geq 1$, then $N - |Z| + L \leq 3$. To estimate the product

$$|(\eta_{Z_1}^{(\varepsilon)} q_\varepsilon(g, a)) \cdots (\eta_{Z_{n-1}}^{(\varepsilon)} q_\varepsilon(g, a))|(k)$$

we can suppose that $|Z_i| \geq |Z_j|$ for $i < j$. If $L = 0$ then we use inequality (3.38) for all the factors and if $L \geq 1$ then we use inequality (3.36), with $1 - \rho < b \leq 1/2$, for the factors

$\eta^{(\varepsilon)}_{Z_i} q_\varepsilon(g,a)$, $1 \leq i \leq L$, and inequality (3.38) for the other factors. This gives

$$\begin{aligned}
&|(\eta^{(\varepsilon)}_{Z_1} q_\varepsilon(g,a)) \cdots (\eta^{(\varepsilon)}_{Z_{n-1}} q_\varepsilon(g,a))|(k) \\
&\quad \leq \Big(C_{\rho,b,N} |a|^{1/2-b} \omega(k))^L \|g\|_{M_{|Z_1|}} \cdots \|g\|_{M_{|Z_L|}} \\
&\qquad \prod_{i=L+1}^{n-1} C_{\rho,|Z_i|} \|g\|_{M_{|Z_i|+3}} |a| (1 + \ln(1+\omega(k)/m)) \Big), \quad |a| \leq 1.
\end{aligned}$$

Using Corollary 2.6 and redefining the constants, we obtain that

$$\begin{aligned}
&|(\eta^{(\varepsilon)}_{Z_1} q_\varepsilon(g,a)) \cdots (\eta^{(\varepsilon)}_{Z_{n-1}} q_\varepsilon(g,a))|(k) \\
&\quad \leq C_N C_0^{n-1} \|g\|^{n-2}_{M_3} \|g\|_{M_N} |a|^{(n-1)(1/2-b)} (1 + \ln(1+\omega(k)/m))^{n-1} \omega(k)^L.
\end{aligned}$$

Since $N - |Z| + L \leq 3$, when $L \geq 1$ and since $0 \leq L \leq 3$, we now obtain from (3.44), by using Leibniz rule,

$$\begin{aligned}
\|e^{n-1}_{h(a)}(g)\alpha\|_{D_N} &\leq C_N C_0^{n-1} ((n-1)!)^{-1} |a|^{(n-1)(1/2-b)} \\
&\qquad \|g\|^{n-2}_{M_3} \|g\|_{M_N} \sum_{\substack{Y \in \Pi' \\ |X|+|Y| \leq N}} \|(1 + \ln(1+\omega(\cdot)/m))^{n-1} (T^{D1}_Y \alpha)^\wedge(\cdot)\|_{L^2}.
\end{aligned}$$

If $0 < s < 1$, then

$$\|(1 + \ln(1+\omega(\cdot)/m))^{n-1} (1+\omega(\cdot)/m)^{-s}\|_{L^\infty} \leq \big(\frac{n-1}{s}\big)^{n-1} \exp(1-(n-1)/s),$$

so using the Stirling formula for $(n-1)!$ we obtain, with new constants,

$$\begin{aligned}
&\|e^{n-1}_{h(a)}(g)\alpha\|_{D_N} \\
&\quad \leq C_N C_0^{n-1} (se^{(1-s)/s})^{-(n-1)} |a|^{(n-1)(1/2-b)} \|g\|^{n-2}_{M_3} \|g\|_{M_N} \|(1-\Delta)^{s/2}\alpha\|_{D_N},
\end{aligned} \tag{3.45}$$

for $N \geq 3$, $1 - \rho < b \leq 1/2$, $n \geq 2$, $0 < s < 1$, where we have used that the norms $\|(1-\Delta)^{s/2} \cdot \|_{D_N}$ and $\|(1+\omega(-i\partial)/m)^s \cdot \|_{D_N}$ are uniformly equivalent for $0 \leq s \leq 1$. For given ρ, b, s such that $1/2 < \rho < 1$, $1 - \rho < b \leq 1$ and $0 < s < 1$, there exists, according to inequality (3.45), $r > 0$ and $C^{(r)}_N \geq 0$, $N \geq 3$, such that

$$\sum_{n \geq 2} \|e^{(n-1)}_{h(a)}(g)(1-\Delta)^{-s/2}\alpha\|_{D_N} \leq C^{(r)}_N |a|^{1/2-b} \|g\|_{M_N} \|\alpha\|_{D_N}, \tag{3.46}$$

for $N \geq 3$, $(g,\alpha) \in E_N$, $\|g\|_{M_3} \leq r$, $|a| \leq 1$.

Let $u = (g,\alpha) \in E_N$, $u_0 = (g_0,\alpha_0) \in E_N$ and $N \geq 3$. According to definition (3.33b) of $e_{h(a)}$, it follows that $e_{h(a)}(g) = e_{h(a)}(g-g_0)e_{h(a)}(g_0)$. The equality

$$\begin{aligned}
e_{h(a)}(g)\alpha - e_{h(a)}(g_0)\alpha_0 = e_{h(a)}&(g_0)(\alpha - \alpha_0) \\
&+ (e_{h(a)}(g-g_0) - I) e_{h(a)}(g_0)(\alpha - \alpha_0) \\
&+ (e_{h(a)}(g-g_0) - I) e_{h(a)}(g_0)(\alpha_0),
\end{aligned}$$

the definitions

$$\begin{aligned}F^1_{h(a)}(u_0; u-u_0) &= (e_{h(a)}(g_0) - I)(\alpha-\alpha_0)\\ &\quad + e^{(1)}_{h(a)}(g-g_0)e_{h(a)}(g_0)\alpha_0,\\ F^n_{h(a)}(u_0; u-u_0) &= e^{(n-1)}_{h(a)}(g-g_0)e_{h(a)}(g_0)(\alpha-\alpha_0)\\ &\quad + e^{(n)}_{h(a)}(g-g_0)e_{h(a)}(g_0)\alpha_0, \quad n \geq 2,\end{aligned}$$

the fact that $\|e_{h(a)}(g)\beta\|_{D_N} \leq C'_N(g_0)\|\beta\|_{D_N}$, which has been established in the proof of statement i), and inequality (3.46), show that if r and $C^{(r)}_N$ are as in (3.46), then the series

$$\sum_{n\geq 1} \|F^n_{h(a)}(\mathcal{J}^s u_0; \mathcal{J}^s(u-u_0))\|_{D_N}$$

converges uniformly for $\|g-g_0\|_{M_3} \leq r$,

$$\begin{aligned}&\sum_{n\geq 1} F^n_{h(a)}(\mathcal{J}^s u_0; \mathcal{J}^s(u-u_0))\\ &\quad = e_{h(a)}(g)(1-\Delta)^{-s/2}\alpha - e_{h(a)}(g_0)(1-\Delta)^{-s/2}\alpha_0 - (1-\Delta)^{-s/2}(\alpha-\alpha_0)\end{aligned}$$

and that

$$\begin{aligned}&\sum_{n\geq 1} \|F^n_{h(a)}(\mathcal{J}^s u_0; \mathcal{J}^s(u-u_0))\|_{D_N} \qquad (3.47)\\ &\quad \leq \|(e_{h(a)}(g_0) - I)(1-\Delta)^{-s/2}(\alpha-\alpha_0)\|_{D_N}\\ &\qquad + C^{(r)}_N|a|^{1/2-b}C'_N(g_0)\|g-g_0\|_{M_N}(\|\alpha-\alpha_0\|_{D_N} + \|\alpha_0\|_{D_N}), \quad |a| \leq 1, N \geq 3.\end{aligned}$$

It follows from (3.38), with $N = 0$, and from (3.42), with $N \geq 3$ and $1-\rho < b \leq 1/2$, that

$$\|(e_{h(a)}(g_0) - I)(1-\Delta)^{-s/2}(\alpha-\alpha_0)\|_{D_N} \to 0$$

uniformly on bounded subsets of elements $\alpha - \alpha_0 \in D_N$, when $a \to 0$. Choosing $1-\rho < b \leq 1/2$ in (3.47), we obtain that

$$\lim_{a\to 0} \sup_{\substack{\|u-u_0\|_{E_N} \leq R\\ \|g-g_0\|_{M_3} \leq r}} \sum_{n\geq 1} \|F^n_{h(a)}(\mathcal{J}^s u_0; \mathcal{J}^s(u-u_0))\|_{D_N} = 0, \qquad (3.48)$$

for all $N \geq 3$, $R > 0$. By definition (3.33b) of $e_{h(a)}$, it follows that

$$S_{h(a)}(u) = U^1_{h(a)^{-1}} U^{(+)}_{h(a)}(u),$$

then $S_{h(a)}(u) = (g, e_{h(a)}(g)\alpha)$, for $u = (g, \alpha)$. Therefore we have proved that

$$S_{h(a)} \circ \mathcal{J}^s : E_N \to E_N, \quad N \geq 3, |a| \leq 1,$$

is a real analytic function and, if

$$S_{h(a)}(u) = \sum_{n \geq 0} S^n_{h(a)}(u_0; u - u_0)$$

is the power series development of $S_{h(a)}(u)$ at u_0, choosing $0 < R \leq r$ in (3.48), we have:

$$\begin{aligned} &\lim_{a \to 0} \|S_{h(a)}(\mathcal{J}^s u) - S_{h(a)}(\mathcal{J}^s u_0) - \mathcal{J}^s(u - u_0)\|_{E_N} \\ &\quad \leq \lim_{a \to 0} \sup_{\|u\|_{E_N} \leq R} \Big(\sum_{n \geq 2} \|S^n_{h(a)}(\mathcal{J}^s u_0; \mathcal{J}^s(u - u_0))\|_{E_N} \\ &\qquad + \|S^1_{h(a)}(\mathcal{J}^s u_0; \mathcal{J}^s(u - u_0)) - \mathcal{J}^s(u - u_0)\|_{E_N} \Big) = 0. \end{aligned} \tag{3.49}$$

Since $U^{(+)}$ is a group representation and since $\mathcal{J}^s$ commutes with $U^{(+)}_{h(a)}$, we can conclude that $S_{h(a)} \circ \mathcal{J}^s : E_N \to E_N$ is real analytic for $a \in \mathbb{R}^4$ and that inequality (3.49) is true with $a \to a_0$, $a_0 \in \mathbb{R}^4$, instead of $a \to 0$. Since $U^{(+)}$ is linear on $SL(2, \mathbb{C})$, this proves statement iii) of the theorem.

To prove statement ii), let $u \in E_N$ be a C^∞-vector for $U^{(+)}$ in E_N, $N \geq 3$. Then $\wp_L(T^{(+)}(u)) < \infty$, $L \geq 0$, according to definition (2.109) of $\wp_L$ and that of C^∞-vectors. It follows from statement iii) of Theorem 3.11 that

$$\|u\|_{E_L} \leq C_L(\|u\|_{E_3})(\wp_L(T^{(+)}(u)) + \|u\|_{E_3}).$$

This proves that $u \in E_\infty$. Now, let u be a C^∞-vector of U^1 in E_N, $N \geq 3$, i.e. $u \in E_\infty$. Then it follows from statement i) of Theorem 3.11 that $\wp_L(T^{(+)}(u)) < \infty$ for $L \geq 0$, which proves that u is a differential vector for $U^{(+)}$.

According to Corollary 3.10 $T^{(+)}_Y : E_\infty \to E_\infty$, $Y \in \Pi'$, is a C^∞ function. Statement iii) of the theorem shows in particular that $(g, u) \mapsto (D^n U^{(+)}_g)(u; u_1, \ldots, u_n)$ is a continuous function from $\mathcal{P}_0 \times E_\infty$ into E_∞ for $u_i \in E_\infty$. Therefore $(D^n(T^{(+)}_Y \circ U^{(+)}_g))(u; u_1, \ldots, u_n)$ is continuous from $\mathcal{P}_0 \times E_\infty$ into E_∞. It then follows from the definition of $T^{(+)}$ that the function $(g, u) \mapsto (U^{(+)}_g)(u)$ from $\mathcal{P}_0 \times E_\infty$ to E_∞ is C^∞. This proves the theorem.

The representation $U^{(+)}$ of $\mathcal{P}_0$ on E^ρ_∞ or on $E^{\circ\rho}_\infty$ is *not linearizable* by a C^2 map. This is a particular case of next theorem:

Theorem 3.13. *Let $1/2 < \rho < 1$. There does not exist a neighbourhood $\mathcal{O}$ of zero in $E^{\circ\rho}_\infty$ and a C^2 map $F : \mathcal{O} \to E^\rho_0$ such that $F(0) = 0$, $(DF)(0; u) = u$ and such that $F(U^{(+)}_g(u)) = U^1_g F(u)$ for all u in a neighbourhood of zero in $E^{\circ\rho}_\infty$ and all g in a neighbourhood of the identity in $\mathcal{P}_0$.*

Proof. Suppose on the contrary that $\mathcal{O}$ and $F\colon \mathcal{O} \to E_0^\rho$ exist and let $F^1u = u$, $F^2(u_1 \otimes u_2) = 1/2(D^2F)(0; u_1, u_2)$ for u, u_1, $u_2 \in E_\infty^{\circ\rho}$. Let $L(B_1, B_2)$ be the linear continuous operators from B_1 to B_2, where B_1 and B_2 are topological vector spaces. $F^2 \in L(E_\infty^{\circ\rho} \hat{\otimes} E_\infty^{\circ\rho}, E_0^\rho)$ since F is a C^2 map and $F^2(u_1 \otimes u_2) = F^2(u_2 \otimes u_1)$. Let $R_g^{(+)2} = U_g^{(+)}(U_{g^{-1}}^1 \otimes U_{g^{-1}}^1)$, $g \in \mathcal{P}_0$. According to statements i) and ii) of Theorem 3.12, $g \mapsto R_g^{(+)2}(u_1 \otimes u_2)$ is C^∞ from $\mathcal{P}_0$ to $E_\infty^{\circ\rho}$ for u_1, $u_2 \in E_\infty^{\circ\rho}$ and according to statement iii) of Theorem 3.12 $R_g^{(+)2} \in L(E_\infty^{\circ\rho} \hat{\otimes} E_\infty^{\circ\rho}, E_0^\rho)$ for $g \in \mathcal{P}_0$. Differentiating twice with respect to u at $u = 0$, the equality $F(U_g^{(+)}(u)) = U_g^1 F(u)$, which by hypothesis is true for all g in a neighbourhood of the identity in $\mathcal{P}_0$ and for all u in a neighbourhood of zero in $E_\infty^{\circ\rho}$, we obtain that $U_g^1 F^2(U_{g^{-1}}^1 \otimes U_{g^{-1}}^1) - F^2 = -R_g^{(+)2}$ for g in a neighbourhood of the identity in $\mathcal{P}_0$. Using the fact that $R^{(+)2}$ is a 1-cocycle for the $\mathcal{P}_0$-module $L(E_\infty^{\circ\rho} \hat{\otimes} E_\infty^{\circ\rho}, E_0^\rho)$ defined by $(g, Q) \mapsto U_g^1 Q(U_{g^{-1}}^1 \otimes U_{g^{-1}}^1)$, i.e. $R_{gh}^{(+)2} = U_g^1 R_h^{(+)2}(U_{g^{-1}}^1 \otimes U_{g^{-1}}^1) + R_g^{(+)2}$, it then follows that

$$U_g^1 F^2(U_{g^{-1}}^1 \otimes U_{g^{-1}}^1) - F^2 = -R_g^{(+)2}, \quad g \in \mathcal{P}_0. \tag{3.50}$$

Since $g \mapsto F^2(U_g^1 u_1 \otimes U_g^1 u_2)$ and $g \mapsto R_g^{(+)2}(U_g^1 u_1 \otimes U_g^1 u_2)$, u_1, $u_2 \in E_\infty^{\circ\rho}$, are C^∞ from $\mathcal{P}_0$ to E_0^ρ, it follows that $F^2(u_1 \otimes u_2) \in E_\infty^\rho$ for u_1, $u_2 \in E_\infty^{\circ\rho}$ and that $F^2 \in L(E_\infty^{\circ\rho} \hat{\otimes} E_\infty^{\circ\rho}, E_\infty^\rho)$.

The definition (1.17a) and (1.17b) of $U^{(+)}$ shows that $R_g^{(+)2} = (0, R_g^{(+)D2})$, where $R_g^{(+)D2} \in L(E_\infty^{\circ\rho} \hat{\otimes} E_\infty^{\circ\rho}, E_\infty^\rho)$ is given by

$$\begin{aligned} &(R_g^{(+)D2}(u_1 \otimes u_2))^\wedge(k) \\ &\quad = \frac{i}{2} \sum_{\varepsilon=\pm} (\varphi_g(U_{g^{-1}}^1 u_1, -\varepsilon k) P_\varepsilon(k) \hat{\alpha}_2(k) + \varphi_g(U_{g^{-1}}^1 u_2, -\varepsilon k) P_\varepsilon(k) \hat{\alpha}_1(k)) \end{aligned} \tag{3.51}$$

and where φ is given by (1.23a) and (1.23b). Let $F^2 = (F^{M2}, F^{D2})$, where $F^{M2} \in L(E_\infty^{\circ\rho} \hat{\otimes} E_\infty^{\circ\rho}, M_\infty^\rho)$ and $F^{D2} \in L(E_\infty^{\circ\rho} \hat{\otimes} E_\infty^{\circ\rho}, D_\infty^\rho)$, let $G^2((f, \dot{f}) \otimes \alpha) = F^{D2}((f, \dot{f}, 0) \otimes (0, 0, \alpha))$ and let $r_g^2((f, \dot{f}) \otimes \alpha) = R_g^{(+)D2}((f, \dot{f}, 0) \otimes (0, 0, \alpha))$. Then $G^2 \in L(M_\infty^{\circ\rho} \hat{\otimes} D_\infty, D_\infty)$ satisfies, according to (3.50), the equality

$$U_g^{1D} G^2(U_{g^{-1}}^{1M} \otimes U_{g^{-1}}^{1D}) - G^2 = -r_g^2, \quad g \in \mathcal{P}_0. \tag{3.52}$$

G^2 is the unique element in $L(M_\infty^{\circ\rho} \hat{\otimes} D_\infty, D_0)$ satisfying (3.52). As a matter of fact, if $H \in L(M_\infty^{\circ\rho} \hat{\otimes} D_\infty, D_0)$ satisfies the equation

$$U_g^{1D} H((f, \dot{f}) \otimes \alpha) = H(U_g^{1M}(f, \dot{f}) \otimes U_g^{1D} \alpha)$$

for $g \in \mathcal{P}_0$, $(f, \dot{f}) \in M_\infty^{\circ\rho}$ and $\alpha \in D_\infty$, it follows that $H((f, \dot{f}) \otimes \alpha) \in D_\infty$ and that $T_X^{1D} H = H S_X$, $X \in \mathfrak{p}$, where $S_X = T_X^{1M} \otimes I + I \otimes T_X^{1D}$. This equality extends to the enveloping algebra, i.e. $T_Y^{1D} H = H S_Y$ for $Y \in U(\mathfrak{p})$. Using $\sum_{0 \le \mu \le 3} T_{P_\mu P^\mu}^{1D} = -m^2$ and $\sum_{0 \le \mu \le 3} T_{P_\mu P^\mu}^{1M} = 0$, we obtain that

$$H\Big(\sum_{0 \le \mu \le 3} T_{P_\mu}^{1M} \otimes T_{P^\mu}^{1D} \Big) = 0 \tag{3.53}$$

Since $|p_1|\omega(p_2) - p_1 \cdot p_2 \geq |p_1|m^2/(2\omega(p_2))$ for $p_1, p_2 \in \mathbb{R}^3$, it follows by studying the expression

$$\sum_{0\leq\mu\leq3} ((T^{1M}_{P_\mu}(f,\dot{f}))^\wedge \otimes (T^{1D}_{P^\mu}\alpha)^\wedge(p_1,p_2)$$

that the map

$$\sum_{0\leq\mu\leq3} T^{1M}_{P_\mu} \otimes T^{1D}_{P^\mu} \colon M^{\circ\rho}_c \hat{\otimes} D_\infty \to M^{\circ\rho}_c \hat{\otimes} D_\infty,$$

where M_c was defined in Theorem 2.9, is surjective. Therefore, since $M^{\circ\rho}_c$ is dense in $M^{\circ\rho}_\infty$, H vanishes on a dense subset of $M^{\circ\rho}_\infty \hat{\otimes} D_\infty$, which by continuity proves that $H = 0$. Hence G^2 is the unique element in $L(M^{\circ\rho}_\infty \hat{\otimes} D_\infty, D_0)$ satisfying (3.52).

Let $(f,\dot{f}) \in M^{\circ\rho}_c$ and let

$$(\Theta_\varepsilon((f,\dot{f})))(k) = \vartheta^\infty(\chi_0 \circ \rho B^{(+)1}, (\omega(k), -\varepsilon k)), \quad \varepsilon = \pm, \tag{3.54}$$

where ϑ^∞, χ_0, ρ and $B^{(+)1}$ are as in (1.22b)–(1.23b). It follows from statement i) of Lemma 3.2 that the function $k \mapsto \omega(k)^{-1}(\Theta_\varepsilon((f,\dot{f})))(k)$ is an element of $L^\infty(\mathbb{R}^3)$, so $\Theta_\varepsilon((f,\dot{f}))\hat{\alpha} \in L^2(\mathbb{R}^3, \mathbb{C}^4)$ for $\alpha \in D_\infty$. Therefore $G^2((f,\dot{f}) \otimes \alpha)$ defined by

$$(G^2((f,\dot{f}) \otimes \alpha))^\wedge = \frac{-i}{2} \sum_{\varepsilon=\pm} (\Theta_\varepsilon((f,\dot{f})))(k) P_\varepsilon(k)\hat{\alpha}(k), \tag{3.55}$$

is an element of D_0 and a direct calculation shows that

$$(U^{1D}_g G^2(U^{1M}_{g^{-1}} \otimes U^{1D}_{g^{-1}}) - G^2 + r^2_g)((f,\dot{f}) \otimes \alpha) = 0,$$

for $g \in \mathcal{P}_0$, $(f,\dot{f}) \in M^{\circ\rho}_c$ and $\alpha \in D_\infty$. Since $M^{\circ\rho}_c$ is dense in $M^{\circ\rho}_\infty$, it follows that the unique solution $G^2 \in L(M^{\circ\rho}_\infty \hat{\otimes} D_\infty, D_0)$ of equation (3.52) is given by continuous extension of G^2 defined in (3.55) for $(f,\dot{f}) \in M^{\circ\rho}_c$. Let $(f,\dot{f}) \in M^{\circ\rho}_\infty$ and let $(f^{(n)}, \dot{f}^{(n)}) \in M^{\circ\rho}_c$, $n \geq 0$, be a sequence which converges in $M^{\circ\rho}_\infty$ to $(f,\dot{f})$. Since $G^2 \in L(M^{\circ\rho}_\infty \hat{\otimes} D_\infty, D_\infty)$, it follows that $G^2((f^{(n)}, \dot{f}^{(n)}) \otimes \alpha)$ converges in D_∞ to $G^2((f,\dot{f}) \otimes \alpha)$ for $\alpha \in D_\infty$, which in particular shows that $(G^2((f^{(n)}, \dot{f}^{(n)}) \otimes \alpha))^\wedge(0)$ converges in $\mathbb{C}^4$ to $(G^2((f,\dot{f}) \otimes \alpha))^\wedge(0)$. For given $\varepsilon = \pm$, we choose $\alpha \in D_\infty$ such that $\hat{\alpha}(0) = P_\varepsilon(0)\hat{\alpha}(0)$ and $|\hat{\alpha}(0)| = 1$. It then follows from (3.55) that

$$(\Theta_\varepsilon((f^{(n)}, \dot{f}^{(n)})))(0) = 2i(\hat{\alpha}(0))^+(G^2((f^{(n)}, \dot{f}^{(n)}) \otimes \alpha))^\wedge(0),$$

which proves that the sequence $\Theta_\varepsilon((f^{(n)}, \dot{f}^{(n)}))(0)$, $n \geq 0$, converges in $\mathbb{R}$, when $n \to \infty$. By definition (3.54) of Θ_ε and definition (1.23b) of ϑ^∞, it follows that

$$(\Theta_\varepsilon((f^{(n)}, \dot{f}^{(n)})))(0) = \int_0^\infty \chi_0(\tau) Q^{(n)}(\tau, 0)d\tau, \tag{3.56}$$

where $Q^{(n)} = Q^{(n)}_1 + Q^{(n)}_2$ and

$$Q^{(n)}_1(\tau, 0) = (\cos(\tau|\nabla|)f^{(n)}_0)(0),$$
$$Q^{(n)}_2(\tau, 0) = |\nabla|^{-1}\sin(\tau|\nabla|)\dot{f}^{(n)}_0)(0).$$

Let $\theta \in S'(\mathbb{R})$ be the inverse Fourier transform of $\chi_0: \mathbb{R} \to \mathbb{R}$, where $\chi_0(\tau) = 0$ for $\tau < 0$. Let $\lambda(\tau) = 1$ for $\tau \geq 0$ and $\lambda(\tau) = 0$ for $\tau < 0$ and let θ_0 be the inverse Fourier transform of $\sqrt{2\pi}\lambda$. Then $\theta_0(s) = i(s + i0)^{-1}$, $\theta - \theta_0$ is an entire analytic function since $\chi_0 - \lambda$ has compact support and $|(\theta - \theta_0)(s)| \leq (1 + |s|)^{-1}$. Equality (3.4b) gives, noting that $0 \notin \operatorname{supp} \hat{f}_0^{(n)}$,

$$\int_0^\infty \chi_0(\tau) Q_1^{(n)}(\tau, 0) d\tau = (2(2\pi)^2)^{-1} \int ((\theta - \theta_0)(|p|) + (\theta - \theta_0)(-|p|) \hat{f}_0^{(n)}(p) dp \tag{3.57a}$$

and

$$\begin{aligned} \int_0^\infty \chi_0(\tau) Q_2^{(n)}(\tau, 0) d\tau = (2\pi)^{-2} \int |p|^{-2} \hat{\dot{f}}_0^{(n)}(p) dp \qquad\qquad (3.57\text{b}) \\ + (2\pi)^{-2} \int (2i|p|)^{-1} ((\theta - \theta_0)(|p|) - (\theta - \theta_0)(-|p|)) \hat{\dot{f}}_0^{(n)}(p) dp. \end{aligned}$$

The properties of $\theta - \theta_0$ show that

$$\begin{aligned} (2(2\pi)^2)^{-1} \int (|(\theta - \theta_0)(|p|)| + |(\theta - \theta_0)(-|p|)|)(|\hat{f}_0^{(n)}(p)| + |p|^{-1} |\hat{\dot{f}}_0^{(n)}(p)|) dp \\ \leq C_\rho \|(f^{(n)}, \dot{f}^{(n)})\|_{M^\rho}, \quad 1/2 < \rho < 1. \end{aligned}$$

This inequality and inequalities (3.56), (3.57a) and (3.57b) give that

$$\begin{aligned} &(\Theta_\varepsilon((f^{(n)}, \dot{f}^{(n)})))(0) \leq C_\rho \|(f^{(n)}, \dot{f}^{(n)})\|_{M^\rho} + (2\pi)^{-1} \int |p|^{-2} \hat{\dot{f}}_0^{(n)}(p) dp, \qquad (3.58) \\ &\big|(\Theta_\varepsilon((f^{(n)}, \dot{f}^{(n)})))(0) - (2\pi)^{-2} \int |p|^{-2} \hat{\dot{f}}_0^{(n)}(p) dp\big| \\ &\qquad\qquad \leq C_\rho \|(f^{(n)}, \dot{f}^{(n)})\|_{M^\rho}, \quad \varepsilon = \pm, 1/2 < \rho < 1. \end{aligned}$$

Let $1 < b < 1/2 + \rho$, $\psi \in C_0^\infty(\mathbb{R}^3)$, $0 \leq \psi(p) \leq 1$ for $p \in \mathbb{R}^3$, $\psi(p) = \psi(-p)$ for $p \in \mathbb{R}^3$, $\psi(p) = 1$ for $|p| \leq 1$ and $\psi(p) = 0$ for $|p| \geq 2$. Let $f_0 = 0$, $\dot{f}_j = 0$ for $1 \leq j \leq 3$, $\hat{\dot{f}}_0(p) = \psi(p)|p|^{-a}$ and $\hat{\dot{f}}_j(p) = -i(p_j/|p|^2)\hat{\dot{f}}_0(p)$ for $1 \leq j \leq 3$. Let $\hat{f}_\mu^{(n)}(p) = (1 - \psi(np))\hat{f}_\mu(p)$ and $\hat{\dot{f}}_\mu^{(n)}(p) = (1 - \psi(np))\hat{\dot{f}}_\mu(p)$, for $0 \leq \mu \leq 3$ and $n \geq 0$. It follows using the equivalence of the norms $\|\cdot\|_{E_l}$ and q_l given by Theorem 2.9 that $(f, \dot{f}) \in M_\infty^\rho$ and $\|(f^{(n)}, \dot{f}^{(n)}) - (f, \dot{f})\|_{M_\infty^\rho} \to 0$ when $n \to \infty$. Obviously $(f^{(n)}, \dot{f}^{(n)}) \in M_c^\rho$. Moreover conditions (1.4c) are satisfied for $(f, \dot{f})$ (resp. $(f^{(n)}, \dot{f}^{(n)})$, $n \geq 0$), so $(f, \dot{f}) \in M_\infty^{\circ\rho}$ (resp. $(f^{(n)}, \dot{f}^{(n)}) \in M_c^{\circ\rho}$, $n \geq 0$). The right-hand side of inequality (3.58) is uniformly bounded in n for the sequence $(f^{(n)}, \dot{f}^{(n)})$ so constructed, but the integral on the left-hand side goes to infinity when n goes to infinity. This proves that the sequence $(\Theta_\varepsilon((f^{(n)}, \dot{f}^{(n)})))(0)$, $n \geq 0$, does not converges in $\mathbb{R}$ when $n \to \infty$. This is in contradiction with the fact that it follows, as we proved from the hypothesis that this sequence converges in $\mathbb{R}$. This shows that the map $F: \mathcal{O} \to E_0^\rho$ does not exist, which proves the theorem.

We shall prove that asymptotic reprensentations $U^{(+)}$ defined by formulas (1.17a), (1.17b) and (1.23a) for different choices of the function χ_0 are equivalent. For $i \in \{1,2\}$ let $\chi_i \in C^\infty([0,\infty[)$, $0 \le \chi_i(\tau) \le 1$ for $\tau \ge 0$, $\chi_i(\tau) = 1$ for $\tau \ge 2$, let $U_i^{(+)}$ be the corresponding asymptotic representation given by formulas (1.17a), (1.17b) and (1.23a) with χ_i replacing χ_0, and let $A_i^{(+)1}$ be defined correspondingly by formula (1.22a) with χ_i replacing χ_0. We introduce the notation $\Theta_\varepsilon((f,\dot f))$ for $\varepsilon = \pm$, $(f,\dot f) \in M^\rho$ by

$$\big(\Theta_\varepsilon((f,\dot f))\big)(k) = \vartheta^\infty(A_1^{(+)1} - A_2^{(+)1}, (\omega(k), -\varepsilon k)). \tag{3.59a}$$

For $u = (f,\dot f,\alpha) \in E^\rho$, let $F(u) = (f,\dot f, F^D(u))$, where

$$(F^D(u))^\wedge(k) = \sum_{\varepsilon=\pm} \exp\big(-i(\Theta_\varepsilon((f,\dot f)))(k)\big) P_\varepsilon(k)\hat\alpha(k). \tag{3.59b}$$

Theorem 3.14. *Let $1/2 < \rho < 1$. If $n, N \in \mathbb{N}$, then $F, F^{-1} \in C^n(E_{N+n}, E_N)$, $F, F^{-1} \in C^\infty(E_\infty, E_\infty)$ and $U_{2g}^{(+)} = F \circ U_{1g}^{(+)} \circ F^{-1}$ for $g \in \mathcal{P}_0$.*

Proof. To prove that $F \in C^n(E_{N+n}, E_N)$ it is sufficient to prove that $F^D \in C^n(E_{N+n}, D_N)$. Let $\chi_0 = \chi_1 - \chi_2$. Then $\chi_0 \in C^\infty([0,\infty[)$ and $\chi_0(\tau) = 0$ for $\tau \ge 2$. Since $\vartheta^\infty(H, \Lambda y) = \vartheta^\infty(\Lambda^{-1}H, y)$, $\Lambda \in SO(3,1)$, the function $k \mapsto \big(\Theta_\varepsilon((f,\dot f))\big)(k)$ is C^∞ on $\mathbb{R}^3$. It follows from (3.59a) and statement ii) of Lemma 3.2 that

$$|\big(\Theta_\varepsilon((f,\dot f))\big)(k)| \le C_\rho \omega(k)^{\rho-1/2} \|(f,\dot f)\|_{M_0^\rho}, \quad (f,\dot f) \in M_c^\rho.$$

The fact that M_c^ρ is dense is M_∞^ρ, according to Theorem 2.9, then shows

$$|\big(\eta_Z^{(\varepsilon)} \Theta_\varepsilon((f,\dot f))\big)(k)| \le C_\rho \omega(k)^{\rho-1/2} \|T_Z^{1M}(f,\dot f)\|_{M_0^\rho}, \tag{3.60}$$

for $Z \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$, $(f,\dot f) \in M_{|Z|}^\rho$. Let $G_\varepsilon(u) = \exp\big(-i\Theta_\varepsilon((f,\dot f))\big)\hat\alpha$ for $u = (f,\dot f,\alpha) \in M_0^\rho$. Inequality (3.60), with $Z = \mathbb{I}$, Lebesgue's dominated convergence theorem and Plancherel theorem show that $\omega^l G_\varepsilon \in C^0(E_l^\rho, L^2(\mathbb{R}^3, \mathbb{C}^4))$ for $l \in \mathbb{N}$. Let $Z \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$. Then $\eta_Z^{(\varepsilon)} G_\varepsilon(u)$ is a linear combination of expressions:

$$(\eta_{Z_1}^{(\varepsilon)})\Theta_\varepsilon((f,\dot f)) \cdots (\eta_{Z_j}^{(\varepsilon)})\Theta_\varepsilon((f,\dot f)) G_\varepsilon((f,\dot f,\beta_{j+1}^{(\varepsilon)})),$$

where $(\beta_{j+1}^{(\varepsilon)})^\wedge = \eta_{Z_{j+1}}^{(\varepsilon)}\hat\alpha$, $Z_i \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$ for $1 \le i \le j+1$, $|Z_1| + \cdots + |Z_{j+1}| = |Z|$, $|Z_i| \ge 1$ for $1 \le i \le j$ and when $j \ge 0$ and the expression is reduced to $G_\varepsilon((f,\dot f,\beta_1^{(\varepsilon)}))$ if $j = 0$. Since $\omega^l G_\varepsilon \in C^0(E_l^\rho, L^2)$ and $\rho - 1/2 \le 1$, it follows from inequality (3.60) and from the equivalence of norms in Theorem 2.9 that $\omega^l \eta_Z^{(\varepsilon)} G_\varepsilon(u) \in C^0(E_{|Z|+l}^\rho, L^2(\mathbb{R}^3, \mathbb{C}^4))$. Using that $(F^D(u))^\wedge = \sum_\varepsilon G_\varepsilon((f,\dot f, P_\varepsilon(-i\partial)\alpha))$ and the equivalence of norms in Theorem 2.9, we obtain

$$T_Y^{1D} F^D \in C^0(E_{|Y|}^\rho, D_0), \quad Y \in \Pi'. \tag{3.61}$$

Derivation of F^D shows that $P_\varepsilon((D^nF^D)(u;u_1,\ldots,u_n))^\wedge$, where $u=(f,\dot{f},\alpha)$ and $u_i=(f_i,\dot{f}_i,\alpha_i)$ is a linear combination of terms:

$$\Theta_\varepsilon((f_{i_1},\dot{f}_{i_1}))\cdots\Theta_\varepsilon((f_{i_n},\dot{f}_{i_n}))P_\varepsilon(F^D((f,\dot{f},\alpha)))^\wedge \tag{3.62a}$$

and

$$\Theta_\varepsilon((f_{i_1},\dot{f}_{i_1}))\cdots\Theta_\varepsilon((f_{i_{n-1}},\dot{f}_{i_{n-1}}))P_\varepsilon(F^D((f,\dot{f},\alpha_{i_n})))^\wedge, \tag{3.62b}$$

where $(i_1,\ldots,i_n)$ is a permutation of $(1,\ldots,n)$. This proves, together with (3.60) and (3.61), that the function $u\mapsto T_Y^{1D}(D^nF^D)(u;u_1,\ldots,u_n)$ is an element of $C^0(E_{|Y|+n},D_0)$, for $Y\in\Pi'$ and $(u_1,\ldots,u_n)\in E_{|Y|+n}$. This shows that the map $u\mapsto(D^nF^D)(u;u_1,\ldots,u_n)$ is an element of $C^0(E_{N+n},D_N)$, for $n,N\in\mathbb{N}$, which then shows that $F^D\in C^n(E_{N+n},D_N)$ and $F^D\in C^\infty(E_\infty,E_\infty)$. Since the inverse of F is given by $(f,\dot{f},\alpha)\mapsto F((-f,-\dot{f},\alpha))$, it also follows that F^{-1} has these properties. A short calculation shows that F intertwines $U_1^{(+)}$ and $U_2^{(+)}$. This proves the theorem.

4. Construction of the approximate solution.

The basic technical tool in the proof of the existence of modified wave operators for the M-D equation is here the same as in [8], namely the *construction of approximate solutions absorbing the slowly decaying part of the solution of the M-D equation.* The existence of the remainder (the difference between the exact and the approximate solution) can then be proved using only standard Sobolev estimates. In order to be able to use the implicit functions theorem in Fréchet spaces for the existence of the inverse of the modified wave operator, we shall establish precise estimates giving the loss of order of seminorms. This is done by the study of the equations for the enveloping algebra after a transformation compensating the long range Fourier phase.

We shall need a lemma which is an analog of Theorem 3.5 in [8] adapted to our spaces E_N. Let first $t \mapsto f_j(t), j = 1, 2,$ be continuous functions from $\mathbb{R}^+$ to D_N and let

$$(G_{\varepsilon,\mu}(f_1, f_2))(t) \tag{4.1a}$$
$$= -\int_t^\infty \frac{\sin((-\Delta)^{1/2}(t-s))}{(-\Delta)^{1/2}} (e^{i\varepsilon\omega(-i\partial)s} f_1(s))^+ \gamma_0\gamma_\mu (e^{i\varepsilon\omega(-i\partial)s} f_2(s)) ds, \quad t \geq 0,$$

and

$$(\dot{G}_{\varepsilon,\mu}(f_1, f_2))(t) \tag{4.1b}$$
$$= -\int_t^\infty \cos((-\Delta)^{1/2}(t-s)) (e^{i\varepsilon\omega(-i\partial)s} f_1(s))^+ \gamma_0\gamma_\mu (e^{i\varepsilon\omega(-i\partial)s} f_2(s)) ds, \quad t \geq 0.$$

Lemma 4.1. *Let $n \geq 0$, $N \geq 0$, $q \geq 0$ be integers, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ a multi-index and let $f_j : \mathbb{R}^+ \to D_N, j = 1, 2,$ be C^q functions. Let G and $\dot{G}$ be given by (4.1a) and (4.1b). If N is chosen sufficiently large, depending on $|\alpha|$ and n, then*

$$(1+t)^{|\alpha|+q-|\beta|+\chi+\rho-1/2} \| x^\beta \big(\frac{d}{dt}\big)^q \partial^\alpha (G(f_1, f_2)(t), \dot{G}(f_1, f_2)(t)) \|_{M_n^\rho} \tag{4.2}$$
$$\leq C_{n,|\alpha|,|\beta|,q,\rho} \sum_{q_1+q_2=q} \sup_{s \geq t} \big((1+s)^{q+\chi} \| \big(\frac{d}{ds}\big)^{q_1} f_1(s) \|_{D_N} \| \big(\frac{d}{ds}\big)^{q_2} f_2(s) \|_{D_N} \big), \quad t \geq 0,$$

where $0 \leq \rho \leq 1$, $\chi + q + |\alpha| - |\beta| + \rho - 1/2 > 0$. Moreover there is N' depending on $|\alpha|$ such that

$$\big| \big(\frac{\partial}{\partial t}\big)^q \partial^\alpha G(f_1, f_2)(t,x) \big| + (1+t+|x|) \big| \big(\frac{\partial}{\partial t}\big)^q \partial^\alpha \dot{G}(f_1, f_2)(t,x) \big| \tag{4.3}$$
$$\leq C_{|\alpha|,q,\chi,\varepsilon} (1+|x|+t)^{-(1+|\alpha|+q+\chi-\varepsilon)} (1+t)^{-\varepsilon}$$
$$\sum_{q_1+q_2=q} \sup_{s \geq t} \big((1+s)^{q+\chi} \| \big(\frac{d}{ds}\big)^{q_1} f_1(s) \|_{D_{N'}} \| \big(\frac{d}{ds}\big)^{q_2} f_2(s) \|_{D_{N'}} \big), \quad t \geq 0, \chi \geq 0, \varepsilon > 0.$$

Proof. After the change of variable $s-t \to s$, we have

$$(G_{\varepsilon,\mu}(f_1,f_2))(t) = \int_0^\infty \frac{\sin((-\Delta)^{1/2}s)}{(-\Delta)^{1/2}} J_\mu(s+t)ds \tag{4.4a}$$

and

$$(\dot{G}_{\varepsilon,\mu}(f_1,f_2))(t) = -\int_0^\infty \cos((-\Delta)^{1/2}s) J_\mu(s+t)ds, \tag{4.4b}$$

where

$$J_\mu(s+t) = (e^{i\varepsilon\omega(-i\partial)(s+t)} f_1(s+t))^+ \gamma_0\gamma_\mu (e^{i\varepsilon\omega(-i\partial)(s+t)} f_2(s+t)). \tag{4.4c}$$

The function $k \mapsto K(k) = |k|^{-1}\sin(|k|)$ is an entire analytic function on $\mathbb{C}^3$. As it is seen by considering its Taylor development for small k,

$$|K_\gamma(k)| \le C_\gamma(1+|k|)^{-1}, \quad k \in \mathbb{R}^3, \tag{4.5}$$

where $K_\gamma(k) = \frac{\partial^\gamma}{\partial k^\gamma} K(k)$.

Therefore, if $\beta = (\beta_1,\beta_2,\beta_3)$ is a multi-index,

$$x^\beta \frac{\sin((-\Delta)^{1/2}s)}{(-\Delta)^{1/2}s} s = \sum_{\gamma_1+\gamma_2=\beta} C_{\gamma_1,\gamma_2} K_{\gamma_2}(-is\partial) x^{\gamma_1} s^{1+|\gamma_2|}, \tag{4.6}$$

where C_{γ_1,γ_2} are constants given by Leibniz formula.

Similarly if $L(k) = \cos(|k|)$, then the entire analytic function L satisfies

$$|L_\gamma(k)| \le C_\gamma, \quad k \in \mathbb{R}^3, L_\gamma(k) = \frac{\partial^\gamma}{\partial k^\gamma} L(k) \tag{4.7}$$

and we obtain

$$x^\beta \cos((-\Delta)^{1/2}s) = \sum_{\gamma_1+\gamma_2=\beta} C_{\gamma_1,\gamma_2} L_{\gamma_2}(-is\partial) x^{\gamma_1} s^{|\gamma_2|}. \tag{4.8}$$

We next estimate the L^p-norm, $1 \le p \le \infty$, of $x^\gamma \partial^\alpha J_\mu(t)$, $|\gamma| \le |\alpha|$, $t \ge 0$. To do this, it is sufficient to estimate the L^p-norm of

$$\Gamma_{\gamma,\alpha}(t) = x^\gamma \partial^\alpha (e^{-i\varepsilon\omega(-i\partial)t} h_1(t))(e^{i\varepsilon\omega(-i\partial)t} h_2(t)), \quad t \ge 0, \tag{4.9}$$

where $h_j(t) \in D_\infty$, $j = 1,2$. According to Theorem A.1 of the appendix there are, for given $n \ge 0$, functions $r_j(t) \in D_\infty$ with $\operatorname{supp} r_j(t) \subset \{x \big| |x| \le t\}$, such that

$$\begin{aligned} &\|x^\gamma \partial^\alpha (e^{i\nu(j)\varepsilon\omega(-i\partial)t} h_j(t) - e^{i\nu(j)\varepsilon m\rho(t)} r_j(t))\|_{L^2} \\ &\quad \le C_{n,|\alpha|} \|h_j(t)\|_{D_{N+2|\alpha|}} t^{-(n+1-|\gamma|)}, \quad t > 0, \end{aligned} \tag{4.10}$$

where $\rho(t)$ is the function $x \mapsto (t^2-|x|^2)^{1/2}$ for $|x| \le t$ and $\rho(t)(x) = 0$ otherwise, and

$$\|\rho(t)^{-l} \partial^\alpha r_j(t)\|_{L^2} \le C_{n,|\alpha|,l} \|h_j(t)\|_{D_{N+2|\alpha|+l}} t^{-|\alpha|-l}, \quad t > 0, l \ge 0, \tag{4.11}$$

where $\rho(t)^{-l}\partial^\alpha r_j(t)$ is defined as being equal to zero for $|x| \geq t$, and where $\nu(1) = -1$, $\nu(2) = 1$ and where N, depending on n, is sufficiently large. Again, according to Theorem A.1 the functions r_j, $j = 1, 2$, satisfy

$$\begin{aligned} &\|x^\gamma \partial^\alpha (e^{i\nu(j)\varepsilon\omega(-i\partial)t} h_j(t) - e^{i\nu(j)\varepsilon m\rho(t)} r_j(t))\|_{L^\infty} \\ &\quad \leq C_{n,|\alpha|} \|h_j(t)\|_{D_{N+2|\alpha|}} t^{-(n+5/2-|\gamma|)}, \quad t > 0, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} &\|\rho(t)^{-l}\partial^\alpha r_j(t)\|_{L^\infty} \\ &\quad \leq C_{n,|\alpha|,l} \|h_j(t)\|_{D_{N+2|\alpha|+l}} t^{-3/2-|\alpha|-l}, \quad t > 0, l \geq 0. \end{aligned} \tag{4.13}$$

If

$$\delta_j(t) = e^{i\nu(j)\varepsilon\omega(-i\partial)t} h_j(t) - e^{i\nu(j)\varepsilon m\rho(t)} r_j(t),$$

then it follows from (4.9) that

$$\begin{aligned} \Gamma_{\gamma,\alpha}(t) = x^\gamma \partial^\alpha \big(&r_1(t) r_2(t) + e^{-im\rho(t)} r_1(t)\delta_2(t) \\ &+ \delta_1(t) e^{im\rho(t)} r_2(t) + \delta_1(t)\delta_2(t)\big). \end{aligned}$$

Leibniz rule and the fact that supp $r_j(t) \subset \{x \big| |x| \leq t\}$ give

$$\begin{aligned} &\|\Gamma_{\gamma,\alpha}(t) - x^\gamma \partial^\alpha (r_1(t) r_2(t))\|_{L^p} \\ &\quad \leq C_{|\alpha|} \sum_{\alpha_1+\alpha_2=\alpha} \Big(t^{|\gamma|} \|(\partial^{\alpha_1} e^{-im\rho(t)} r_1(t))(\partial^{\alpha_2}\delta_2(t))\|_{L^p} \\ &\qquad + t^{|\gamma|} \|(\partial^{\alpha_1}\delta_1(t))(\partial^{\alpha_2} e^{im\rho(t)} r_2(t))\|_{L^p} \\ &\qquad + \|x^\gamma (\partial^{\alpha_1}\delta_1(t))(\partial^{\alpha_2}\delta_2(t))\|_{L^p}\Big), \quad t > 0, 1 \leq p \leq \infty. \end{aligned} \tag{4.14}$$

Since $|\gamma| \leq |\alpha|$ we can write (for given α_1, α_2 with $\alpha_1 + \alpha_2 = \alpha$) $\gamma = \gamma_1 + \gamma_2$ with $|\gamma_j| \leq |\alpha_j|, j = 1, 2$. Doing this we obtain from (4.10) and (4.12) that

$$\begin{aligned} &\|x^\gamma (\partial^{\alpha_1}\delta_1(t))(\partial^{\alpha_2}\delta_2(t))\|_{L^p} \\ &\quad \leq \|x^{\gamma_1}\partial^{\alpha_1}\delta_1(t)\|_{L^2}^{1/p} \|x^{\gamma_1}\partial^{\alpha_1}\delta_1(t)\|_{L^\infty}^{(p-1)/p} \|x^{\gamma_2}\partial^{\alpha_2}\delta_2(t)\|_{L^2}^{1/p} \|x^{\gamma_2}\partial^{\alpha_2}\delta_2(t)\|_{L^\infty}^{(p-1)/p} \\ &\quad \leq C_{n,|\alpha|} t^{-(2n+2+3(p-1)/p-|\gamma|} \|h_1(t)\|_{D_{N+|\alpha|}} \|h_2(t)\|_{D_{N+|\alpha|}}, \quad t > 0, 1 \leq p \leq \infty. \end{aligned} \tag{4.15}$$

We have used here the fact that

$$\|fg\|_{L^p} \leq \|f\|_{L^2}^{1/p} \|f\|_{L^\infty}^{(p-1)/p} \|g\|_{L^2}^{1/p} \|g\|_{L^\infty}^{(p-1)/p}, \quad 1 \leq p \leq \infty.$$

It follows directly by induction that there are polynomials in (t, x), $F_\gamma^{(l)}$ of degree not higher than $|\gamma|$, such that, if $|x| < t$,

$$\partial^\gamma e^{im\rho} = e^{im\rho} \sum_{l=0}^{|\gamma|-1} \frac{1}{\rho^{|\gamma|+l}} F_\gamma^{(l)}, \quad |\gamma| \geq 1. \tag{4.16}$$

Equality (4.16) gives

$$|\partial^\gamma e^{im\rho(t,x)}| \leq C_{|\gamma|}(1+t)^{|\gamma|} \sum_{l=0}^{|\gamma|} \rho(t,x)^{-(|\gamma|+l)}, \quad |x| \leq t, t>0, |\gamma| \geq 0. \tag{4.17}$$

Since the support of $r_j(t)$ is contained in $\{x \big| |x| \leq t\}$, $t>0$, it follows from (4.17) and Schwarz inequality that

$$\begin{aligned}
&\sum_{\alpha_1+\alpha_2=\alpha} \|(\partial^{\alpha_1} e^{-im\rho(t)} r_1(t))(\partial^{\alpha_2}\delta_2(t))\|_{L^p} \\
&\quad \leq C_{|\alpha|} \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \|(\partial^{\alpha_1} e^{-im\rho(t)})(\partial^{\alpha_2} r_1(t))(\partial^{\alpha_3}\delta_2(t))\|_{L^p} \\
&\quad \leq C'_{|\alpha|} \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} (1+t)^{|\alpha_1|} \sum_{l=0}^{|\alpha_1|} \|\rho(t)^{-(|\alpha_1|+l)}(\partial^{\alpha_2} r_1(t))(\partial^{\alpha_3}\delta_2(t))\|_{L^p} \\
&\quad \leq C'_{|\alpha|} \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \sum_{l=0}^{|\alpha_1|} (1+t)^{|\alpha_1|} \|\rho(t)^{-(|\alpha_1|+l)} \partial^{\alpha_2} r_1(t)\|_{L^2}^{1/p} \\
&\qquad \|\rho(t)^{-(|\alpha_1|+l)} \partial^{\alpha_2} r_1(t)\|_{L^\infty}^{(p-1)/p} \|\partial^{\alpha_3}\delta_2(t)\|_{L^2}^{1/p} \|\partial^{\alpha_3}\delta_2(t)\|_{L^\infty}^{(p-1)/p}, \quad t>0, 1 \leq p \leq \infty.
\end{aligned}$$

It now follows from inequalities (4.10), (4.11), (4.12) and (4.13) that

$$\begin{aligned}
&\sum_{\alpha_1+\alpha_2=\alpha} \|(\partial^{\alpha_1} e^{-im\rho(t)} r_1(t))(\partial^{\alpha_2}\delta_2(t))\|_{L^p} \\
&\quad \leq C_{n,|\alpha|} \|h_1(t)\|_{D_{N'}} \|h_2(t)\|_{D_{N'}} \\
&\qquad \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \sum_{l=0}^{|\alpha_1|} (1+t)^{|\alpha_1|} t^{-(n+1+3(p-1)/p+|\alpha_1|+|\alpha_2|+l)}, \quad t>0, 1 \leq p \leq \infty,
\end{aligned}$$

where N' depends only on n and $|\alpha|$.

For $t \geq 1$, we get

$$\begin{aligned}
&\sum_{\alpha_1+\alpha_2=\alpha} \|(\partial^{\alpha_1} e^{-im\rho(t)} r_1(t))(\partial^{\alpha_2}\delta_2(t))\|_{L^p} \\
&\quad \leq C_{n,|\alpha|} t^{-(n+1+3(p-1)/p)} \|h_1(t)\|_{D_{N'}} \|h_2(t)\|_{D_{N'}}, \quad t \geq 1, 1 \leq p \leq \infty,
\end{aligned} \tag{4.18}$$

for some constant $C_{n,|\alpha|}$.

Inequalities (4.11) and (4.13) give similarly

$$\begin{aligned}
&\sum_{\alpha_1+\alpha_2=\alpha} \|(\partial^{\alpha_1} r_1(t))(\partial^{\alpha_2} r_2(t))\|_{L^p} \\
&\quad \leq C_{n,|\alpha|} \|h_1(t)\|_{D_{N+2|\alpha|}} \|h_2(t)\|_{D_{N+2|\alpha|}} t^{-(|\alpha|+3(p-1)/p)}, \quad t>0, 1 \leq p \leq \infty.
\end{aligned} \tag{4.19}$$

Inequalities (4.14), (4.15) and (4.18) give for $t \geq 1$,

$$\begin{aligned} &\|\Gamma_{\gamma,\alpha}(t) - x^{\gamma}\partial^{\alpha}(r_1(t)r_2(t))\|_{L^p} \\ &\quad \leq C_{n,|\alpha|}\|h_1(t)\|_{D_{N'}}\|h_2(t)\|_{D_{N'}}(t^{-2(n+1)} + t^{-(n+1)})t^{|\gamma|-3(p-1)/p} \\ &\quad \leq C_{n,|\alpha|}\|h_1(t)\|_{D_{N'}}\|h_2(t)\|_{D_{N'}}t^{-(n+1-|\gamma|+3(p-1)/p)}, \quad t \geq 1, 1 \leq p \leq \infty. \end{aligned} \tag{4.20}$$

Choosing $n = |\alpha| - 1$ for $|\alpha| \geq 1$ and $n = 0$ for $|\alpha| = 0$ in (4.20), and using (4.19), we obtain

$$\begin{aligned} &\|\Gamma_{\gamma,\alpha}(t)\|_{L^p} \\ &\quad \leq C_{|\alpha|}\|h_1(t)\|_{D_N}\|h_2(t)\|_{D_N}t^{-(|\alpha|-|\gamma|+3(p-1)/p)}, \quad t \geq 1, |\gamma| \leq |\alpha|, 1 \leq p \leq \infty, \end{aligned} \tag{4.21}$$

where N is redefined and depends only on $|\alpha|$. Since

$$\|x^{\gamma}\partial^{\alpha}e^{-i\omega(-\partial)t}h_1(t)\|_{L^2} \leq C_{|\alpha|}\|h_1(t)\|_{D_{|\alpha|}}, \quad 0 \leq t \leq 1, |\gamma| \leq |\alpha| \tag{4.22}$$

and

$$\|x^{\gamma}\partial^{\alpha}e^{i\omega(-i\partial)t}h_2(t)\|_{L^\infty} \leq C_{|\alpha|}\|h_2(t)\|_{D_{|\alpha|+2}}, \quad 0 \leq t \leq 1, |\gamma| \leq |\alpha|,$$

which is obtained by using $\|f\|_{L^\infty} \leq C\|(1-\Delta)f\|_{L^2}$, we obtain from (4.9) and Schwarz inequality that

$$\begin{aligned} &\|\Gamma_{\gamma,\alpha}(t)\|_{L^p} \\ &\quad \leq C'_{|\alpha|}\|h_1(t)\|_{D_{|\alpha|+2}}\|h_2(t)\|_{D_{|\alpha|+2}}, \quad 0 \leq t \leq 1, |\gamma| \leq |\alpha|, 1 \leq p \leq \infty. \end{aligned} \tag{4.23}$$

Inequalities (4.21) and (4.23) give (with new $C_{|\alpha|}$)

$$\begin{aligned} &\|\Gamma_{\gamma,\alpha}(t)\|_{L^p} \\ &\quad \leq C_{|\alpha|}\|h_1(t)\|_{D_N}\|h_2(t)\|_{D_N}(1+t)^{-(|\alpha|-|\gamma|+3(p-1)/p)}, \quad t \geq 0, 1 \leq p \leq \infty, \end{aligned} \tag{4.24}$$

for some N depending only on $|\alpha|$.

It follows from the expression (4.4a) of $G_{\varepsilon,\mu}$ and equality (4.6) that

$$\begin{aligned} &x^{\beta}\partial^{\alpha}(G_{\varepsilon,\mu}(f_1,f_2))(t) \\ &\quad = \sum_{\gamma_1+\gamma_2=\beta} C_{\gamma_1,\gamma_2}\int_0^{\infty} K_{\gamma_2}(-is\partial)x^{\gamma_1}s^{1+|\gamma_2|}\partial^{\alpha}J_{\mu}(s+t)ds, \quad t \geq 0. \end{aligned} \tag{4.25}$$

According to definition (4.4c) of J_μ and estimate (4.24) for $\Gamma_{\gamma,\alpha}$ defined by (4.9), we have

$$\begin{aligned} &\|x^{\gamma}\partial^{\alpha}J_{\mu}(t)\|_{L^p} \\ &\quad \leq C_{|\alpha|}\|f_1(t)\|_{D_N}\|f_2(t)\|_{D_N}(1+t)^{-(|\alpha|-|\gamma|+3(p-1)/p)}, \quad t \geq 0, |\alpha| \geq |\gamma|, 1 \leq p \leq \infty. \end{aligned} \tag{4.26}$$

Since $|p|^\rho |K_\gamma(p)| \leq C_{|\gamma|}(1+|p|)^{-(1-\rho)}$ according to (4.5) we have

$$\begin{aligned} t^\rho \||\nabla|^\rho K_\gamma(-it\partial)f\|_{L^2} &= \||-it\partial|^\rho K_\gamma(-it\partial)f\|_{L^2} \\ &\leq C_{|\gamma|}\|f\|_{L^2}, \quad t\geq 0, 0\leq\rho\leq 1. \end{aligned} \tag{4.27}$$

Inequalities (4.25), (4.26) and (4.27) give for $|\beta|\leq|\alpha|$ (with a new $C_{|\alpha|}$)

$$\begin{aligned} &\||\nabla|^\rho x^\beta \partial^\alpha (G_{\varepsilon,\mu}(f_1,f_2))(t)\|_{L^2} \\ &\quad\leq C_{|\alpha|}\sum_{\gamma_1+\gamma_2=\beta}\int_0^\infty s^{1-\rho}s^{|\gamma_2|}\|f_1(s+t)\|_{D_N}\|f_2(s+t)\|_{D_N}(1+s+t)^{-(3/2+|\alpha|-|\gamma_1|)}ds \\ &\quad\leq C'_{\rho,|\alpha|}\sup_{s\geq t}((1+s)^q\|f_1(s)\|_{D_N}\|f_2(s)\|_{D_N})(1+t)^{-(\rho-1/2+|\alpha|-|\beta|+q)}, \end{aligned} \tag{4.28}$$

$t\geq 0$, $|\beta|\leq|\alpha|$, $q+|\alpha|-|\beta|+\rho-1/2>0$.

It follows from (4.4b), (4.7) and (4.8) that

$$\begin{aligned} &\||\nabla|^{\rho-1}x^\beta\partial^\alpha(\dot G_{\varepsilon,\mu}(f_1,f_2))(t)\|_{L^2} \\ &\quad\leq C_{|\alpha|}\sum_{\gamma_1+\gamma_2=\beta}\int_0^\infty s^{|\gamma_2|}\||\nabla|^{\rho-1}x^{\gamma_1}\partial^\alpha J_\mu(s+t)\|_{L^2}ds, \quad |\beta|\leq|\alpha|. \end{aligned} \tag{4.29}$$

According to (2.67) we obtain, since $C_0^\infty(\mathbb{R}^3)$ is dense in L^p, $1\leq p<\infty$, that

$$\||\nabla|^{\rho-1}x^\gamma\partial^\alpha J_\mu(t)\|_{L^2}\leq C\|x^\gamma\partial^\alpha J_\mu(t)\|_{L^p}, \quad t\geq 0,$$

where $p=6(5-2\rho)^{-1}$, $0\leq\rho\leq 1$. We obtain from (4.26) that

$$\begin{aligned} &\||\nabla|^{\rho-1}x^\gamma\partial^\alpha J_\mu(t)\|_{L^2} \\ &\quad\leq C_{|\alpha|}\|f_1(t)\|_{D_N}\|f_2(t)\|_{D_N}(1+t)^{-(|\alpha|-|\gamma|+1/2+\rho)}, \quad t\geq 0, 0\leq\rho\leq 1, |\gamma|\leq|\alpha|. \end{aligned} \tag{4.30}$$

Inequalities (4.29) and (4.30) show that (with a new $C_{|\alpha|}$)

$$\begin{aligned} &\||\nabla|^{\rho-1}x^\beta\partial^\alpha(\dot G_{\varepsilon,\mu}(f_1,f_2))(t)\|_{L^2} \\ &\quad\leq C_{|\alpha|}\sup_{s\geq 0}(\|f_1(s)\|_{D_N}\|f_2(s)\|_{D_N}) \\ &\qquad\qquad\sum_{\gamma_1+\gamma_2=\beta}\int_0^\infty(1+s+t)^{|\gamma_2|}(1+t+s)^{-(|\alpha|-|\gamma_1|+1/2+\rho)}ds \\ &\quad\leq C'_{|\alpha|}\sup_{s\geq t}\big((1+s)^q\|f_1(s)\|_{D_N}\|f_2(s)\|_{D_N}\big)(1+t)^{-(|\alpha|-|\gamma|-1/2+\rho+q)}, \quad t\geq 0, \end{aligned} \tag{4.31}$$

where $0\leq\rho\leq 1$, $|\beta|\leq|\alpha|$ and $|\alpha|-|\gamma|+\rho-1/2+q>0$. Inequalities (4.28) and (4.31) prove inequality (4.2) of the lemma in the case where $q=0$. The case $q\geq 1$ follows by differentiation of expressions (4.4a) and (4.4b) with respect to t, and then by using inequalities (4.28) and (4.31) for $q\geq 1$. This proves inequality (4.2).

To prove inequality (4.3), introduce

$$I_{\gamma,\alpha,\mu}(t) = x^\gamma \partial^\alpha J_\mu(t), \quad t \geq 0, \tag{4.32}$$

and let $v_j(t)$ be the stationary phase development of $e^{i\varepsilon\omega(-i\partial)t} f_j(t)$ up to order n. We observe that for $t > 0$, Plancherel theorem gives

$$\begin{aligned} \|(I_{\gamma,\alpha,\mu}(t))^\wedge\|_{L^1} &\leq \Big(\int (1+|k|^2t^2)^{-2} dk\Big)^{1/2} \|(1-t^2\Delta) I_{\gamma,\alpha,\mu}(t)\|_{L^2} \\ &= Ct^{-3/2} \|(1-t^2\Delta) I_{\gamma,\alpha,\mu}(t)\|_{L^2}, \quad t > 0. \end{aligned} \tag{4.33}$$

Commutation of Δ and x^γ gives

$$\begin{aligned} &\|(1-t^2\Delta) I_{\gamma,\alpha,\mu}(t)\|_{L^2} \\ &\quad \leq C_{|\gamma|,|\alpha|}\Big(\|I_{\gamma,\alpha,\mu}(t)\|_{L^2} + t^2 \sum_{|\delta|=|\gamma|-2} \|I_{\delta,\alpha,\mu}(t)\|_{L^2} \\ &\qquad + t^2 \sum_{\substack{|\delta|=|\gamma|-1 \\ |\lambda|=|\alpha|+1}} \|I_{\delta,\lambda,\mu}(t)\|_{L^2} + t^2 \sum_{|\lambda|=|\alpha|+2} \|I_{\gamma,\lambda,\mu}(t)\|_{L^2}\Big), \end{aligned} \tag{4.34}$$

where sum over δ is absent if the upper bound of $|\delta|$ is strictly negative.

According to (4.26) and (4.32) we now get

$$\begin{aligned} &\|(1-t^2\Delta) I_{\gamma,\alpha,\mu}(t)\|_{L^2} \\ &\quad \leq C_{|\alpha|} \|f_1(t)\|_{D_N} \|f_2(t)\|_{D_N} (1+t)^{-(|\alpha|-|\gamma|+3/2)}, \quad t \geq 0, |\gamma| \leq |\alpha|, \end{aligned}$$

which together with (4.33), shows that (with a new $C_{|\alpha|}$)

$$\begin{aligned} &\|(I_{\gamma,\alpha,\mu}(t))^\wedge\|_{L^1} \\ &\quad \leq C_{|\alpha|} \|f_1(t)\|_{D_N} \|f_2(t)\|_{D_N} t^{-(|\alpha|-\gamma|+3)}, \quad t \geq 1, |\gamma| \leq |\alpha|, \end{aligned} \tag{4.35a}$$

where N depends only on $|\alpha|$. Similarly, we obtain using (4.20)

$$\|\big(I_{\gamma,\alpha,\mu}(t) - x^\gamma \partial^\alpha (v_1^+(t)\gamma_0\gamma_\mu v_2(t))\big)^\wedge\|_{L^1} \leq C_{|\alpha|,n} \|f_1(t)\|_{D_N} \|f_2(t)\|_{D_N} t^{-(n+2-|\gamma|)}. \tag{4.35b}$$

On the other hand, we also have

$$\|(I_{\gamma,\alpha,\mu}(t))^\wedge\|_{L^1} \leq C \|(1-\Delta) I_{\gamma,\alpha,\mu}(t)\|_{L^2}, \quad t \geq 0, \tag{4.36}$$

which, by the method used from (4.33) to (4.35), gives

$$\|(I_{\gamma,\alpha,\mu}(t))^\wedge\|_{L^1} \leq C_{|\alpha|} \|f_1(t)\|_{D_N} \|f_2(t)\|_{D_N}, \quad t \geq 0, |\gamma| \leq |\alpha|. \tag{4.37}$$

Inequalities (4.35) and (4.37) give (with a new $C_{|\alpha|}$)

$$\|(I_{\gamma,\alpha,\mu}(t))^\wedge\|_{L^1} \leq C_{|\alpha|}\|f_1(t)\|_{D_N}\|f_2(t)\|_{D_N}(1+t)^{-(|\alpha|-|\gamma|+3)}, \quad t \geq 0, |\gamma| \leq |\alpha|, \quad (4.38)$$

where N depends only on $|\alpha|$.

Since $\|f\|_{L^\infty} \leq \|\hat{f}\|_{L^1}$ it follows from inequality (4.5), equality (4.25) and definition (4.32), that

$$\begin{aligned}&\|x^\beta\partial^\alpha(G_{\varepsilon,\mu}(f_1,f_2))(t)\|_{L^\infty}\\ &\quad\leq C_{|\alpha|}\sum_{\gamma_1+\gamma_2=\beta}\int_0^\infty s^{1+|\gamma_2|}\|(I_{\gamma_1,\alpha,\mu}(t+s))^\wedge\|_{L^1}ds, \quad t\geq 0, |\beta|\leq|\alpha|.\end{aligned}$$

It now follows from (4.38) that (with a new $C_{|\alpha|}$)

$$\begin{aligned}&\|x^\beta\partial^\alpha(G_{\varepsilon,\mu}(f_1,f_2))(t)\|_{L^\infty} \qquad (4.39)\\ &\quad\leq C_{|\alpha|}\sum_{\gamma_1+\gamma_2=\beta}\int_0^\infty s^{1+|\gamma_2|}(1+s+t)^{-(|\alpha|-|\gamma_1|+3)}\|f_1(t+s)\|_{D_N}\|f_2(t+s)\|_{D_N}ds\\ &\quad\leq C'_{|\alpha|}\sup_{s\geq 0}(\|f_1(s)\|_{D_N}\|f_2(s)\|_{D_N})(1+t)^{-(|\alpha|-|\beta|+1)}, \quad t\geq 0, |\beta|\leq|\alpha|,\end{aligned}$$

where N depends only on $|\alpha|$.

The proof of

$$\begin{aligned}&\|x^\beta\partial^\alpha(\dot{G}_{\varepsilon,\mu}(f_1,f_2))(t)\|_{L^\infty} \qquad (4.40)\\ &\quad\leq C'_{|\alpha|}\sup_{s\geq 0}(\|f_1(s)\|_{D_N}\|f_2(s)\|_{D_N})(1+t)^{-(|\alpha|-|\beta|+2)}, \quad t\geq 0, |\beta|\leq|\alpha|,\end{aligned}$$

is so similar to the proof of (4.39) that we omit it.

Let $|t| \leq |x|$, then it follows from inequalities (2.60a) and (2.60b) of Theorem (2.12), with $1/2 < \rho < 1$, that

$$\begin{aligned}&(1+|x|+t)^{3/2-\rho+|\alpha|}(|\partial^\alpha G(f_1,f_2)(t,x)| + (1+|x|+|t|)|\partial^\alpha\dot{G}(f_1,f_2)(t,x)|) \qquad (4.41)\\ &\quad\leq C_{|\alpha|}(1+|x|)^{3/2-\rho+|\alpha|}(|\partial^\alpha G(f_1,f_2)(t,x)| + (1+|x|)|\partial^\alpha\dot{G}(f_1,f_2)(t,x)|)\\ &\quad\leq C'_{|\alpha|}\|(G(f_1,f_2)(t),\dot{G}(f_1,f_2)(t)\|_{M^\rho_{|\alpha|+2}}, \quad t\geq 0, |x|\leq t.\end{aligned}$$

Inequality (4.2) now shows that

$$\begin{aligned}&(1+|x|+t)^{3/2-\rho+|\alpha|}(|\partial^\alpha G(f_1,f_2)(t,x)| + (1+|x|+t)|\partial^\alpha\dot{G}(f_1,f_2)(t,x)|) \qquad (4.42)\\ &\quad\leq C_{|\alpha|}\sup_{s\geq 0}(\|f_1(s)\|_{D_{N'}}\|f_2(s)\|_{D_{N'}})(1+t)^{-\rho+1/2}, \quad |x|\leq t, t\geq 0,\end{aligned}$$

where N' depends only on $|\alpha|$. Inequalities (4.39), (4.40) and (4.42) with $\varepsilon = \rho - 1/2$, prove the last statement of the lemma in the case where $q = 0$ and $\chi = 0$. The cases

$q > 0$ or $\chi > 0$ are obtained by derivation of expressions (4.4a) and (4.4b) and by using the formulas (4.5) to (4.8). This proves the lemma since D_∞ a dense in D_N.

The proof of Lemma 4.1 has a useful corollary giving the decay rate of the rest term of $G_{\varepsilon,\mu}$ after substraction of the main contribution given by stationary phase development of $e^{i\varepsilon\omega(-i\partial)t}f_j(t)$, $j = 1, 2$, in (4.1). To state the result let, s being a fixed parameter,

$$\sum_{0\le l\le n} g_{j,\varepsilon,l}(s,t,x)e^{i\varepsilon\rho(t,x)m}, \quad t > 0,$$

be the stationary phase development up to order n of $e^{i\varepsilon\omega(-i\partial)t}f_j(s)$, $j = 1, 2$, $f_j(s) \in D_\infty$, $s \ge 0$, defined by (A.1a) and (A.1b). We define

$$r_{j,\varepsilon,n}(t,x) = \sum_{0\le l\le n} g_{j,\varepsilon,l}(t,t,x), \quad t > 0, \tag{4.43}$$

and

$$\begin{aligned}
&(R_{\varepsilon,\mu,n}(f_1,f_2))(t) && (4.44a)\\
&\quad = -\int_t^\infty \frac{\sin((-\Delta)^{1/2}(t-s))}{(-\Delta)^{1/2}}(r_{1,\varepsilon,n}(s))^+\gamma_0\gamma_\mu r_{2,\varepsilon,n}(s)ds, \quad t > 0,\\
&(\dot{R}_{\varepsilon,\mu,n}(f_1,f_2))(t) && (4.44b)\\
&\quad = -\int_t^\infty \cos((-\Delta)^{1/2}(t-s))(r_{1,\varepsilon,n}(s))^+\gamma_0\gamma_\mu r_{2,\varepsilon,n}(s)ds, \quad t > 0.
\end{aligned}$$

Corollary 4.2. *Let $(G, \dot{G})$ and $(R, \dot{R})$ be defined by (4.1) and (4.44) respectively and let $f_j : \mathbb{R}^+ \to D_N$, $j = 1, 2$, be C^0 functions. Let $n \ge 0$ be an integer and let α, β be multi-indices. If N, depending on n, $|\alpha|$ and $|\beta|$, is chosen sufficiently large, then*

$$\begin{aligned}
&\|x^\beta\partial^\alpha(G_{\varepsilon,\mu}(f_1,f_2) - R_{\varepsilon,\mu,n}(f_1,f_2))(t)\|_{L^2} && (4.45a)\\
&\quad + t\|x^\beta\partial^\alpha(\dot{G}_{\varepsilon,\mu}(f_1,f_2) - \dot{R}_{\varepsilon,\mu,n}(f_1,f_2))(t)\|_{L^2}\\
&\qquad \le C_{n,|\alpha|}\sup_{s\ge 0}(\|f_1(s)\|_{D_N}\|f_2(s)\|_{D_N})t^{-(n-|\beta|+1/2)}, \quad t \ge 1, n \ge |\beta|,
\end{aligned}$$

$$\begin{aligned}
&\|\partial^\alpha(G_{\varepsilon,\mu}(f_1,f_2) - R_{\varepsilon,\mu,n}(f_1,f_2))(t)\|_{L^\infty} && (4.45b)\\
&\quad + t\|\partial^\alpha(\dot{G}_{\varepsilon,\mu}(f_1,f_2) - \dot{R}_{\varepsilon,\mu,n}(f_1,f_2))(t)\|_{L^\infty}\\
&\qquad \le C_{n,|\alpha|}\sup_{s\ge 0}(\|f_1(s)\|_{D_N}\|f_2(s)\|_{D_N})t^{-(n+2+|\alpha|)}, \quad t \ge 1, n \ge 0,
\end{aligned}$$

$$\begin{aligned}
&t^{|\alpha|-|\beta|+\rho-1/2}\|x^\beta\partial^\alpha(R_{\varepsilon,n}(f_1,f_2)(t), \dot{R}_{\varepsilon,n}(f_1,f_2)(t))\|_{M^\rho} && (4.46)\\
&\quad \le C_{n,|\alpha|}\sup_{s\ge 0}(\|f_1(s)\|_{D_N}\|f_2(s)\|_{D_N}), \quad t \ge 1, |\alpha| \ge |\beta|, 1/2 < \rho < 1,
\end{aligned}$$

and

$$\begin{aligned}(1+|x|+|t|)^{3/2-\rho+|\alpha|}(|\partial^\alpha R_{\varepsilon,\mu,n}(f_1,f_2)(t,x)| \\ +(1+|x|+|t|)|\partial^\alpha \dot{R}_{\varepsilon,\mu,n}(f_1,f_2)(t,x)|) \\ \le C_{n,|\alpha|}\sup_{s\ge 0}(\|f_1(s)\|_{D_N}\|f_2(s)\|_{D_N})t^{-\rho+1/2}, \quad t\ge 1.\end{aligned} \tag{4.47}$$

Proof. Inequality (4.45a) follows from the estimate (4.20):

$$\begin{aligned}&\|\Gamma_{\gamma,\alpha}(t)-x^\gamma\partial^\alpha(r_1(t)r_2(t))\|_{L^2} \\ &\quad\le C_{n,|\alpha|}\|h_1(t)\|_{D_{N'}}\|h_2(t)\|_{D_{N'}}t^{-(1+n)}t^{|\gamma|-3/2}, \quad t>0,\end{aligned}$$

and by observing that

$$\begin{aligned}&\Big\|\int_0^\infty \frac{\sin((-\Delta)^{1/2}s)}{(-\Delta)^{1/2}}\big(\Gamma_{\gamma,\alpha}(t+s)-x^\gamma\partial^\alpha(r_1(s+t)r_2(s+t))\big)ds\Big\|_{L^2} \\ &\quad+t\Big\|\int_0^\infty \cos((-\Delta)^{1/2}s)\big(\Gamma_{\gamma,\alpha}(t+s)-x^\gamma\partial^\alpha(r_1(s+t)r_2(s+t))\big)ds\Big\|_{L^2} \\ &\qquad\le C'_{n,|\alpha|}\sup_{s\ge 0}(\|h_1(s)\|_{D_{N'}}\|h_2(s)\|_{D_{N'}})t^{-(n-|\gamma|+1/2)}, \quad t\ge 1.\end{aligned}$$

The proofs of (4.46) and (4.47) are contained in the proof of (4.2) and (4.3) because of the bound (4.14). Inequality (4.45b) follows from (4.35) with n replaced by $n+1+|\alpha|$ and by estimating all terms in $v_1^+(t)\gamma_0\gamma_\mu v_2(t)$ of order higher than n. This proves the corollary.

We can now prove an analog of Theorem 3.5 of [8] adapted to the spaces E_N. Let $t\mapsto f_j(t)$, $j=1,2$, be C^2 functions from $\mathbb{R}^+$ to D_N, let $m_1,m_2>0$, $\varepsilon_1,\varepsilon_2=\pm 1$, $M=|\varepsilon_1 m_1-\varepsilon_2 m_2|>0$, let ε be the sign of $-\varepsilon_1 m_1+\varepsilon_2 m_2$ and let $\omega_M(k)=(M^2+|k|^2)^{1/2}$, $k\in\mathbb{R}^3$. Introduce

$$(H_\mu(f_1,f_2))(t)=-\int_t^\infty \frac{\sin((-\Delta)^{1/2}(t-s))}{(-\Delta)^{1/2}}(I_\mu(f_1,f_2))(s)ds, \quad t\ge 0, \tag{4.48a}$$

$$(\dot{H}_\mu(f_1,f_2))(t)=-\int_t^\infty \cos((-\Delta)^{1/2}(t-s))(I_\mu(f_1,f_2))(s)ds, \quad t\ge 0, \tag{4.48b}$$

where

$$(I_\mu(f_1,f_2))(t)=(e^{i\varepsilon_1\omega_{m_1}(-i\partial)t}f_1(t))^+\gamma_0\gamma_\mu(e^{i\varepsilon_2\omega_{m_2}(-i\partial)t}f_2(t)). \tag{4.48c}$$

If $t\mapsto f_j(t)$, $j=1,2$, are bounded and C^∞ from $\mathbb{R}^+$ to D_∞, then it turns out that the asymptotic behaviour of H_μ is that of $e^{i\varepsilon\omega_M(-i\partial)t}h(t)$, where $k\mapsto(h(t))^\wedge(k)$ is regular outside $k=0$. This fact, which follows from the proof of the next lemma, was already proved in [8] in a different setting.

Lemma 4.3. *Let $(H, \dot{H})$ be defined by (4.48) and let $f_j : \mathbb{R}^+ \to D_N$, $j = 1, 2$, be C^3 functions. Let $n \geq 0$ be an integer. If N is chosen sufficiently large, then*

i) $$\sum_{|\alpha| \leq n} \|\partial^\alpha((H(f_1, f_2))(t), (\dot{H}(f_1, f_2))(t))\|_{M^\rho}$$
$$\leq C_{n,\rho} b_N(f_1, f_2)(t)(1+t)^{-(\rho+1/2)}, \quad -1/2 < \rho \leq 1, t \geq 0,$$

ii) $$\sum_{|\alpha| \leq n} \|\partial^\alpha |\nabla|^{\rho-1}((H(f_1, f_2))(t), (\dot{H}(f_1, f_2))(t))\|_{L^2}$$
$$\leq C_{n,\rho} b_N(f_1, f_2)(t)(1+t)^{-(\rho+1/2)}, \quad 1/2 < \rho \leq 1, t \geq 0,$$

iii) $$|((H(f_1, f_2))(t)(x)| + |((\dot{H}(f_1, f_2))(t))(x)|$$
$$\leq C b_N(f_1, f_2)(t)(1+t+|x|)^{-3}, \quad t \geq 0,$$

where

$$b_N(f_1, f_2)(t) = \prod_{j=1}^{2} \sum_{l=0}^{3} \sup_{s \geq t} \big((1+s)^l \|\frac{d^l}{ds^l} f_j(s)\|_{D_N}\big),$$

and where N depends on n.

Proof. Let $h_j \in S(\mathbb{R}^3, \mathbb{C})$, $j = 1, 2$, and let

$$\Gamma(t) = (e^{-i\varepsilon_1 \omega_{m_1}(-i\partial)t} h_1)(e^{i\varepsilon_2 \omega_{m_2}(-i\partial)t} h_2), \quad t \geq 0, \tag{4.49}$$

and let $r_j(t) = \sum_{q=0}^{n} r_j^{(q)}(t)$ be the stationary phase development of $e^{-i\varepsilon_1 \omega(-i\partial)t} h_j, j = 1, 2$ up to order n in Theorem A.1. In complete analogy with the derivation of inequality (4.20), by inequalities (4.14), (4.15), (4.18) and (4.19) we obtain for N sufficiently large, depending on $|\alpha|$ and n, and for $|\beta| \leq n$,

$$\begin{aligned} &\|x^\beta \partial^\alpha(\Gamma(t) - e^{i\varepsilon M\rho(t)} r_1(t) r_2(t))\|_{L^p} \\ &\quad \leq C_n \|h_1\|_{D_N} \|h_2\|_{D_N} t^{-(n+1+3(p-1)/p-|\beta|)}, \quad t \geq 1, 1 \leq p \leq \infty, \end{aligned} \tag{4.50}$$

where supp $r_j(t) \subset \{|x| \leq t\}$,

$$\|\rho(t)^{-l} \partial^\alpha r_j^{(q)}(t)\|_{L^2} \leq C_{n,|\alpha|,l} \|h_j\|_{D_{N+2|\alpha|+l}} t^{-|\alpha|-l-q}, \tag{4.51a}$$

and

$$\|\rho(t)^{-l} \partial^\alpha r_j^{(q)}(t)\|_{L^\infty} \leq C_{n,|\alpha|,l} \|h_j\|_{D_{N+2|\alpha|+l}} t^{-|\alpha|-l-3/2-q}, \tag{4.51b}$$

$t > 0$, $0 \leq q \leq n$, $l \geq 0$, $|\alpha| \geq 0$. The function $(t, x) \mapsto r_j^{(q)}(t, x)$ is homogeneous of degree $-3/2 - q$.

The functions $r_{(q)}$, $0 \leq q \leq n$ defined by

$$r_{(q)}(t,x) = t^{3/2+q} \sum_{q_1+q_2=q} r_1^{(q_1)}(t,x) r_2^{(q_2)}(t,x), \quad t > 0, \tag{4.52}$$

vanish outside the forward light cone and are homogeneous of degree $-3/2$. We can therefore apply Theorem A.2 which gives, using (4.51a), (4.51b) and (4.52) for $|\beta| \leq L$,

$$\begin{aligned} &\|x^\beta \partial^\alpha (e^{i\varepsilon M\rho(t)} r_{(q)}(t) - \sum_{0\leq l \leq L} t^{-l} e^{i\varepsilon\omega_M(-i\partial)t} g_{q,l})\|_{L^2} \qquad (4.53a)\\ &\qquad \leq C_{L,|\alpha|} \|h_1\|_{D_{N'}} \|h_2\|_{D_{N'}} t^{-(L+1-|\beta|)}, \quad t > 0, |\beta| \leq L,\\ &\|x^\beta \partial^\alpha (e^{i\varepsilon M\rho(t)} r_{(q)}(t) - \sum_{0\leq l \leq L} t^{-l} e^{i\varepsilon\omega_M(-i\partial)t} g_{q,l})\|_{L^\infty} \qquad (4.53b)\\ &\qquad \leq C_{L,|\alpha|} \|h_1\|_{D_{N'}} \|h_2\|_{D_{N'}} t^{-(L+5/2-|\beta|)}, \quad t > 0, |\beta| \leq L, \end{aligned}$$

and

$$\|g_{q,l}\|_{D_j} \leq C_{L,j} \|h_1\|_{D_{N'+2j}} \|h_2\|_{D_{N'+2j}}, \tag{4.54}$$

where N' depends on L and $|\alpha|$ and where $g_{q,l}$ stands for the fonctions f_l of theorem A.2. Defining

$$g'(t) = \sum_{q,l\geq 0} t^{-(3/2+q)} g_{q,l}, \quad t > 0, \tag{4.55}$$

using definition (4.52), interpolating between the L^2- and L^∞-estimate in (4.53) and using (4.54), we obtain for $2 \leq p \leq \infty$,

$$\begin{aligned} &\|x^\beta \partial^\alpha (e^{i\varepsilon M\rho(t)} r_1(t) r_2(t) - e^{i\varepsilon\omega_M(-i\partial)t} g'(t))\|_{L^p} \\ &\qquad \leq C_{L,|\alpha|} \|h_1\|_{D_{N'}} \|h_2\|_{D_{N'}} t^{-(L+5/2-|\beta|+3(p-2)/2p)}, \quad t \geq 1, \end{aligned} \tag{4.56}$$

$$\|(\frac{d}{dt})^l g'(t)\|_{D_j} \leq C_{L,j} \|h_1\|_{D_{N'+2j}} \|h_2\|_{D_{N'+2j}} t^{-3/2-l}, \quad t \geq 1, \tag{4.57}$$

where $l \geq 0$, $|\beta| \leq L$, $L \geq 0$ and where N' depends on L and $|\alpha|$.

Since $\|f\|_{L^1} \leq C \|(1+|x|)^2 f\|_{L^2}$, we obtain, with a new L and a new $C_{L,|\alpha|}$,

$$\begin{aligned} &\|x^\beta \partial^\alpha (e^{i\varepsilon M\rho(t)} r_1(t) r_2(t) - e^{i\varepsilon\omega_M(-i\partial)t} g'(t)\|_{L^p} \\ &\qquad \leq C_{L,|\alpha|} \|h_1\|_{D_{N'}} \|h_2\|_{D_{N'}} t^{-(L+5/2-|\beta|)}, \quad t \geq 1, 1 \leq p \leq \infty, |\beta| \leq L-2, \end{aligned} \tag{4.58}$$

N' depends on $|\alpha|$ and L.

According to (4.50) and (4.58) we have, choosing L and N sufficiently large,

$$\begin{aligned} &\|x^\beta \partial^\alpha (\Gamma(t) - e^{i\varepsilon\omega_M(-i\partial)t} g'(t))\|_{L^p} \\ &\qquad \leq C_n \|h_1\|_{D_N} \|h_2\|_{D_N} t^{-(n+1+3(p-1)/p-|\beta|)}, \quad t \geq 1, 1 \leq p \leq \infty. \end{aligned} \tag{4.59}$$

We define

$$g(t) = \chi(t) g'(t), \quad t \in \mathbb{R}, \tag{4.60}$$

where $\chi \in C^\infty(\mathbb{R})$, $0 \leq \chi(t) \leq 1$, $\chi(t) = 0$ for $t \leq 1$, $\chi(t) = 1$ for $t \geq 2$. It then follows from (4.57) and (4.59) that

$$\begin{aligned} &\|x^\beta \partial^\alpha (\Gamma(t) - e^{i\varepsilon\omega_M(-i\partial)t} g(t))\|_{L^p} \\ &\qquad \leq C_n \|h_1\|_{D_N} \|h_2\|_{D_N} (1+t)^{-(n+1+3(p-1)/p-|\beta|)}, \quad t \geq 0, 1 \leq p \leq \infty, \end{aligned} \tag{4.61}$$

and

$$\|\big(\frac{d}{dt}\big)^l g(t)\|_{D_j} \leq C_{n,j,l} \|h_1\|_{D_{N+2j}} \|h_2\|_{D_{N+2j}} (1+t)^{-3/2-l}, \quad t \geq 0, j \geq 0, l \geq 0, \tag{4.62}$$

where N depends on $n \geq 0$. In fact $\|\Gamma(t)\|_{L^p} \leq C \|h_1\|_{D_N} \|h_2\|_{D_N}$ for $0 \leq t \leq 2$, if N is sufficiently large.

Let $f_j \colon \mathbb{R}^+ \to D_\infty$ be C^k and let

$$B_\mu(t,s) = (e^{i\varepsilon_1\omega_{m_1}(-i\partial)t} f_1(s))^+ \gamma_0 \gamma_\mu (e^{i\varepsilon_2\omega_{m_2}(-i\partial)t} f_2(s)), \quad t, s \geq 0.$$

According to (4.48c) we have

$$(I_\mu(f_1, f_2))(t) = B_\mu(t,t), \quad t \geq 0.$$

Application of inequalities (4.61) and (4.62) for $B_\mu(t,s)$, with s fixed, proves that there is a C^k function $g \colon \mathbb{R}^+ \to D_\infty$ such that

$$\begin{aligned} &\|x^\beta \partial^\alpha ((I_\mu(f_1,f_2))(t) - e^{i\varepsilon\omega_M(-i\partial)t} g_\mu(t))\|_{L^p} \\ &\qquad \leq C_{n,\alpha} \|f_1(t)\|_{D_N} \|f_2(t)\|_{D_N} (1+t)^{-(n+1+3(p-1)/p-|\beta|)}, \quad t \geq 0, 1 \leq p \leq \infty, \end{aligned} \tag{4.63}$$

and

$$\begin{aligned} &\|\big(\frac{d}{dt}\big)^l g(t)\|_{D_j} \\ &\quad \leq C_{n,j,l} \sum_{0 \leq q \leq l} \|(1+t)^q \big(\frac{d}{dt}\big)^q f_1(t)\|_{D_N} \|(1+t)^{l-q} \big(\frac{d}{dt}\big)^{l-q} f_2(t)\|_{D_N} (1+t)^{-3/2-l}, \end{aligned} \tag{4.64}$$

$t \geq 0$, $n \geq 0$, $j \geq 0$, $q \geq l \geq 0$, where N depends on n, j, $|\alpha|$ and $|\beta| \leq n-2$.

Let

$$\Delta^{(n)}_{1,\mu}(t) = -\int_t^\infty \frac{\sin((-\Delta)^{1/2}(t-s))}{(-\Delta)^{1/2}} \big((I_\mu(f_1,f_2))(s) - e^{i\varepsilon\omega_M(-i\partial)s} g_\mu(s)\big) ds, \tag{4.65a}$$

$$\dot{\Delta}^{(n)}_{1,\mu}(t) = -\int_t^\infty \cos((-\Delta)^{1/2}(t-s)) \big((I_\mu(f_1,f_2))(s) - e^{i\varepsilon\omega_M(-i\partial)s} g_\mu(s)\big) ds, \tag{4.65b}$$

$t \geq 0$, where g satisfies (4.63) and (4.64). It follows from (4.6a) and (4.6b) that, for $-1/2 < \rho \leq 1$,

$$\sum_{|\alpha| \leq q} \|x^\beta \partial^\alpha (\Delta_1^{(n)}(t), \dot{\Delta}_1^{(n)}(t))\|_{M^\rho}$$
$$\leq \sum_{0 \leq \mu \leq 3} \sum_{|\alpha| \leq q} \int_t^\infty \||\nabla|^{\rho-1} (s^{|\beta|} + |x|^{|\beta|}) \partial^\alpha \big((I_\mu(f_1, f_2))(s) - e^{i\varepsilon\omega_M(-i\partial)s} g_\mu(s) \big)\|_{L^2} ds.$$

Since $\||\nabla|^{\rho-1} f\|_{L^2} \leq C_p \|f\|_{L^p}$, $p = 6(5-2\rho)^{-1}$, $1 < p \leq 2$, i.e. $-1/2 < \rho \leq 1$, inequality (4.63) gives

$$\sum_{|\alpha| \leq q} \|x^\beta \partial^\alpha (\Delta_1^{(n)}(t), \dot{\Delta}_1^{(n)}(t))\|_{M^\rho} \tag{4.66}$$
$$\leq C_{n,q,\rho} \sup_{s \geq 0} (\|f_1(s)\|_{D_N} \|f_2(s)\|_{D_N})(1+t)^{-n-|\beta|}, \quad n \geq 1 + |\beta|, t \geq 0, q \geq 0,$$

where N depends on n and k.

For $n \geq 1$, let

$$H_\mu^{(n)}(t) = -\int_t^\infty \frac{\sin((-\Delta)^{1/2}(t-s))}{(-\Delta)^{1/2}} e^{i\varepsilon\omega_M(-i\partial)s} g_\mu(s) ds, \tag{4.67a}$$
$$\dot{H}_\mu^{(n)}(t) = -\int_t^\infty \cos((-\Delta)^{1/2}(t-s)) e^{i\varepsilon\omega_M(-i\partial)s} g_\mu(s) ds, \quad t \geq 0, \tag{4.67b}$$

where g satisfies (4.63) and (4.64). We observe that

$$\frac{\sin(|k|(t-s))}{|k|} e^{i\varepsilon\omega_M(k)s} = \frac{\partial^l}{\partial s^l} K_l(t,s,k), \quad l \geq 0, \tag{4.68a}$$

and

$$\cos(|k|(t-s)) e^{i\varepsilon\omega_M(k)s} = \frac{\partial^l}{\partial s^l} \dot{K}_l(t,s,k), \quad l \geq 0, \tag{4.68b}$$

where

$$K_l(t,s,k) = \Big(\frac{(\varepsilon\omega_M(k) + |k|)^l - (\varepsilon\omega_M(k) - |k|)^l}{i^l 2i|k| M^{2l}} \cos(|k|(t-s)) \tag{4.69a}$$
$$+ \frac{(\varepsilon\omega_M(k) + |k|)^l + (\varepsilon\omega_M(k) - |k|)^l}{i^l 2 M^{2l}} \frac{\sin(|k|(t-s))}{|k|} \Big) e^{i\varepsilon\omega_M(k)s},$$

and

$$\dot{K}_l(t,s,k) = \frac{\partial}{\partial t} K_l(t,s,k). \tag{4.69b}$$

The functions $k \mapsto K_l(t,s,k)$ and $k \mapsto \dot{K}_l(t,s,k)$ are real analytic functions on $\mathbb{R}^3$.

Partial integration three times in (4.67) gives, using (4.68),

$$H_\mu^{(n)}(t) = K_1(t,t,-i\partial)g_\mu(t) - K_2(t,t,-i\partial)\frac{d}{dt}g_\mu(t) + \Delta_{2,\mu}^{(n)}(t), \tag{4.70a}$$

and

$$\dot{H}_\mu^{(n)}(t) = \dot{K}_1(t,t,-i\partial)g_\mu(t) - \dot{K}_2(t,t,-i\partial)\frac{d}{dt}g_\mu(t) + \dot{\Delta}_{2,\mu}^{(n)}(t), \tag{4.70b}$$

where

$$\Delta_{2,\mu}^{(n)}(t) = K_3(t,t,-i\partial)\frac{d^2}{dt^2}g_\mu(t) + \int_t^\infty K_3(t,s,-i\partial)\frac{d^3}{ds^3}g_\mu(s)ds, \tag{4.71a}$$

and

$$\dot{\Delta}_{2,\mu}^{(n)}(t) = \dot{K}_3(t,t,-i\partial)\frac{d^2}{dt^2}g_\mu(t) + \int_t^\infty \dot{K}_3(t,s,-i\partial)\frac{d^3}{ds^3}g_\mu(s)ds. \tag{4.71b}$$

According to definition (4.69) of K, we have

$$\begin{aligned} &|K_l(t,s,k)| \leq C_l(\omega_M(k))^l|k|^{-1}, \quad l \geq 0,\\ &|\dot{K}_l(t,s,k)| \leq C_l(\omega_M(k))^l, \quad l \geq 0, \end{aligned}$$

and

$$\begin{aligned} &|K_l(t,t,k)| \leq lC_l(\omega_M(k))^{l-1}, \quad l \geq 0,\\ &|\dot{K}_l(t,t,k)| \leq C_l(\omega_M(k))^l, \quad l \geq 0, \end{aligned}$$

$t, s \in \mathbb{R}^+, k \in \mathbb{R}^3$.

These inequalities for K, similar inequalities for the derivatives in the third argument of K and (4.71) give, for $k \in \mathbb{N}$,

$$\begin{aligned} &\sum_{|\alpha|\leq k} \|x^\beta \partial^\alpha(\Delta_2^{(n)}(t), \dot{\Delta}_2^{(n)}(t))\|_{M^\rho}\\ &\quad \leq C_k \sum_{|\alpha|\leq k+3} \||x|^{|\beta|}\partial^\alpha|\nabla|^{\rho-1}\frac{d^2}{dt^2}g(t)\|_{L^2}\\ &\qquad + C_k \sum_{|\alpha|\leq k+3} \int_t^\infty \||x|^{|\beta|}\partial^\alpha|\nabla|^{\rho-1}\frac{d^3}{ds^3}g(s)\|_{L^2}, \quad |\beta| \leq 3. \end{aligned}$$

Since $\||\nabla|^{\rho-1}f\|_{L^2} \leq C_p\|f\|_{L^p}$, $p = 6(5-2\rho)^{-1}$, $1 < p \leq 2$, and since $\|f\|_{L^p} \leq C'\|(1+|x|)^2 f\|_{L^2}$ for $1 \leq p \leq 2$, inequality (4.64) now gives for $-1/2 < \rho \leq 1$ and $|\beta| \leq 3$

$$\begin{aligned} &\sum_{|\alpha|\leq k} \|x^\beta \partial^\alpha(\Delta_2^{(n)}(t), \dot{\Delta}_2^{(n)}(t))\|_{M^\rho} &(4.72)\\ &\quad \leq C_{n,k,\rho}\prod_{j=1}^{2}\sum_{l=0}^{3}\sup_{s\geq t}\big(\|(1+s)^l\frac{d^l}{ds^l}f_j(s)\|_{D_N}\big)(1+t)^{-7/2+|\beta|}, \quad t \geq 0, \end{aligned}$$

where N depends on n and k.

Let

$$\begin{aligned}\Delta^{(n)}_\mu &= \Delta^{(n)}_{1,\mu} + \Delta^{(n)}_{2,\mu}, \\ \dot\Delta^{(n)}_\mu &= \dot\Delta^{(n)}_{1,\mu} + \dot\Delta^{(n)}_{2,\mu}.\end{aligned} \tag{4.73}$$

For $n \geq 4$ it follows from inequalities (4.66) and (4.72) that

$$\begin{aligned}&\sum_{|\alpha|\leq k} \|x^\beta \partial^\alpha (\Delta^{(n)}(t), \dot\Delta^{(n)}(t))\|_{M^\rho} \\ &\qquad \leq C_{n,k,\rho} \prod_{j=1}^{2} \sum_{l=0}^{3} \sup_{s\geq t} \big((1+s)^l \|\frac{d^l}{ds^l} f_j(s)\|_{D_N}\big)(1+t)^{-7/2+|\beta|},\end{aligned} \tag{4.74}$$

where $t \geq 0$, $-1/2 < \rho \leq 1$, $n \geq 4$ and N depends on n and k.

It follows from definition (4.48) of $(H, \dot H)$, (4.65) of $(\Delta_1^{(n)}, \dot\Delta_1^{(n)})$, (4.67) of $(H^{(n)}, \dot H^{(n)})$, (4.71) of $(\Delta_2^{(n)}, \dot\Delta_2^{(n)})$, (4.73) of $(\Delta^{(n)}, \dot\Delta^{(n)})$ and from expression (4.70) that

$$H_\mu(t) = \Delta^{(n)}_\mu(t) + K_1(t,t,-i\partial) g_\mu(t) - K_2(t,t,-i\partial)\frac{d}{dt} g_\mu(t),$$

and

$$\dot H_\mu(t) = \dot\Delta^{(n)}_\mu(t) + \dot K_1(t,t,-i\partial) g_\mu(t) - \dot K_2(t,t,-i\partial)\frac{d}{dt} g_\mu(t).$$

It now follows from the explicit expression (4.69) of K and $\dot K$, that

$$H_\mu(t) = \Delta^{(n)}_\mu(t) - M^{-2} e^{i\varepsilon\omega_M(-i\partial)t} g_\mu(t) - 2i\varepsilon\omega_M(-i\partial) M^{-4} e^{i\varepsilon\omega_M(-i\partial)t}\frac{d}{dt} g_\mu(t), \tag{4.75a}$$

$$\begin{aligned}\dot H_\mu(t) = \dot\Delta^{(n)}_\mu(t) &- i\varepsilon\omega_M(-i\partial) M^{-2} e^{i\varepsilon\omega_M(-i\partial)t} g_\mu(t) \\ &- (2(\omega_M(-i\partial))^2 - M^2) M^{-4} e^{i\varepsilon\omega_M(-i\partial)t}\frac{d}{dt} g_\mu(t).\end{aligned} \tag{4.75b}$$

Inequality (4.64) and equality (4.75) give,

$$\begin{aligned}&\sum_{\|\alpha|\leq k} \|\partial^\alpha (H(t) - \Delta^{(n)}(t), 0)\|_{M^\rho} \\ &\qquad \leq C_{n,k,\rho} \prod_{j=1}^{2} \sum_{l=0}^{1} \sup_{s\geq 0} \big(\|\frac{d^l}{ds^l} f_j(s)\|_{D_N}\big)(1+t)^{-3/2}, \quad t \geq 0, k \geq 0,\end{aligned} \tag{4.76}$$

where $0 \leq \rho \leq 1$, $n \geq 4$, and N depends on k and n.

We get from (4.75) and $\||\nabla|^{\rho-1} f\|_{L^2} \leq C_p \|f\|_{L^p}$, $p = 6(5-2\rho)^{-1}$, $1 < p \leq 2$,

$$\begin{aligned}&\sum_{|\alpha|\leq k} |\partial^\alpha |\nabla|^{\rho-1} (H(t) - \Delta^{(n)}(t), \dot H(t) - \dot\Delta^{(n)}(t))\|_{L^2} \\ &\qquad \leq C_\rho \sum_{|\alpha|\leq k+2} \big(\|\partial^\alpha e^{i\varepsilon\omega_M(-i\partial)t} g(t)\|_{L^p} + \|\partial^\alpha e^{i\varepsilon\omega_M(-i\partial)t}\frac{d}{dt} g(t)\|_{L^p}\big), \quad -1/2 < \rho \leq 1.\end{aligned}$$

It follows using for example Corollary 2.2 of [12], that

$$\|e^{i\varepsilon\omega_M(-i\partial)t}f\|_{L^p} \le C_p\|f\|_{D_J}(1+|t|)^{3/p-3/2},$$

for $1 \le p \le 2$ for some $J > 0$. Since $3/p - 3/2 = 1-\rho$ we obtain, using (4.64),

$$\begin{aligned}\sum_{|\alpha|\le k} &\|\partial^\alpha|\nabla|^{\rho-1}(H(t)-\Delta^{(n)}(t),\dot H(t)-\dot\Delta^{(n)}(t))\|_{L^2} \\ &\le C_{n,k,\rho}\prod_{j=1}^{2}\sum_{l=0}^{1}\sup_{s\ge 0}(\|\frac{d^l}{ds^l}f_j(s)\|_{D_N})(1+t)^{-(\rho+1/2)},\end{aligned}\tag{4.77}$$

for $t \ge 0$, $-1/2 < \rho \le 1$, $n \ge 4$, $k \ge 0$, where N depends on k and n.

Using that $|(e^{i\varepsilon\omega_M(-i\partial)t}f)(t,x)| \le C(1+t+|x|)^{-3/2}\|f\|_{D_J}$, for some $J \ge 0$, we obtain from inequality (4.64) and equality (4.75) that

$$\begin{aligned}&|(H(t)-\Delta^{(n)}(t),\dot H(t)-\dot\Delta^{(n)}(t))(x)| \\ &\le C_n\prod_{j=1}^{2}\sum_{l=0}^{1}\sup_{s\ge t}\big(\|\frac{d^l}{ds^l}f_j(s)\|_{D_{N_0}}\big)(1+t+|x|)^{-3}, \quad t\ge 0, n\ge 4\end{aligned}\tag{4.78}$$

for some N_0.

When $f_j\colon \mathbb{R}^+ \to D_\infty$ is a C^3 function, statement ii) of the lemma now follows from inequalities (4.74) and (4.77) with $n = 4$. In fact, for $1/2 < \rho \le 1$:

$$\begin{aligned}\|\partial^\alpha|\nabla|^{\rho-1}(H(t),\dot H(t))\|_{L^2} &\le \|\partial^\alpha(\Delta^{(4)}(t),0)\|_{M^{\rho-1}} + \|\partial^\alpha(0,\dot\Delta^{(4)}(t))\|_{M^\rho} \\ &\quad + \|\partial^\alpha|\nabla|^{\rho-1}(H(t)-\Delta^{(4)}(t),\dot H(t)-\dot\Delta^{(4)}(t))\|_{L^2},\end{aligned}$$

where the first two terms can be estimated by (4.74) since $-1/2 < \rho - 1 \le 1$, and the last term by (4.77). Statement i) follows from inequalities (4.74), (4.76) and (4.77) for $0 \le \rho \le 1$. For $-1/2 < \rho < 0$, it follows from (4.74) and (4.77), since in this case,

$$\begin{aligned}\|\partial^\alpha(H(t),\dot H(t))\|_{M^\rho} &\le \|\partial^\alpha(\Delta^{(4)}(t),\dot\Delta^{(4)}(t))\|_{M^\rho} + \|\partial^\alpha|\nabla|^{\rho}(H(t)-\Delta^{(4)}(t))\|_{L^2} \\ &\quad + \|\partial^\alpha|\nabla|^{\rho-1}(\dot H(t)-\dot\Delta^{(4)}(t))\|_{L^2}\end{aligned}$$

and, since $-1/2 < \rho < 0$ and $0 < \rho + 1 < 1$, (4.77) can be applied to the last two terms. Statement iii) of the lemma follows from (4.74) and (4.78), since

$$\begin{aligned}&\|\nu(t)^3(H(t),\dot H(t))\|_{L^\infty} \\ &\quad\le \|\nu(t)^3(H(t)-\Delta^{(4)}(t),\dot H(t)-\dot\Delta^{(4)}(t))\|_{L^\infty} \\ &\qquad + C\sum_{|\alpha|\le 2}\|\partial^\alpha\nu(t)^3(\Delta^{(4)}(t),\dot\Delta^{(4)}(t))\|_{L^2}, \quad (\nu(t))(x) = (1+t+|x|^2)^{1/2},\end{aligned}$$

where we have used $\|f\|_{L^\infty} \le C\|f\|_{W^{2,2}}$. Finally it follows by continuous extension that statements i), ii) and iii) are valid for C^3 functions $f_j\colon \mathbb{R}^+ \to D_N$ having finite b_N. This ends the proof of the lemma.

We remark that statement ii) of the lemma is still true for $-1/2 < \rho \leq 1$, which follows by a slight change in the proof. However this result will not be used in this article.

To prove *the existence of a modified wave operator* for $t \to +\infty$, we first use an asymptotic condition slightly different from that defined by $s_\varepsilon^{(+)}$ in (1.17c), where $s_\varepsilon^{(+)}$ is given by (1.18). We introduce

$$\Theta'(t)u = \Big(f, \dot{f}, \sum_{\varepsilon=\pm}(\varepsilon\omega(-i\partial)t + s'_\varepsilon(t,-i\partial))P_\varepsilon(-i\partial)\alpha\Big), \quad u = (f,\dot{f},\alpha) \in E_N, \tag{4.79}$$

for some N sufficiently large, where

$$s'_\varepsilon(t,k) = -\vartheta(A', t, -\varepsilon kt/\omega(k)), \quad t \geq 0, \tag{4.80}$$

ϑ is given by (1.19) and A' being an electromagnetic potential which we shall determine later.

To estimate ϑ, we introduce the following representation of the Poincaré Lie algebra $\mathfrak{p}$ on functions of (t,x):

$$\xi_{P_0} = \frac{\partial}{\partial t}, \quad \xi_{P_i} = \frac{\partial}{\partial x_i}, \quad 1 \leq i \leq 3, \tag{4.81a}$$

$$\xi_{M_{0i}} = x_i\frac{\partial}{\partial t} + t\frac{\partial}{\partial x_i}, \quad 1 \leq i \leq 3, \tag{4.81b}$$

$$\xi_{M_{ij}} = -x_i\frac{\partial}{\partial x_j} + x_j\frac{\partial}{\partial x_i}, \quad 1 \leq i < j \leq 3. \tag{4.81c}$$

We also define the representation ξ^D (resp. ξ^M) on the space of Dirac fields (resp. vector fields) by

$$\xi^D_{P_\mu} = \xi_{P_\mu}, \qquad \xi^M_{P_\mu} = \xi_{P_\mu}, \quad 0 \leq \mu \leq 3 \tag{4.81d}$$

$$\xi^D_{M_{\mu\nu}} = \xi_{M_{\mu\nu}} + \sigma_{\mu\nu}, \quad \xi^M_{M_{\mu\nu}} = \xi_{M_{\mu\nu}} + n_{\mu\nu}, \quad 0 \leq \mu < \nu \leq 3. \tag{4.81e}$$

Lemma 4.4. *Let $A'\colon \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^4$ be a C^k function, $k \geq 0$, and let $1/2 < \rho \leq 3/2$, then*

i) $|\vartheta(A',t,x)| \leq ((1+t+|x|)^{\rho-1/2} - 1)C(\rho-1/2)^{-1}\sup_{s,y}((1+s+|y|)^{3/2-\rho}|A'(s,y)|)$

ii) $$|\frac{\partial}{\partial t}\vartheta(A',t,x)| + \sum_{i=1}^{3}|\frac{\partial}{\partial x_i}\vartheta(A',t,x)|$$
$$\leq C(1+t+|x|)^{\rho-3/2}(\rho-1/2)^{-1}$$
$$\sum_{0\leq\nu\leq3}\sup_{s,y}\Big((1+s+|y|)^{3/2-\rho}\big(|A'_\nu(s,y)| + |A''_\nu(s,y)| + \sum_{0\leq\mu\leq3}|\xi_{M_{\mu\nu}}A'_\mu(s,y)|\big)\Big),$$

where

$$A_0''(t,x) = \Big(\frac{\partial}{\partial t}A_0'(t,x) - \sum_{1\le i\le 3}\partial_i A_i'(t,x)\Big)t,$$

$$A_j''(t,x) = \Big(\frac{\partial}{\partial t}A_0'(t,x) - \sum_{1\le i\le 3}\partial_i A_i'(t,x)\Big)(-x_j), \quad 1\le j\le 3 \textit{ and } k\ge 1,$$

iii)
$$\sum_{|\alpha|+l=n+n'} |\frac{\partial^l}{\partial t^l}\partial^\alpha \vartheta(A',t,x)|$$
$$\le C(\rho-1/2)^{-1}(1+t+|x|)^{-1}(1+\big|t-|x|\big|)^{-n+\rho+1/2}$$
$$\sum_{0\le\nu\le 3}\sum_{|\beta|+r=n-1+n'}\sup_{s,y}\Big((1+s+|y|)(1+\big|s-|y|\big|)^{n-\rho-1/2}$$
$$(|\frac{\partial^r}{\partial s^r}\partial^\beta A_\nu'(s,y)| + |\frac{\partial^r}{\partial s^r}\partial^\beta A_\nu''(s,y)| + \sum_{0\le\mu\le 3}|\frac{\partial^r}{\partial s^r}\partial^\beta \xi_{M_{\mu\nu}}A_\mu'(s,y)|)\Big),$$

if $k\ge n\ge 2$, $n'\ge 0$.

In the three cases the supremum is taken over $(s,y)\in\mathbb{R}^+\times\mathbb{R}^3$.

Proof. Statement i) follows directly from

$$|\vartheta(A',t,x)| \le \int_0^1 \Big(|t||A_0(s(t,x))| + |x|\sum_{i=1}^3 |A_i(s(t,x))|\Big)ds$$
$$\le \sum_{0\le\mu\le 3}\sup_{(s,y)\in\mathbb{R}^+\times\mathbb{R}^3}\Big((1+s+|y|)^{3/2-\rho}|A_\mu(s,y)|\int_0^1 \frac{t+|x|}{(1+s(t+|x|))^{3/2-\rho}}ds\Big),$$

where $1/2<\rho\le 3/2$. To prove statement ii) we use covariant notation and summation convention over repeated contravariant and covariant indices. Let $y^0=t$, $y^i=x_i$, $1\le i\le 3$. We note that using the gauge condition we obtain

$$\xi_{P_\nu} y_\mu A'^\mu(sy) = (\xi_{M_{\mu\nu}}A'^\mu)(sy) + A_\nu'(sy) + A_\nu''(sy), \tag{4.82}$$

where $A_\nu''(z) = z_\nu \frac{\partial}{\partial z^\mu}A'^\mu(z)$. This shows that (not using summation conventions)

$$|\frac{\partial}{\partial t}\vartheta(A',t,x)| + \sum_{i=1}^3 |\frac{\partial}{\partial x_i}\vartheta(A',t,x)|$$
$$\le \sum_{0\le\nu\le 3}\sup_{y\in\mathbb{R}^+\times\mathbb{R}^3}\Big((1+|\vec{y}|+y^0)^{3/2-\rho}\big(|A_\nu'(y)| + |A_\nu''(sy)| + \sum_{0\le\mu\le 3}|\xi_{M_{\mu\nu}}A_\mu'(y)|\big)\Big)$$
$$\int_0^1 (1+s(t+|x|)^{\rho-3/2}ds, \quad y=(y^0,\vec{y}).$$

Since the last integral is bounded by $C(\rho-1/2)^{-1}(1+t+|x|)^{\rho-3/2}$, $\rho>1/2$, this proves statement ii).

To prove statement iii) let $|\alpha|+l=n+n'=N$, $n\geq 2$. It follows from equality (4.82) and definition (4.81a) of ξ_{P_μ} that

$$\begin{aligned}&\xi_{P_{\nu_1}}\cdots\xi_{P_{\nu_N}}y_\mu A'^{\mu}(sy) \\ &\quad = s^{N-1}\big(\xi_{P_{\nu_1}}\cdots\xi_{P_{\nu_{N-1}}}\xi_{M_{\mu\nu_N}}A'^{\mu}+\xi_{P_{\nu_1}}A'_{\nu_N}+\xi_{P_{\nu_1}}\cdots\xi_{P_{\nu_{N-1}}}A''_{\nu_N}\big)(sy),\end{aligned} \tag{4.83}$$

which gives

$$\sum_{|\alpha|+l=N}\Big|\frac{\partial^l}{\partial t^l}\partial^\alpha\vartheta(A',t,x)\Big|\leq Q_{n,N}(A')I_n(t,x), \tag{4.84}$$

where

$$I_n(t,x)=\int_0^1 s^{n-1}(1+s(t+|x|))^{-1}(1+s\big|t-|x|\big|)^{\rho-n+1/2}ds$$

and

$$\begin{aligned}Q_{n,N}(A')=\sum_{0\leq\nu\leq 3}\sup_{y\in\mathbb{R}^+\times\mathbb{R}^3}&\Big((1+y^0+|\vec{y}|)(1+\big|y^0-|\vec{y}|\big|)^{n-1/2-\rho}\\ &\big(|\nabla^{N-1}A'_\nu(y)|+|\nabla^{N-1}A''_\nu(y)|+\sum_{0\leq\mu\leq 3}|\nabla^{N-1}(\xi_{M_{\mu\nu}}A'_\mu)(y)|\big)\Big),\end{aligned}$$

$$\nabla=\Big(\frac{\partial}{\partial y^0},\frac{\partial}{\partial y^1},\frac{\partial}{\partial y^2},\frac{\partial}{\partial y^3}\Big),\quad y=(y^0,\vec{y}).$$

To estimate the integral $I_n(t,x)$, $n\geq 2$, $(t,x)\in\mathbb{R}^+\times\mathbb{R}^3$, we observe that

$$I_n(t,x)\leq 2^{n+1/2-\rho}\int_0^1 s^{n-1}(1+s(1+t+|x|))^{-1}(1+s(1+\big|t-|x|\big|))^{\rho-n+1/2}ds.$$

This gives, for $n\geq 2$ and $1/2<\rho\leq 3/2$,

$$\begin{aligned}I_n(t,x)&\leq 2^{n+1/2-\rho}(1+t+|x|)^{-1}(1+\big|t-|x|\big|)^{\rho+1/2-n}\\ &\qquad\int_0^1\frac{s(1+t+|x|)}{1+s(1+t+|x|)}\Big(\frac{s(1+\big|t-|x|\big|)}{1+s(1+\big|t-|x|\big|)}\Big)^{n-1/2-\rho}s^{\rho-3/2}ds\\ &\leq(\rho-1/2)^{-1}2^{n+1/2-\rho}(1+t+|x|)^{-1}(1+\big|t-|x|\big|)^{\rho+1/2-n},\end{aligned}$$

which together with (4.84) proves statement iii) of the lemma.

There is an analog of Lemma 4.4 with L^2-estimates:

Lemma 4.5. *Let $\xi^M_Y A'\in C^0(\mathbb{R}^+,M^\rho)$ for $Y\in\Pi'$, let A'' be defined as in Lemma 4.4, let $B_\mu=-\partial_\mu\vartheta(A')$, $0\leq\mu\leq 3$, let $Z\in\Pi'\cap U(\mathbb{R}^4)$, let $L\in U(\mathfrak{sl}(2,\mathbb{C}))$ and let $F_\mu(y)=\int_0^1 A_\mu(sy)ds$, $y\in\mathbb{R}^+\times\mathbb{R}^3$.*

i) *If* $1/2 < \rho \leq 1$, *then*

$$\begin{aligned}
&\|(\xi^M_{ZL}B, \xi^M_{P_0ZL}B)(t)\|_{M^\rho} \\
&\quad \leq C(1+t)^{-a} \sup_{0\leq s\leq t} \Big((1+s)^b(\|(\xi^M_{ZL}A', \xi^M_{P_0ZL}A')(s)\|_{M^\rho} \\
&\qquad + \|(\xi^M_{ZL}A'', \xi^M_{P_0ZL}A'')(s)\|_{M^\rho} + \sum_{X\in\Pi\cap\mathfrak{sl}(2,\mathbb{C})} \|(\xi^M_{ZXL}A', \xi^M_{P_0ZXL}A')(s)\|_{M^\rho}\Big),
\end{aligned}$$

where $a = |Z| + \rho - 1/2$ *if* $b > |Z| + \rho - 1/2$, $a = b$ *if* $b < |Z| + \rho - 1/2$ *and* $b \in \mathbb{R}$. *The constant* C *depends only on* ρ, a, b.

ii) *If* $0 \leq \rho < 1$, *then*

$$\begin{aligned}
&\|(1+t+|\cdot|)^{-1}(\xi^M_{ZY}F)(t)\|_{L^2(\mathbb{R}^3,\mathbb{R}^4)} + \|(\xi^M_{P_\mu ZY}F)(t)\|_{L^2(\mathbb{R}^3,\mathbb{R}^4)} \\
&\quad \leq C\Big((1+t)^{-a_1} \sup_{0\leq s\leq t} \big((1+s)^{b_1}\||\nabla|^\rho(\xi^M_{ZY}A')(s)\|_{L^2}\big) \\
&\qquad + (1+t)^{-a_2} \sup_{0\leq s\leq t} \big((1+s)^{b_2}\|(\xi^M_{ZY}A', \xi^M_{P_0ZY}A')(s)\|_{M^1}\big)\Big), \quad Y \in \Pi', 0 \leq \mu \leq 3, t \geq 0,
\end{aligned}$$

for each $\tau \in]0,1[$, *such that* $|Z| + 1/2 - (1-\rho)\tau > 0$, *where*

$$\tau a_1 + (1-\tau)a_2 = |Z| + 1/2,$$

if $\tau b_1 + (1-\tau)b_2 > |Z| + 1/2 - (1-\rho)\tau$,

$$\tau a_1 + (1-\tau)a_2 = (1-\rho)\tau + \tau b_1 + (1-\tau)b_2,$$

if $\tau b_1 + (1-\tau)b_2 < |Z| + 1/2 - (1-\rho)\tau$, *and where* $a_2 = |Z| + 1/2$ *if* $b_2 > |Z| + 1/2$ *and* $a_2 = b_2$ *if* $b_2 < |Z| + 1/2$. *The constant* C *depends only on* a_1, a_2, b_1, b_2, ρ.

Proof. Since $(\xi^M_L B)_\mu = -\partial_\mu \vartheta(\xi^M_L A')$, $0 \leq \mu \leq 3$, for $L \in U(\mathfrak{sl}(2,\mathbb{C}))$, and since

$$|(\xi^M_{Z'Y}F)(y)| \leq \int_0^1 s^{|Z'|}|(\xi^M_{Z'Y}A')(sy)|ds, \quad Z' \in U(\mathbb{R}^4), y \in \mathbb{R}^+ \times \mathbb{R}^3,$$

it is enough to consider the case where $L = Y = \mathbb{I}$, the identity element in $U(\mathfrak{p})$.

It follows from equality (4.83) with $N = |Z| + 1$, since $\xi^M_Y = \xi_Y$ for $Y \in U(\mathbb{R}^4)$, that

$$\begin{aligned}
&\big|(|\nabla|^\rho(\xi^M_Z B)_\mu(t))(x)\big| \\
&\quad \leq \int_0^1 s^{\rho+|Z|}\Big(\sum_{0\leq\alpha\leq 3} \big|(|\nabla|^\rho \xi_{ZM_{\alpha\mu}}A'_\alpha)(st,sx)\big| \\
&\qquad + \big|(|\nabla|^\rho \xi_Z A'_\mu)(st,sx)\big| + \big|(|\nabla|^\rho \xi_Z A''_\mu)(st,sx)\big|\Big)ds
\end{aligned} \tag{4.85a}$$

and

$$\begin{aligned}&\big|\big(|\nabla|^{\rho-1}(\xi^M_{P_0Z}B)_\mu(t)\big)(x)\big| \\ &\quad\le \int_0^1 s^{\rho+|Z|}\Big(\sum_{0\le\alpha\le3}\big|(|\nabla|^{\rho-1}\xi_{P_0ZM_{\alpha\mu}}A'_\alpha)(st,sx)\big| \\ &\qquad + \big|(|\nabla|^{\rho-1}\xi_{P_0}A'_\mu)(st,sx)\big| + \big|(|\nabla|^{\rho-1}\xi_{P_0Z}A''_\mu)(st,sx)\big|\Big)ds.\end{aligned} \tag{4.85b}$$

We prove next the following result. If $1\le p\le\infty$, $f\in C(\mathbb{R}^+,L^p(\mathbb{R}^3))$

$$(g_t(s))(x)=s^\varepsilon(f(st))(sx),\quad \varepsilon>3/p-1, \tag{4.86a}$$

where $t,s\in\mathbb{R}^+$, $x\in\mathbb{R}^3$, then

$$\Big\|\int_0^1 g_t(s)ds\Big\|_{L^p}\le C(1+t)^{3/p-\varepsilon-1}\sup_{0\le s\le t}(1+s)^\lambda\|f(s)\|_{L^p}, \tag{4.86b}$$

for $\lambda>\varepsilon-3/p+1$, and

$$\Big\|\int_0^1 g_t(s)ds\Big\|_{L^p}\le C(1+t)^{-\lambda}\sup_{0\le s\le t}(1+s)^\lambda\|f(s)\|_{L^p}, \tag{4.86c}$$

for $\lambda<\varepsilon-3/p+1$. C is a constant depending on p,λ and ε. In fact, since

$$\Big\|\int_0^1 g_t(s)ds\Big\|_{L^p}\le\int_0^1\|g_t(s)\|_{L^p}ds$$

and

$$\|g_t(s)\|_{L^p}=s^{\varepsilon-3/p}\|f(st)\|_{L^p},$$

it follows that

$$\Big\|\int_0^1 g_t(s)ds\Big\|_{L^p}\le\int_0^1 s^{\varepsilon-3/p}(1+st)^{-\lambda}ds\sup_{0\le s'\le t}(1+s')^\lambda\|f(s')\|_{L^p}.$$

Since $\varepsilon-3/p>-1$, the integral in this expression exists and is bounded by $C(1+t)^{-\varepsilon+3/p-1}$ if $\varepsilon-3/p-\lambda<-1$ and by $C(1+t)^{-\lambda}$ if $\varepsilon-3/p-\lambda>-1$, which is seen by making the substitution $s'=st$. This proves estimates (4.86b) and (4.86c).

The inequality in statement i) of the lemma follows by applying (4.86a) and (4.86b), with $p=2$ and $\varepsilon=\rho+|Z|$, to the integrands in (4.85a) and (4.85b), since $\varepsilon>-1/2$.

To prove statement ii), we note that

$$\begin{aligned}&|(1+t+|x|)^{-1}(\xi_ZF)(t,x)|+|(\xi_{P_\mu Z}F)(t,x)| \\ &\quad\le(1+t+|x|)^{-1}(v(t))(x)+(u(t))(x),\quad t\in\mathbb{R}^+,x\in\mathbb{R}^3,\end{aligned}$$

where

$$(v(t))(x)=\int_0^1 s^{|Z|}|(\xi_ZA')(st,sx)|ds$$

and

$$(u(t))(x) = \sum_{\substack{Z' \in \Pi' \cap U(\mathbb{R}^4) \\ |Z'|=|Z|+1}} \int_0^1 s^{|Z|+1} |(\xi_{Z'} A')(st, sx)| ds.$$

Let $0 < \tau \leq 1$ and let $q^{-1} = \tau p^{-1} + (1-\tau)/6$, where $p = 6/(3-2\rho)$, $0 \leq \rho < 1$. Then $p \leq q < 6$ and $\|(1+t+|\cdot|)^{-1}\|_{L^{q'}} \leq C_q (1+t)^{3/q'-1}$, where $q' = 2q/(q-2) > 3$. Hölder inequality gives

$$\|(1+t+|\cdot|)^{-1} v(t)\|_{L^2} \leq C_q (1+t)^{1/2-3/q} \|v(t)\|_{L^q}.$$

Estimating $\|v(t)\|_{L^q}$ by inequalities (4.86b) and (4.86c), we obtain

$$\|(1+t+|\cdot|)^{-1} v(t)\|_{L^2} \leq C(1+t)^{1/2-3/q-\lambda'} \sup_{0 \leq s \leq t} (1+s)^{\lambda} \|f(s)\|_{L^q},$$

where q is chosen such that $|Z| > 3/q - 1$ and where $\lambda' = \lambda$ if $\lambda < |Z| - 3/q + 1$ and $\lambda' = |Z| - 3/q + 1$ for $\lambda > |Z| - 3/q + 1$. It follows, by the choice of q and τ that $\|f(s)\|_{L^q} \leq \|f(s)\|_{L^p}^{\tau} \|f(s)\|_{L^6}^{1-\tau}$, which together with the Sobolev inequality (2.61) give

$$\begin{aligned} &\|(1+t+|\cdot|)^{-1} v(t)\|_{L^2} \\ &\quad \leq C'(1+t)^{1/2-3/q-\lambda'} \sup_{0 \leq s \leq t} \big((1+s)^{\lambda} \||\nabla|^{\rho} (\xi_Z A')(s)\|_{L^2}^{\tau} \||\nabla|(\xi_Z A')(s)\|_{L^2}^{1-\tau}\big). \end{aligned}$$

Introduce $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that $\lambda' + 3/q - 1/2 = \tau a_1 + (1-\tau) a_2$ and $\lambda = \tau b_1 + (1-\tau) b_2$. Substitution of $1/q = \tau/p + (1-\tau)/6$, $1/p = (3-2\rho)/6$ and the relation between λ and λ', give that

$$\tau a_1 + (1-\tau) a_2 = |Z| + 1/2,$$

if $\tau b_1 + (1-\tau) b_2 > |Z| + 1/2 - (1-\rho)\tau$, and

$$\tau a_1 + (1-\tau) a_2 = (1-\rho)\tau + \tau b_1 + (1-\tau) b_2,$$

if $\tau b_1 + (1-\tau) b_2 < |Z| + 1/2 - (1-\rho)\tau$. Moreover $\tau \in]0,1]$ and $|Z| + 1/2 - (1-\rho)\tau > 0$. By the definition of a_1, a_2, b_1, b_2 we obtain that

$$\begin{aligned} \|(1+t+|\cdot|)^{-1} v(t)\|_{L^2} \leq C' \big((1+t)^{-a_1} &\sup_{0 \leq s \leq t} (1+s)^{b_1} \||\nabla|^{\rho} (\xi_Z A')(s)\|_{L^2}\big)^{\tau} \\ &\big((1+t)^{-a_2} \sup_{0 \leq s \leq t} (1+s)^{b_2} \||\nabla|(\xi_Z A')(s)\|_{L^2}\big)^{1-\tau}. \end{aligned}$$

Using that $r_1^{\tau} r_2^{1-\tau} \leq \tau r_1 + (1-\tau) r_2$, where $r_1, r_2 \geq 0$ we obtain that $\|(1+t+|\cdot|)^{-1} v(t)\|_{L^2}$ is bounded by the right-hand side of the inequality in statement ii) of the lemma. This is also the case for $\|u(t)\|_{L^2}$, which is seen by estimating the terms in the sum defining $u(t)$ by inequalities (4.86b) and (4.86c), with $\varepsilon = |Z| + 1$, $\lambda = b_2$ and $\lambda' = a_2$. This is possible since $\tau > 0$, so a_1 and b_1 are determined by the above conditions when τ is fixed. We note

that the condition $\varepsilon > 3/p-1$ is satisfied for $p=2$ and that $a_2=b_2$ if $b_2<|Z|+1/2$ and that $a_2=|Z|+1/2$ if $b_2>|Z|+1/2$. This proves the lemma.

In the next corollary, we define

$$\begin{aligned} A'_\mu(t) = &\cos((-\Delta)^{1/2}t)f_\mu + (-\Delta)^{-1/2}\sin((-\nabla)^{1/2}t)\dot f_\mu \\ &+ \sum_{\varepsilon=\pm}(G_{\varepsilon,\mu}(\beta_1^{(\varepsilon)},\beta_2^{(\varepsilon)}))(t), \quad 0\le\mu\le 3, \end{aligned} \tag{4.87a}$$

where $(f,\dot f)\in M_N^\rho$ and $\beta_j^{(\varepsilon)}\in C^0(\mathbb{R}^+,D_N)$, $\varepsilon=\pm$, $j=1,2$, for some N sufficiently large and where $G_{\varepsilon,\mu}$ is defined by (4.1a). We suppose that the free field part of A'_μ satisfies the Lorentz gauge condition, i.e.

$$\dot f_0 - \sum_{1\le i\le 3}\partial_i f_i = 0, \quad \Delta f_0 - \sum_{1\le i\le 3}\partial_i \dot f_i = 0. \tag{4.87b}$$

Corollary 4.6. *Let $1/2<\rho<1$, let $(f,\dot f)\in M_N^{\circ\rho}$ and let $\beta_j^{(\varepsilon)}\in C^\infty(\mathbb{R}^+,D_N)$. If A' is defined by (4.87a), B defined as in Lemma 4.5 and N is sufficiently large, then*

i) $\|(B(t),\dot B(t))\|_{M^\rho}$
$$\le C\|(f,\dot f)\|_{M_1^\rho} + C_\rho\sum_{\varepsilon=\pm}\sup_{0\le s}\big(\|\beta_1^{(\varepsilon)}(s)\|_{D_N}\|\beta_2^{(\varepsilon)}(s)\|_{D_N}\big),$$
$t\ge 0$,

ii) $|\vartheta(A',t,x)|$
$$\le ((1+t+|x|)^{\rho-1/2}-1)C_\rho\Big(\|(f,\dot f)\|_{M_2^\rho} + \sum_{\varepsilon=\pm}\sup_{0\le s}\big(\|\beta_1^{(\varepsilon)}(s)\|_{D_N}\|\beta_2^{(\varepsilon)}(s)\|_{D_N}\big)\Big),$$
$t\ge 0$,

iii) $|\frac{\partial}{\partial t}\vartheta(A',t,x)| + |\frac{\partial}{\partial x_i}\vartheta(A',t,x)|$
$$\le (1+t+|x|)^{\rho-3/2}C_\rho\Big(\|(f,\dot f)\|_{M_3^\rho} + \sum_{\varepsilon=\pm}\sup_{0\le s}\big(\|\beta_1^{(\varepsilon)}(s)\|_{D_N}\|\beta_2^{(\varepsilon)}(s)\|_{D_N}\big)\Big),$$
$t\ge 0$,

iv) $\displaystyle\sum_{|\alpha|+l=n+n'}|\big(\frac{\partial}{\partial t}\big)^l\partial^\alpha\vartheta(A',t,x)|$
$$\begin{aligned}&\le (1+t+|x|)^{-1}(1+\big|t-|x|\big|)^{-n+\rho+1/2}C_{\rho,n}\Big(\|(f,\dot f)\|_{M^\rho_{n+1+n'}} \\ &\quad+\sum_{\substack{\varepsilon=\pm\\ l_1+l_2=l-1\\ |\alpha_1|+|\alpha_2|\le n'}}\sup_{0\le s}(1+s)^{l_1-1}\big(\|\big(\frac{d}{ds}\big)^{l_1}\partial^{\alpha_1}\beta_1^{(\varepsilon)}(s)\|_{D_N}\|\big(\frac{d}{ds}\big)^{l_2}\partial^{\alpha_2}\beta_2^{(\varepsilon)}(s)\|_{D_N}\big)\Big),\end{aligned}$$
$t\ge 0$, $n\ge 2$, $n'\ge 0$, where N depends only on n,

v) *if moreover* $(f,\dot f)=0$ *and* $\chi\geq 0$, *then*

$$\sum_{|\alpha|+l\leq n+n'}|\big(\frac{\partial}{\partial t}\big)^l\partial^\alpha\vartheta(A',t,x)|$$
$$\leq S_{n,\chi}(t,x)\sum_{\varepsilon=\pm}\sum_{\substack{l_1+l_2+|\alpha_1|+|\alpha_2|\leq n\\ l_1'+l_2'+|\alpha_1'|+|\alpha_2'|\leq n'}}$$
$$\sup_{0\leq s}\Big((1+s)^{l_1+l_2+\chi}\|\big(\frac{d}{ds}\big)^{l_1+l_1'}\partial^{\alpha_1}\beta_1^{(\varepsilon)}(s)\|_{D_N}\|\big(\frac{d}{ds}\big)^{l_2+l_2'}\partial^{\alpha_2}\beta_2^{(\varepsilon)}(s)\|_{D_N}\Big)$$

where $t\geq 0$, $x\in\mathbb{R}^3$, $n\geq 1$, $n'\geq 0$, *and where*

$$S_{n,\chi}(t,x)\leq C_\chi(1+t+|x|)^{-n}\quad \textit{for } \chi>0$$

and

$$S_{n,0}(t,x)\leq C'_\delta(1+t+|x|)^{-n+\delta}\quad \textit{for } \chi=0,\delta>0.$$

The constants $C_\chi,\chi>0$ *and* $C'_\delta,\delta>0$ *depend on* n *and* n', *and* N *depends only on* n.

Proof. It follows from Lemma 4.1 that the hypotheses of Lemma 4.5 are satisfied for N sufficiently large. We shall bound the terms on the right-hand side of the inequality in statement i) of Lemma 4.5. It follows from Lemma 4.1 and definitions (4.87a) and (4.87b) of A' that, for N sufficiently large,

$$\|(A'(t),\frac{d}{dt}A'(t))\|_{M^\rho}\leq \|(f,\dot f)\|_{M^\rho}+(1+t)^{1/2-\rho}C_\rho\sum_{\varepsilon=\pm}\sup_{0\leq s}\big(\|\beta_1^{(\varepsilon)}(s)\|_{D_N}\|\beta_2^{(\varepsilon)}(s)\|_{D_N}\big), \tag{4.88a}$$

$$\|(A''(t),\frac{d}{dt}A''(t))\|_{M^\rho}\leq (1+t)^{1/2-\rho}C_\rho\sum_{\varepsilon=\pm}\sup_{0\leq s}\big(\|\beta_1^{(\varepsilon)}(s)\|_{D_N}\|\beta_2^{(\varepsilon)}(s)\|_{D_N}\big), \tag{4.88b}$$

where A'' is defined in Lemma 4.4, and

$$\sum_X\|(\xi_XA')(t),\frac{d}{dt}\xi_XA'(t))\|_{M^\rho}\leq C\|(f,\dot f)\|_{M_1^\rho}+(1+t)^{1/2-\rho}C_\rho\sum_{\varepsilon=\pm}\sup_{0\leq s}\big(\|\beta_1^{(\varepsilon)}(s)\|_{D_N}\|\beta_2^{(\varepsilon)}(s)\|_{D_N}\big), \tag{4.88c}$$

where the sum is taken over $X\in\mathfrak{sl}(2,\mathbb{C})\cap\Pi$. Statement i) now follows from Lemma 4.5 and inequalities (4.88).

It follows directly from (4.87a), Proposition 2.15 with $n=0$ and inequality (4.3) of Lemma 4.1 with $|\alpha|=0$, that

$$(1+t+|x|)^{3/2-\rho}|A'_\mu(t,x)|\leq C_\rho\Big(\|(f,\dot f)\|_{M_2^\rho}+(1+t)^{1/2-\rho}\sum_{\varepsilon=\pm}\sup_{0\leq s}\big(\|\beta_1^{(\varepsilon)}(s)\|_{D_N}\|\beta_2^{(\varepsilon)}(s)\|_{D_N}\big)\Big),\quad t\geq 0. \tag{4.89}$$

Statement ii) of the corollary follows from (4.89) and statement i) of Lemma 4.4. The proof of statements iii) and iv) is so similar to that of ii) that we omit it.

To prove statement v) we note that

$$\frac{d}{dt}(G_{\varepsilon,\mu}(f_1,f_2))(t) = G_{\varepsilon,\mu}(i\varepsilon\omega(-i\partial)f_1+\dot{f}_1,f_2)+G_{\varepsilon,\mu}(f_1,i\varepsilon\omega(-i\partial)f_2+\dot{f}_2),$$
$$\dot{f}_j(t)=\frac{d}{dt}f_j(t),\quad j=1,2,$$

and that

$$\partial_i G_{\varepsilon,\mu}(f_1,f_2)=G_{\varepsilon,\mu}(\partial_i f_1,f_2)+G_{\varepsilon,\mu}(f_1,\partial_i f_2),\quad 1\le i\le 3,$$

where $\varepsilon=\pm$, $0\le\mu\le 3$. These two equalities, inequality (4.3) of Lemma 4.1 and equality (4.83) give, with $l+|\alpha|=n+n'$, $n\ge 1$, $n'\ge 0$,

$$\begin{aligned}
&|\big(\frac{\partial}{\partial t}\big)^l\partial^\alpha\vartheta(A',t,x)| \\
&\le \sum_{\varepsilon=\pm}\sum_{\substack{l_1+l_2+|\alpha_1|+|\alpha_2|=n\\ l_1'+l_2'+|\alpha_1'|+|\alpha_2'|\le n'}}\int_0^1 s^{n+n'-1}(1+st+|sx|)^{-(n+\chi-\delta)}(1+st)^{-\delta}ds\\
&\qquad C_{n,n',\chi,\delta}\sup_{s'\ge 0}\Big((1+s')^{l_1+l_2+\chi}\|\big(\frac{d}{ds}\big)^{l_1+l_1'}\partial^{\alpha_1'}\beta_1^{(\varepsilon)}(s')\|_{D_N}\|\big(\frac{d}{ds}\big)^{l_2+l_2'}\partial^{\alpha_2'}\beta_2^{(\varepsilon)}(s')\|_{D_{N'}}\Big),
\end{aligned}$$

where $\chi\ge 0$, $\delta>0$ and where N depends on n. If $\chi>0$ then we choose $\delta=\chi/2$. The integral in the last estimate is then bounded by

$$\int_0^1 s^{n-1}(1+s(t+|x|))^{-(n+\chi/2)}ds\le C_\chi(1+t+|x|)^{-n},$$

which proves statement v) of the corollary for $\chi>0$. If $\chi=0$ then the integral is bounded by

$$\int_0^1 s^{n-1}(1+s(t+|x|))^{-n+\delta}ds\le C'_\delta(1+t+|x|)^{-n+\delta},\quad 0<\delta<1,$$

which proves the statement for $\chi=0$. This proves the corollary.

In order to *construct approximate solutions of the M-D equations,* we introduce the Banach space $\mathcal{D}^{d,\chi}_{n,j}$, $n\ge 0$, $j\ge 0$, $d\ge 0$, $\chi\ge 0$ as the subspace of elements $(\beta^{(+)},\beta^{(-)})\in C^n(\mathbb{R}^+,D_j\oplus D_j)$ with finite norm

$$\begin{aligned}
\|(\beta^{(+)},\beta^{(-)})\|_{\mathcal{D}^{d,\chi}_{n,j}} = \sum_{\varepsilon=\pm}\sum_{0\le l\le n}\Big(&\sup_{t\ge 0}\big((1+t)^{\chi+l}\|\big(\frac{d}{dt}\big)^l P_\varepsilon(-i\partial)\beta^{(\varepsilon)}(t)\|_{D_j}\big)\\
&+\sup_{t\ge 0}\big((1+t)^{d+\chi+l}\|\big(\frac{d}{dt}\big)^l P_{-\varepsilon}(-i\partial)\beta^{(\varepsilon)}(t)\|_{D_j}\big)\Big).
\end{aligned}\tag{4.90}$$

We also introduce the Fréchet spaces $\mathcal{D}^{d,\chi}_{\infty,\infty}=\cap_{n\ge 0,j\ge 0}\mathcal{D}^{d,\chi}_{n,j}$.

For $(f,\dot f)\in M^{\circ\rho}_{\infty}$, for $\beta_j=(\beta_j^{(+)},\beta_j^{(-)})\in\mathcal{D}^{0,\chi_j}_{\infty,\infty}$, $j=1,2$, and for $\beta_3=(\beta_3^{(+)},\beta_3^{(-)})\in\mathcal{D}^{d,\chi_3}_{\infty,\infty}$ we introduce the polynomial $((f,\dot f),\beta_1,\beta_2,\beta_3)\mapsto\lambda^{(\varepsilon)}(t)((f,\dot f),\beta_1,\beta_2,\beta_3)$ by the following expression, where we have omitted the arguments $(f,\dot f),\beta_1,\beta_2,\beta_3$:

$$\lambda^{(\varepsilon)}(t) \tag{4.91}$$
$$= ie^{-i\varepsilon\omega(-i\partial)t}\int_t^\infty e^{i(t-s)\mathcal{D}}(A'_\mu(s)+B_\mu(s))\gamma^0\gamma^\mu e^{i\varepsilon\omega(-i\partial)s}\beta_3^{(\varepsilon)}(s)ds,\quad t\ge 0,\varepsilon=\pm,$$

where A', defined in (4.87a), is a function of $(f,\dot f)$, $(\beta_1^{(+)},\beta_1^{(-)})$ and $(\beta_2^{(+)},\beta_2^{(-)})$, and where B_μ is defined as in Lemma 4.5, i.e.

$$(B_0(t))(x)=-\frac{\partial}{\partial t}\vartheta(A',t,x),\quad (B_i(t))(x)=-\frac{\partial}{\partial x_i}\vartheta(A',t,x),\quad 1\le i\le 3.$$

The integral in (4.91) has to be interpreted in the sense of the strong improper Riemann integral in L^2. It follows from the next proposition that $\lambda^{(\varepsilon)}$ exists for certain choices of χ_1, χ_2, χ_3 and d:

Proposition 4.7. *Let $1/2<\rho<1$, $0\le d\le 1$, $q\in\mathbb{N}$ and let $\chi=\chi_1+\chi_2+\chi_3$, $\chi_j\ge 0$. Let $(f,\dot f)\in M^{\circ\rho}_{\infty}$, $(\beta_j^{(+)},\beta_j^{(-)})\in\mathcal{D}^{0,\chi_j}_{\infty,\infty}$ for $j=1,2$, and let $(\beta_3^{(+)},\beta_3^{(-)})\in\mathcal{D}^{d,\chi_3}_{\infty,\infty}$. Then*
i) If $(\beta_j^{(+)},\beta_j^{(-)})=0$ for $j=1$ or $j=2$ and $d-\chi_3>\rho-1/2$ then

$$\sum_{\substack{Y\in\Pi'\cap U(\mathbb{R}^4)\\|Y|\le q}}\|(\xi_Y(\lambda^{(+)},\lambda^{(-)})\|_{\mathcal{D}^{d',\chi'}_{n,l}}$$
$$\le C\sum_{\substack{Y_1,Y_2\in\Pi'\cap U(\mathbb{R}^4)\\|Y_1|+|Y_2|\le q}}\|T^{M1}_{Y_1}(f,\dot f)\|_{M^\rho_N}\|\xi_{Y_2}(\beta_3^{(+)},\beta_3^{(-)})\|_{\mathcal{D}^{d,\chi_3}_{L,N}},\quad n,l\ge 0,$$

where $d'=1-d$, $\chi'=1/2-\rho+\chi_3+d$ and $L=n+l+1$. Here C depends on n, l, d, q and χ_3, while N depends on n and l.
ii) If $(f,\dot f)=0$ and $\chi_3+d>0$, then

$$\sum_{\substack{Y\in\Pi'\cap U(\mathbb{R}^4)\\|Y|\le q}}\|\xi_Y(\lambda^{(+)},\lambda^{(-)})\|_{\mathcal{D}^{d',\chi'}_{n,l}}$$
$$\le C\sum_{\substack{Y_i\in\Pi'\cap U(\mathbb{R}^4)\\|Y_1|+|Y_2|+|Y_3|\le q}}\|\xi_{Y_1}(\beta_1^{(+)},\beta_1^{(-)})\|_{\mathcal{D}^{0,\chi_1}_{L,N}}\|\xi_{Y_2}(\beta_2^{(+)},\beta_2^{(-)})\|_{\mathcal{D}^{0,\chi_2}_{L,N}}\|\xi_{Y_3}(\beta_3^{(+)},\beta_3^{(-)})\|_{\mathcal{D}^{0,\chi_3}_{L,N}},$$

where $l,n\ge 0$, $d'=1-d$, $\chi'=\chi+d-\delta$ and $L=n+l+1$. Here C depends on n, l, d, q and χ, while N depends on n and l. Moreover $\delta=0$ if $\chi_1+\chi_2>0$ and $\delta>0$ if $\chi_1+\chi_2=0$.

Proof. Since the case $q > 0$ is so similar to the case $q = 0$, we only consider $q = 0$. Let $W = \Pi \cup \{D\}$ be a basis of the 11-dimensional Weyl Lie algebra $\mathfrak{w}$ satisfying

$$[D, P_\mu] = -P_\mu, \quad [D, M_{\alpha\beta}] = 0, \quad 0 \leq \mu \leq 3, 0 \leq \alpha < \beta \leq 3,$$

let W_1 be the corresponding basis of the 7-dimensional subalgebra $\mathfrak{w}_1 = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{R}$ and let

$$(\xi_D f)(t, x) = t\frac{\partial}{\partial t} f(t, x) + \sum_{0 \leq i \leq 3} x_i \frac{\partial}{\partial x_i} f(t, x). \tag{4.93}$$

Let $H \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^3)$ and let $F_s(t, x) = H(s, sx/t)$, $t > 0$. Using definition (4.81) of ξ_X, $X \in \Pi$, and definition (4.93) of ξ_D, we obtain

$$(\xi_{P_i} F_S)(t, x) = \frac{s}{t}(\xi_{P_i} H)(s, sx/t), \quad 1 \leq i \leq 3, \tag{4.94a}$$

$$(\xi_{P_0} F_s)(t, x) = \frac{s}{t}(\xi_{P_0} H)(s, sx/t) - \frac{1}{t}(\xi_D H)(s, sx/t),$$

$$(\xi_{M_{0i}} F_s)(t, x) = (\xi_{M_{0i}} H)(s, sx/t) - \frac{x_i}{t}(\xi_D H)(s, sx/t), \quad 0 \leq i \leq 3, \tag{4.94b}$$

$$(\xi_{M_{ij}} F_s)(t, x) = (\xi_{M_{ij}} H)(s, sx/t), \quad 1 \leq i < j \leq 3, \tag{4.94c}$$

$$\xi_D F_s = 0. \tag{4.94d}$$

Repeated use of (4.94b) and (4.94c) shows that if $Y \in \Pi' \cap U(\mathfrak{sl}(2, \mathbb{C})$ then there are elements $Z \in W'$, the standard basis of the enveloping algebra of $\mathfrak{w}_1$ build on W_1, and polynomials q_Z such that

$$(\xi_Y F_s)(t, x) = \sum_{\substack{Z \in W_1' \\ |Z| \leq |Y|}} q_Z(x/t)(\xi_Z H)(s, sx/t), \quad s \geq 0, t \geq 0.$$

Since

$$\Big(\frac{d}{ds} F_s\Big)(t, x) = s^{-1}(\xi_D H)(s, sx/t), \tag{4.94e}$$

we obtain

$$\|\Big(\xi_Y \frac{d^q}{ds^q} F_s\Big)(1, \cdot)\|_{L^\infty(B)} \leq C_{|Y|,q}\ s^{-q} \sum_{\substack{Z \in W' \\ |Z| \leq |Y|+q}} \sup_{|x| \leq 1} |(\xi_Z H)(s, sx)|, \tag{4.95a}$$

where $Y \in \Pi' \cap U(\mathfrak{sl}(2, \mathbb{C}))$ and B is the unit ball in $\mathbb{R}^3$.

It follows by induction in $|X| + q \geq 0$, $q \geq 0$, $X \in \Pi' \cap U(\mathbb{R}^4)$, from (4.94a) and (4.94e), that

$$\begin{aligned} &\Big(\xi_X \frac{d^q}{ds^q} F_s\Big)(t, x) \qquad\qquad (4.95b) \\ &= \sum_{\substack{|Y|+l \leq |X|+q \\ |Y| \leq |X|}} (\xi_Y \xi_D^l H)(s, xs/t) t^{-|X|} s^{|Y|-q} C_{X,Y,l,q}, \quad s > 0, t > 0, Y \in \Pi' \cap U(\mathbb{R}^4), \end{aligned}$$

for some constants $C_{X,Y,l,q}$. This gives, for $0 < t/2 \le s \le 2t$,

$$\|\big(\xi_Y \frac{d^q}{ds^q} F_s\big)(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \le C_{|Y|,q}\ s^{-q} \sum_{\substack{Z \in W' \\ |Z| \le |Y|+q}} \|(\xi_Z H)(s,\cdot)\|_{L^\infty(\mathbb{R}^3)}, \tag{4.96}$$

where $Y \in \Pi' \cap U(\mathbb{R}^4)$.

It follows from definition (1.19) of ϑ and by partial integration, that

$$\vartheta(\xi_D A', y) = y_\mu A'^\mu(y) - \vartheta(A', y),$$

which, together with (1.20b), give

$$\begin{aligned}\xi_D \frac{\partial}{\partial y^\mu}\vartheta(A', y) &= -\frac{\partial}{\partial y^\mu}\vartheta(A', y) + \frac{\partial}{\partial y^\mu}(y^\nu A'_\nu(y)) \\ &= \frac{\partial}{\partial y^\mu}\vartheta(\xi_D A', y),\end{aligned} \tag{4.97}$$

$0 \le \mu \le 3$. If $\xi^M_{M_{\mu\nu}} = \xi_{M_{\mu\nu}} + n_{\mu\nu}$, $0 \le \mu < \nu \le 3$, where $n_{\mu\nu}$ is as in (1.5) and if $\xi^M_D = \xi_D$, then we obtain as in (4.97) that

$$(\xi^M_X B(y))_\mu = -\frac{\partial}{\partial y^\mu}\vartheta(\xi^M_X A', y), \quad X \in \mathfrak{w}_1, 0 \le \mu \le 3,$$

which extends to the enveloping algebra:

$$((\xi^M_Y B)(y))_\mu = -\frac{\partial}{\partial y^\mu}\vartheta(\xi^M_Y A', y), \quad Y \in U(\mathfrak{w}_1), 0 \le \mu \le 3. \tag{4.98}$$

Let $(\beta_i^{(+)}, \beta_i^{(-)}) = 0$ for $i = 1$ or $i = 2$. Then it follows from definition (4.87a) of A', that A' is a free field with initial conditions $(f, \dot f) \in M^{\circ\rho}_\infty$. Then $\xi^M_X A'$ is also a free field with initial conditions $T^{M1}_X(f, \dot f) \in M^{\circ\rho}_\infty$, where T^{M1}_X is defined by (1.5) if $X \in \Pi$ and $(T^{M1}_D(f, \dot f))(x) = (f, \dot f)(x) + \sum_{0 \le i \le 3} x_i \partial_i (f, \dot f)(x)$. Let $H_\mu(f, \dot f) = A'_\mu + B_\mu$. Then it follows from (4.98) and the fact that the initial condition for $\xi^M_Y A'$, $Y \in W'$, is $T^{M1}_Y(f, \dot f) \in M^\rho_\infty$, that $\xi^M_Y H(f, \dot f) = H(T^{M1}_Y(f, \dot f))$. This gives according to Proposition 2.15 and statement iii) of Corollary 4.6

$$|(\xi^M_Y H(f, \dot f))(x, t)| \le C_\rho (1 + |x| + |t|)^{\rho - 3/2} \|(f, \dot f)\|_{M^\rho_{3+|Y|}}, \quad Y \in W'.$$

By the definition of ξ^M, we obtain that

$$\sum_{\substack{Z \in W' \\ |Z| \le i}} |(\xi_Z (A' + B))(t, x)| \le C_\rho (1 + |x| + |t|)^{\rho - 3/2} \|(f, \dot f)\|_{M^\rho_{3+i}}, \quad i \ge 0, \tag{4.99}$$

when $(\beta_j^{(+)}, \beta_j^{(-)}) = 0$ for $j = 1$ or $j = 2$.

When $(f, \dot{f}) = 0$ in definition (4.87a) of A', then we estimate $(\xi_Z(A' + B))(t, x)$ by inequality (4.3) of Lemma 4.1 and by statement v) of Corollary 4.6 for $|x| \leq 2t$. This gives, with $P^\alpha = P_0^{\alpha_0} P_1^{\alpha_1} P_2^{\alpha_2} P_3^{\alpha_3}$,

$$\begin{aligned}
&\sum_{\substack{Z \in W' \\ |Z| \leq i}} |(\xi_Z(A' + B))(t, x)| \qquad (4.100)\\
&\quad \leq C_i \sum_{|\alpha| \leq i} |t|^{|\alpha|} |(\xi_{P^\alpha}(A' + B))(t, x)| \\
&\quad \leq C_i'(1 + t) \sum_{\substack{\varepsilon = \pm \\ l_1 + l_2 \leq i}} \sup_{0 \leq s \leq t} \Big((1 + s)^{l_1 + l_2 + \chi_1 + \chi_2} \| \frac{d^{l_1}}{ds^{l_1}} \beta_1^{(\varepsilon)}(s) \|_{D_N} \| \frac{d^{l_2}}{ds^{l_2}} \beta_2^{(\varepsilon)}(s) \|_{D_N} \Big),
\end{aligned}$$

$|x| \leq 2t$, $t \geq 0$, for some N depending on $i \geq 0$, where $\delta = 0$ if $\chi_1 + \chi_2 > 0$ and $\delta > 0$ if $\chi_1 + \chi_2 = 0$. In the last case C' depends on δ.

Let $H = A' + B$. It follows from inequalities (4.99) and (4.100) that

$$\begin{aligned}
&\sum_{\substack{Z \in W' \\ |Z| \leq i}} |(\xi_Z H)(t, x)| \qquad (4.101)\\
&\quad \leq C_i \Big((1 + t)^{\rho - 3/2} \|(f, \dot{f})\|_{M^\rho_{3+i}} + (1 + t)^{-1+\delta} \\
&\qquad \sum_{\substack{\varepsilon = \pm \\ l_1 + l_2 \leq i}} \sup_{0 \leq s \leq t} \big((1 + s)^{l_1 + l_2 + \chi_1 + \chi_2} \| \frac{d^{l_1}}{ds^{l_1}} \beta_1^{(\varepsilon)}(s) \|_{D_N} \| \frac{d^{l_2}}{ds^{l_2}} \beta_2^{(\varepsilon)}(s) \|_{D_N} \big) \Big),
\end{aligned}$$

$|x| \leq 2t$, $t \geq 0$, for some N depending on $i \geq 0$ and C_i depending on ρ, $1/2 < \rho < 1$.

The function $\lambda^{(\varepsilon)}$, $\varepsilon = \pm$ can now be written as

$$\lambda^{(\varepsilon)}(t) = i e^{-i\varepsilon\omega(-i\partial)t} \int_t^\infty e^{i(t-s)\mathcal{D}} \sum_{0 \leq \mu \leq 3} F_{s,\mu}(s) \gamma^0 \gamma^\mu e^{i\varepsilon\omega(-i\partial)s} \beta_3^{(\varepsilon)}(s) ds, \qquad (4.102)$$

where $(F_{s,\mu}(t))(x) = F_{s,\mu}(t, x) = H_\mu(s, sx/t)$, $t > 0$. According to Theorem A.3 there are bilinear forms $g^{(\varepsilon)}_{s,\mu,j}$ of $F_{s,\mu}$ and $\beta_3^{(\varepsilon)}(s)$ such that

$$\begin{aligned}
&\| \big(\frac{d}{dt}\big)^p \big(\frac{d}{ds}\big)^q \big(e^{-i\varepsilon\omega(-i\partial)t} F_{s,\mu}(t) e^{i\varepsilon\omega(-i\partial)t} \beta_3^{(\varepsilon)}(s) - \sum_{0 \leq j \leq n} t^{-j} g^{(\varepsilon)}_{s,\mu,j} \big) \|_{D_L} \qquad (4.103)\\
&\quad \leq C_{N,q} t^{-n-p-1} \sum_{q_1 + q_2 = q} \Big(\sum_{\substack{|\alpha| \leq L \\ a \leq p}} \| \big(\frac{d}{dt}\big)^a \big(\frac{d}{ds}\big)^{q_1} \partial^\alpha F_{s,\mu}(t) \|_{L^\infty(\mathbb{R}^3)} \\
&\qquad + \sum_{Y \in D(N)} \| \big(\xi_Y \big(\frac{d}{ds}\big)^{q_1} F_{s,\mu} \big)(1, \cdot) \|_{L^\infty(B)} \Big) \| \big(\frac{d}{ds}\big)^{q_2} \beta_3^{(\varepsilon)}(s) \|_{D_N},
\end{aligned}$$

$t \geq 1$, $n \geq 1$, $p \geq 0$, $q \geq 0$, where $D(N) = \{Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C})) \big| |Y| \leq N\}$ and where N depends on n, p and L. Here we have used the Leibniz rule for $(d/ds)^q$ and the fact that $g^{(\varepsilon)}_{s,\mu,j}$ is a bilinear form of $F_{s,\mu}$ and $\beta_3^{(\varepsilon)}(s)$. According to (A.36) and (A.37) we have

$$\widehat{g^{(\varepsilon)}}_{s,\mu,0}(k) = F_{s,\mu}(1, -\varepsilon k/\omega(k))(\beta_3^{(\varepsilon)}(s))^\wedge(k) \tag{4.104}$$

and

$$\begin{aligned} &\|\big(\frac{d}{ds}\big)^q g^{(\varepsilon)}_{s,\mu,l}\|_{D_j} \\ &\quad \leq C_{j,l,q} \sum_{\substack{Y \in D(M) \\ q_1+q_2=q}} \|\big(\xi_Y \big(\frac{d}{ds}\big)^{q_1} F_{s,\mu}\big)(1,\cdot)\|_{L^\infty(B)} \|\big(\frac{d}{ds}\big)^{q_2} \beta_3^{(\varepsilon)}(s)\|_{D_M}, \end{aligned} \tag{4.105}$$

where M depends on j and l.

We introduce for $s > 0$, $q \geq 0$, $N \geq 0$, $1/2 < \rho < 3/2$, $\delta \geq 0$, $\chi_1 \geq 0$ and $\chi_2 \geq 0$:

$$\begin{aligned} \Gamma^{\rho,\chi_1,\chi_2,\delta}_{i,q,N}(s) = \Big(&(1+s)^{\rho-3/2} \|(f,\dot{f})\|_{M^\rho_{3+i+q}} + (1+s)^{-1+\delta} \\ &\sum_{\substack{l_1+l_2 \leq i+q \\ \varepsilon = \pm}} \sup_{0 \leq \tau \leq s} \big((1+\tau)^{l_1+l_2+\chi_1+\chi_2} \|\frac{d^{l_1}}{d\tau^{l_1}} \beta_1^{(\varepsilon)}(\tau)\|_{D_N} \|\frac{d^{l_2}}{d\tau^{l_2}} \beta_1^{(\varepsilon)}(\tau)\|_{D_N}\big)\Big). \end{aligned} \tag{4.106}$$

It follows from (4.95) and (4.101) that

$$\|\big(\xi_Y \frac{d^q}{ds^q} F_{s,\mu}\big)(1,\cdot)\|_{L^\infty(B)} \leq C_{|Y|,q} s^{-q} \Gamma^{\rho,\chi_1,\chi_2,\delta}_{|Y|,q,N}(s), \quad s > 0, \tag{4.107}$$

for $Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$, and it follows from (4.96) and (4.101) that

$$\|\big(\xi_Y \frac{d^q}{ds^q} F_{s,\mu}\big)(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leq C_{|Y|,q} s^{-q} \Gamma^{\rho,\chi_1,\chi_2,\delta}_{|Y|,q,N}(s), \tag{4.108}$$

where $Y \in \Pi' \cap U(\mathbb{R}^4)$ and $0 < t/2 \leq s \leq 2t$. Inequalities (4.103), (4.107) and (4.108) give, since Γ is increasing in q,

$$\begin{aligned} &\|\frac{d^p}{dt^p} \frac{d^q}{ds^q} \Big(e^{-i\varepsilon\omega(-i\partial)t} F_{s,\mu}(t) e^{i\varepsilon\omega(-i\partial)t} \beta_3^{(\varepsilon)}(s) - \sum_{0 \leq j \leq n} t^{-j} g^{(\varepsilon)}_{s,\mu,j}\Big)\|_{D_L} \\ &\quad \leq C_{n,L,p,q} s^{-n-p-q-1} \Gamma^{\rho,\chi_1,\chi_2,\delta}_{N,q,N}(s) \sum_{q' \leq q} (1+s)^{q'} \|\frac{d^{q'}}{ds^{q'}} \beta_3^{(\varepsilon)}(s)\|_{D_N}, \end{aligned} \tag{4.109}$$

where $t \geq 1$, $t/2 \leq s \leq 2t$, $p \geq 0$, $q \geq 0$, $n \geq 1$, $L \geq 0$ and where N, depending on n, p, q and L, is sufficiently large. Inequalities (4.105) and (4.107) give

$$\|\frac{d^q}{ds^q} g^{(\varepsilon)}_{s,\mu,j}\|_{D_l} \leq C_{l,j,q} s^{-q} \Gamma^{\rho,\chi_1,\chi_2,\delta}_{M,q,M}(s) \sum_{q' \leq q} (1+s)^{q'} \|\frac{d^{q'}}{ds^{q'}} \beta_3^{(\varepsilon)}(s)\|_{D_M}, \tag{4.110}$$

where $s > 0$, $q \geq 0$, $j \geq 0$, $l \geq 0$ and where M, depending on l, j, q, is sufficiently large.

It follows from (4.109) from the definition of $F_{\mu,s}(t)$ (cf. (4.102)) and from Leibniz rule that

$$\begin{aligned}&\|\frac{d^p}{ds^p}\Big(e^{-i\varepsilon\omega(-i\partial)s}H_\mu(s)e^{i\varepsilon\omega(-i\partial)s}\beta_3^{(\varepsilon)}(s)-\sum_{0\leq j\leq n}s^{-j}g^{(\varepsilon)}_{s,\mu,j}\Big)\|_{D_L} \\ &\qquad\leq C_{n,L,p}s^{-n-p-1}\Gamma^{\rho,\chi_1,\chi_2,\delta}_{N,P,N}(s)\sum_{q\leq p}(1+s)^q\|\frac{d^q}{ds^q}\beta_3^{(\varepsilon)}(s)\|_{D_N},\end{aligned}\tag{4.111}$$

where $s \geq 1$, $p \geq 0$, $n \geq 1$, $L \geq 0$ and where N, depending on n, p and L, is sufficiently large.

Since $H_\mu = A'_\mu + B_\mu$, we have according to (1.20b) and (4.92) that $tH_0(t,x)+\sum_{1\leq i\leq 3} x_iH_i(t,x) = 0$. The formula (see statement (i) of Lemma 3.13 in [8])

$$\sum_{0\leq\mu\leq 3}P_\varepsilon(k)\gamma^0\gamma^\mu f_\mu P_\varepsilon(k)=\Big(f_0+\sum_{1\leq i\leq 3}\frac{-\varepsilon k_i}{\omega(k)}f_i\Big)P_\varepsilon(k),\quad f_\mu\in\mathbb{C}, 0\leq\mu\leq 3,\tag{4.112}$$

then gives that

$$\sum_{0\leq\mu\leq 3}P_\varepsilon(k)\gamma^0\gamma^\mu H_\mu(t,-\varepsilon kt/\omega(k))P_\varepsilon(k)=0,\quad t\geq 0, k\in\mathbb{R}^3.\tag{4.113}$$

It follows from (4.104), the definition of $F_{s,\mu}$ and (4.113) that

$$\sum_{0\leq\mu\leq 3}P_\varepsilon(-i\partial)\gamma^0\gamma^\mu g^{(\varepsilon)}_{s,\mu,0}=\sum_{0\leq\mu\leq 3}P_\varepsilon(-i\partial)\gamma^0\gamma^\mu h^{(\varepsilon)}_{s,\mu,0},\tag{4.114}$$

where

$$(h^{(\varepsilon)}_{s,\mu,0})^\wedge(k)=F_{s,\mu}(1,-\varepsilon k/\omega(k))(P_{-\varepsilon}(-i\partial)\beta_3^{(\varepsilon)}(s))^\wedge(k).\tag{4.115}$$

Since $h^{(\varepsilon)}_{s,\mu,0}$ is the value of the bilinear form $g^{(\varepsilon)}_{s,\mu,0}$ applied to $F_{s,\mu}$ and $P_{-\varepsilon}(-i\partial)\beta_3^{(\varepsilon)}(s)$, inequality (4.110) gives

$$\begin{aligned}&\|\frac{d^p}{ds^p}h^{(\varepsilon)}_{s,\mu,0}\|_{D_l}\\ &\qquad\leq C_{l,p}s^{-p}\Gamma^{\rho,\chi_1,\chi_2,\delta}_{M,p,M}(s)\sum_{q\leq p}(1+s)^q\|\frac{d^q}{ds^q}P_{-\varepsilon}(-i\partial)\beta_3^{(\varepsilon)}(s)\|_{D_M},\quad s>0,\end{aligned}$$

where M depends on $l \leq 0$ and $p \geq 0$. By definitions (4.90) of the norm in $\mathcal{D}^{d,\chi}_{n,j}$, we obtain from this inequality and inequality (4.114) that

$$\begin{aligned}&\|\frac{d^p}{ds^p}P_\varepsilon(-i\partial)\sum_{0\leq\mu\leq 3}\gamma^0\gamma^\mu g^{(\varepsilon)}_{s,\mu,0}\|_{D_l}\\ &\quad\leq C_{l,p}\Big(s^{\rho-3/2-d-p-\chi_3}\|(f,\dot f)\|_{M^\rho_N}\|(\beta_3^{(+)},\beta_3^{(-)})\|_{\mathcal{D}^{d,\chi_3}_{p,N}}\\ &\qquad+s^{-1-d-p+\delta-\chi_3}\|(\beta_1^{(+)},\beta_1^{(-)})\|_{\mathcal{D}^{0,\chi_1}_{p,N}}\|(\beta_2^{(+)},\beta_2^{(-)})\|_{\mathcal{D}^{0,\chi_2}_{p,N}}\|(\beta_3^{(+)},\beta_3^{(-)})\|_{\mathcal{D}^{d,\chi_3}_{p,N}}\Big),\quad s>0,\end{aligned}\tag{4.116}$$

where N depends on $p \geq 0$ and $l \geq 0$.

It follows from (4.110) that

$$\|\frac{d^p}{ds^p} g^{(\varepsilon)}_{s,\mu,j}\|_{D_l} \leq C_{l,j,p}\Big(s^{\rho-3/2-p-\chi_3}\|(f,\dot f)\|_{M^\rho_N}\|(\beta^{(+)}_3,\beta^{(-)}_3)\|_{\mathcal{D}^{0,\chi_3}_{p,N}} + s^{-1-p+\delta-\chi_3}\prod_{1\leq i\leq 3}\|(\beta^{(+)}_i,\beta^{(-)}_i)\|_{\mathcal{D}^{0,\chi_i}_{p,N}}\Big), \quad s>0, \tag{4.117}$$

where N depends on $p \geq 0$, $j \geq 0$ and $l \geq 0$. Proposition 2.15, (4.3) of Lemma 4.1 and statement iv) of Corollary 4.6 show that

$$\|\frac{d^p}{ds^p} e^{-i\varepsilon\omega(-i\partial)s} H_\mu(s) e^{i\varepsilon\omega(-i\partial)s}\|_{D_l} \leq C_{l,p}\Big(\|(f,\dot f)\|_{M^\rho_N}\|\beta^{(+)}_3,\beta^{(-)}_3)\|_{\mathcal{D}^{0,0}_{p,N}} + \prod_{1\leq i\leq 3}\|(\beta^{(+)}_i,\beta^{(-)}_i)\|_{\mathcal{D}^{0,0}_{p,N}}\Big), \quad 0\leq s\leq 2, \tag{4.118}$$

where N depends on $p \geq 0$ and $l \geq 0$.

It follows from inequalities (4.106), (4.111) with $n=1$, (4.116), (4.117) with $j=1$ and (4.118), that

$$\begin{aligned}&\|\frac{d^p}{ds^p} P_\varepsilon(-i\partial)\sum_{0\leq\mu\leq 3}\gamma^0\gamma^\mu e^{-i\varepsilon\omega(-i\partial)s} H_\mu(s) e^{i\varepsilon\omega(-i\partial)s}\beta^{(\varepsilon)}_3(s)\|_{D_l} \\ &\leq C_{l,p}\Big((1+s)^{\rho-3/2-d-p-\chi_3}\|(f,\dot f)\|_{M^\rho_N}\|(\beta^{(+)}_3,\beta^{(-)}_3)\|_{\mathcal{D}^{d,\chi_3}_{p,N}} \\ &\quad + (1+s)^{-d-1-p-\chi_3+\delta}\|(\beta^{(+)}_1,\beta^{(-)}_1)\|_{\mathcal{D}^{0,\chi_1}_{p,N}}\|(\beta^{(+)}_2,\beta^{(-)}_2)\|_{\mathcal{D}^{0,\chi_2}_{p,N}}\|(\beta^{(+)}_3,\beta^{(-)}_3)\|_{\mathcal{D}^{d,\chi_3}_{p,N}}\Big),\end{aligned} \tag{4.119}$$

$s \geq 0$, $\chi=\chi_1+\chi_2+\chi_3$, where N depends on $p \geq 0$ and $l \geq 0$. It follows from inequalities (4.106), (4.111) with $n=0$, (4.117) with $j=0$ and (4.118), that

$$\begin{aligned}&\|\frac{d^p}{ds^p} P_{-\varepsilon}(-i\partial)\sum_{0\leq\mu\leq 3}\gamma^0\gamma^\mu e^{-i\varepsilon\omega(-i\partial)s} H_\mu(s) e^{i\varepsilon\omega(-i\partial)s}\beta^{(\varepsilon)}_3(s)\|_{D_l} \\ &\leq C_{l,p}\Big((1+s)^{\rho-3/2-p-\chi_3}\|(f,\dot f)\|_{M^\rho_N}\|(\beta^{(+)}_3,\beta^{(-)}_3)\|_{\mathcal{D}^{0,\chi_3}_{p,N}} \\ &\quad + (1+s)^{-1-p-\chi_3+\delta}\prod_{1\leq i\leq 3}\|(\beta^{(+)}_i,\beta^{(-)}_i)\|_{\mathcal{D}^{0,\chi_i}_{p,N}}\Big), \quad s\geq 0, \chi=\chi_1+\chi_2+\chi_3,\end{aligned} \tag{4.120}$$

where N depends on $p \geq 0$ and $l \geq 0$.

Since $P_\varepsilon(-i\partial)\mathcal{D} = i\varepsilon\omega(-i\partial)P_\varepsilon(-i\partial)$ (cf. (1.2b) and (1.16)), we obtain according to (4.102) that

$$\begin{aligned}&P_\varepsilon(-i\partial)\lambda^\varepsilon(t) \\ &= i\int_t^\infty P_\varepsilon(-i\partial)\sum_{0\leq\mu\leq 3}\gamma^0\gamma^\mu e^{-i\varepsilon\omega(-i\partial)s} H_\mu(s) e^{i\varepsilon\omega(-i\partial)s}\beta^{(\varepsilon)}_3(s)ds\end{aligned} \tag{4.121a}$$

and

$$P_{-\varepsilon}(-i\partial)\lambda^{(\varepsilon)}(t) \tag{4.121b}$$
$$= i\int_t^\infty e^{-2i\varepsilon\omega(-i\partial)(t-s)}P_{-\varepsilon}(-i\partial)\sum_{0\le\mu\le 3}\gamma^0\gamma^\mu e^{-i\varepsilon\omega(-i\partial)s}H_\mu(s)e^{i\varepsilon\omega(-i\partial)s}\beta_3^{(\varepsilon)}(s)ds,$$

$t \geq 0$. It follows from (4.119) and (4.121a) that:

a) if $(\beta_i^{(+)}, \beta_i^{(-)}) = 0$ for $i = 1$ or $i = 2$, then

$$\|\frac{d^p}{dt^p}P_\varepsilon(-i\partial)\lambda^{(\varepsilon)}(t)\|_{D_l} \tag{4.122a}$$
$$\le C_{l,p}(1+t)^{\rho-1/2-d-p-\chi_3}\|(f,\dot f)\|_{M_N^\rho}\|(\beta_3^{(+)},\beta_3^{(-)})\|_{\mathcal{D}_{p,N}^{d,\chi_3}},$$

b) if $(f, \dot f) = 0$, then

$$\|\frac{d^p}{dt^p}P_\varepsilon(-i\partial)\lambda^{(\varepsilon)}(t)\|_{D_l} \tag{4.122b}$$
$$\le C_{l,p}(1+t)^{-1-d-p-\chi_3+\delta}\|(\beta_1^{(+)},\beta_1^{(-)})\|_{\mathcal{D}_{p,N}^{0,\chi_1}}\|(\beta_2^{(+)},\beta_2^{(-)})\|_{\mathcal{D}_{p,N}^{0,\chi_2}}\|(\beta_3^{(+)},\beta_3^{(-)})\|_{\mathcal{D}_{p,N}^{d,\chi_3}},$$

where $t \geq 0, \chi = \chi_1 + \chi_2 + \chi_3$ and N depends on $p \geq 0$ and $l \geq 0$.

Let

$$q_\varepsilon(t) = P_{-\varepsilon}(-i\partial)\sum_{0\le\mu\le 3}\gamma^0\gamma^\mu e^{-i\varepsilon\omega(-i\partial)t}H_\mu(t)e^{i\varepsilon\omega(-i\partial)t}\beta_3^{(\varepsilon)}(t), \quad t \ge 0. \tag{4.123}$$

Repeated integration by parts gives according to (4.121b)

$$\frac{d^p}{dt^p}P_{-\varepsilon}(-i\partial)\lambda^{(\varepsilon)}(t) \tag{4.124}$$
$$= i\sum_{0\le j\le n}(-2i\varepsilon\omega(-i\partial))^{-j-1}\frac{d^{j+p}}{dt^{j+p}}q_\varepsilon(t)$$
$$+ i\int_t^\infty(-2i\varepsilon\omega(-i\partial))^{-n-1}e^{-2i\varepsilon\omega(-i\partial)(t-s)}\frac{d^{n+p+1}}{ds^{n+p+1}}q_\varepsilon(s)ds, \quad t\ge 0, p\ge 0, n\ge 0.$$

It follows from the definition of the norm $\|\cdot\|_{D_l}$, that

$$\|e^{i\varepsilon\omega(-i\partial)t}\alpha\|_{D_l} \le C_l(1+t)^l\|\alpha\|_{D_l}, \quad l\ge 0.$$

Let $n \geq l$, then (4.124) gives

$$\|\frac{d^p}{dt^p}P_{-\varepsilon}(-i\partial)\lambda^{(\varepsilon)}(t)\|_{D_l} \tag{4.125}$$
$$\le C_l\sum_{0\le j\le n}\|\frac{d^{j+p}}{dt^{j+p}}q_\varepsilon(t)\|_{D_l} + C_l\int_t^\infty(1+s)^l\|\frac{d^{n+p+1}}{ds^{n+p+1}}q_\varepsilon(s)\|ds, \quad t\ge 0, p\ge 0, n\ge l\ge 0.$$

According to inequality (4.120) and definition (4.123) of q_ε we obtain from the last inequality with $n = l$, that:

c) if $(\beta_i^{(+)}, \beta_i^{(-)}) = 0$ for $i = 1$ or $i = 2$, then

$$\|\frac{d^p}{dt^p} P_{-\varepsilon}(-i\partial)\lambda^{(\varepsilon)}(t)\|_{D_l} \tag{4.126a}$$
$$\leq C_{l,p}(1+t)^{\rho-3/2-p-\chi_3}\|(f,\dot f)\|_{M_N^\rho}\|(\beta_3^{(+)},\beta_3^{(-)})\|_{\mathcal{D}^{0,\chi_3}_{p+l+1,N}},$$

d) if $(f, \dot f) = 0$, then

$$\|\frac{d^p}{dt^p} P_{-\varepsilon}(-i\partial)\lambda^{(\varepsilon)}(t)\|_{D_l} \tag{4.126b}$$
$$\leq C_{l,p}(1+t)^{-1-p-\chi}\prod_{1\leq i\leq 3}\|(\beta_i^{(+)},\beta_i^{(-)})\|_{\mathcal{D}^{0,\chi_i}_{p+l+1,N}},$$

where $t \geq 0$, $\chi = \chi_1 + \chi_2 + \chi_3$ and N depends on $p \geq 0$ and $l \geq 0$.

It follows from (4.122a) and (4.126a) that if $(\beta_i^{(+)}, \beta_i^{(-)}) = 0$ for $i = 1$ or $i = 2$, then

$$\sum_{\varepsilon=\pm}\sum_{0\leq p\leq n}\Big(\sup_{t\geq 0}\big((1+t)^{1/2-\rho+d+p+\chi_3}\|\frac{d^p}{dt^p}P_\varepsilon(-i\partial)\lambda^{(\varepsilon)}(t)\|_{D_l}\big) \tag{4.127a}$$
$$+\sup_{t\geq 0}\big((1+t)^{3/2-\rho+p+\chi_3}\|\frac{d^p}{dt^p}P_{-\varepsilon}(-i\partial)\lambda^{(\varepsilon)}(t)\|_{D_l}\big)\Big)$$
$$\leq C_{l,n}\|(f,\dot f)\|_{M_N^\rho}\|(\beta_3^{(+)},\beta_3^{(-)})\|_{\mathcal{D}^{d,\chi_3}_{n+l+1,N}},\quad \chi_3 + d > \rho - 1/2,$$

where N depends on $n \geq 0$ and $l \geq 0$. This proves statement i) of the proposition.

It follows from (4.122b) and (4.126b) that, if $(f, \dot f) = 0$, then

$$\sum_{\varepsilon=\pm}\sum_{0\leq p\leq n}\Big(\sup_{t\geq 0}\big((1+t)^{d+p+\chi_3-\delta}\|\frac{d^p}{dt^p}P_\varepsilon(-i\partial)\lambda^{(\varepsilon)}(t)\|_{D_l}\big) \tag{4.127b}$$
$$+\sup_{t\geq 0}\big((1+t)^{1+p+\chi_3-\delta}\|\frac{d^p}{dt^p}P_{-\varepsilon}(-i\partial)\lambda^{(\varepsilon)}(t)\|_{D_l}\big)\Big)$$
$$\leq C_{l,n}\|(\beta_1^{(+)},\beta_1^{(-)})\|_{\mathcal{D}^{0,\chi_1}_{l+n+1,N}}\|(\beta_2^{(+)},\beta_2^{(-)})\|_{\mathcal{D}^{0,\chi_2}_{l+n+1,N}}\|(\beta_3^{(+)},\beta_3^{(-)})\|_{\mathcal{D}^{d,\chi_3}_{l+n+1,N}},$$

where $\chi + d > 0$ and N depends on $n \geq 0$ and $l \geq 0$. This proves statement ii) of the proposition.

As in [8] (see formula (3.34) in [8]) we can now construct an approximate solution of the M-D equations by iterating the functional $\lambda = (\lambda^{(+)}, \lambda^{(-)})$ a finite number of times. More precisely, let $(f, \dot f) \in M_\infty^{\circ\rho}$, $1/2 < \rho < 1$, $\alpha \in D_\infty$ and let $\beta_n = (\beta_n^{(+)}, \beta_n^{(-)})$, $n \geq 0$, be the sequence given by

$$\beta_0^{(\varepsilon)}(t) = P_\varepsilon(-i\partial)\alpha,\quad \varepsilon = \pm, t \geq 0, \tag{4.128a}$$
$$\beta_{n+1}^{(\varepsilon)}(t) = \beta_0^{(\varepsilon)}(t) + (\lambda^{(\varepsilon)}((f,\dot f),\beta_n,\beta_n,\beta_n))(t),\quad \varepsilon = \pm, t\geq 0, \tag{4.128b}$$

where $\lambda^{(\varepsilon)}$ is given by (4.91). We remind that $M_\infty^{\circ\rho}$ is the subspace of M_∞^ρ satisfying the gauge condition (4.87b).

Proposition 4.8. *Let $1/2 < \rho < 1$, $Y \in \Pi' \cap U(\mathbb{R}^4)$ and let the sequence (χ_n, d_n), $n \geq 0$, be defined by*

$$\chi_{2p} = 2p(1-\rho), \quad d_{2p} = 1 - 2p(1-\rho) \quad \textit{for } 2(p-1)(1-\rho) < \rho - 1/2, \textit{ and } p \geq 0,$$
$$\chi_{2p+1} = 3/2 - \rho, \quad d_{2p+1} = 2p(1-\rho) \quad \textit{for } 2p(1-\rho) < \rho - 1/2 \textit{ and } p \geq 0,$$

and $\chi_n = 3/2 - \rho$, $d_n = \rho - 1/2$ otherwise. This sequence satisfies $\chi_n \geq 0$, $0 \leq d_n \leq 1$ and $\chi_n + d_n > \rho - 1/2$. If $u = (f, \dot{f}, \alpha) \in E_\infty^{\circ\rho}$ then the element β_n, $n \geq 0$, in the sequence given by (4.128a) and (4.128b) defines a continuous polynomial $u \mapsto \beta_n$ from $E_\infty^{\circ\rho}$ to $\mathcal{D}_{\infty,\infty}^{3/2-\rho,0}$ which satisfies, with $n \geq 0$, $j \geq 0$, $l \geq 0$, $|Y| \geq 0$,

i) $\|\xi_Y \beta_n\|_{\mathcal{D}_{j,l}^{3/2-\rho,0}} \leq C(1 + \|u\|_{E_N^\rho})^N \|u\|_{E_{N+|Y|}^\rho}$,

ii) $\|\xi_Y(\beta_{n+1} - \beta_n)\|_{\mathcal{D}_{j,l}^{d_n,\chi_n}} \leq C(1 + \|u\|_{E_N^\rho})^N \|u\|_{E_N^\rho}^{n+1} \|u\|_{E_{N+|Y|}^\rho}$,

where C depends on j, l, n, ρ,$|Y|$, and N depends on j, l and n. Moreover if $\alpha = 0$, then $\beta_n = 0$ for $n \geq 0$.

Proof. We prove the two statements by induction. It follows from definition (4.90) of the norm in $\mathcal{D}_{j,l}^{d,\chi}$ and from definition (4.128a) of β_0, that

$$\|\xi_Y \beta_0\|_{\mathcal{D}_{j,l}^{3/2-\rho,0}} \leq \|\alpha\|_{D_{l+|Y|}} \leq \|u\|_{E_{l+|Y|}^\rho}, \quad Y \in \Pi' \cap U(\mathbb{R}^4) \tag{4.129}$$

since T_X^{D1} commutes with $P_\varepsilon(-i\partial)$, $X \in \mathfrak{p}$. This proves that statement i) is true for $n = 0$. Suppose it is true for $n \geq 0$. According to definition (4.128b) of β_{n+1} and according to (4.129) we have

$$\begin{aligned}\|\xi_Y \beta_{n+1}\|_{\mathcal{D}_{j,l}^{3/2-\rho,0}} &\leq \|\alpha\|_{D_{l+|Y|}} + \|\xi_Y \lambda((f,\dot{f}), \beta_n, \beta_n, \beta_n)\|_{\mathcal{D}_{j,l}^{3/2-\rho,0}} \\ &\leq \|u\|_{E_{l+|Y|}^\rho} + \|\xi_Y \lambda((f,\dot{f}), \beta_n, \beta_n, \beta_n)\|_{\mathcal{D}_{j,l}^{\rho-1/2, 2(1-\rho)}},\end{aligned}$$

since it follows from definition (4.90) that $\mathcal{D}_{j,l}^{d,\chi} \subset \mathcal{D}_{j,l}^{d',\chi'}$ (topologically) if $\chi \geq \chi'$ and $\chi + d \geq d' + \chi'$. It follows from the last inequality and from statements i) and ii) of Proposition 4.7, since $3/2 - \rho > \rho - 1/2 > 0$, that

$$\begin{aligned}&\|\xi_Y \beta_{n+1}\|_{\mathcal{D}_{j,l}^{3/2-\rho,0}} \\ &\quad\leq \|u\|_{E_{l+|Y|}^\rho} + C' \sum_{|Y_1|+|Y_2| \leq |Y|} \|(f,\dot{f})\|_{M_{N'+|Y_1|}^\rho} \|\xi_{Y_2}\beta_n\|_{\mathcal{D}_{L,N'}^{3/2-\rho,0}} \\ &\qquad + C' \sum_{|Y_1|+|Y_2|+|Y_3| \leq |Y|} \|\xi_{Y_1}\beta_n\|_{\mathcal{D}_{L,N'}^{0,0}} \|\xi_{Y_2}\beta_n\|_{\mathcal{D}_{L,N'}^{0,0}} \|\xi_{Y_3}\beta_n\|_{\mathcal{D}_{L,N'}^{3/2-\rho,0}} \\ &\quad\leq \|u\|_{E_{N'+|Y|}^\rho} + C' \sum_{|Y_1|+|Y_2| \leq |Y|} \|u\|_{E_{N'+|Y_1|}^\rho} \|\xi_{Y_2}\beta_n\|_{\mathcal{D}_{L,N'}^{3/2-\rho,0}} \\ &\qquad + C' \sum_{|Y_1|+|Y_2|+|Y_3| \leq |Y|} \|\xi_{Y_1}\beta_n\|_{\mathcal{D}_{L,N'}^{3/2-\rho,0}} \|\xi_{Y_2}\beta_n\|_{\mathcal{D}_{L,N'}^{3/2-\rho,0}} \|\xi_{Y_3}\beta_n\|_{\mathcal{D}_{L,N'}^{3/2-\rho,0}},\end{aligned} \tag{4.130}$$

where the summation is over $Y_j \in \Pi' \cap U(\mathbb{R}^4)$, where L and N' depend on j, l, n and where C' depends on j, l, n, ρ. It now follows from the induction hypothesis and from (4.130) with L and N' instead of j and L and, by choosing N'' sufficiently large, that

$$\begin{aligned}\|\beta_{n+1}\|_{\mathcal{D}^{3/2-l,0}_{j,l}} &\leq \|\alpha\|_{D_{N'}} + C'\big(\|u\|_{E^\rho_{N'}} + C^2(1+\|u\|_{E^\rho_N})^{2N}\|\alpha\|_{D_N}\big)C(1+\|u\|_{E^\rho_N})^N\|\alpha\|_{D_N}\\ &\leq C''(1+\|u\|_{E^\rho_{N''}})^{N''}\|\alpha\|_{D_{N''}}.\end{aligned}$$

This proves statement i) of the proposition.

To prove statement ii) of the proposition we first observe that according to definition (4.128a) and (4.128b), we have that

$$\|\beta_1-\beta_0\|_{\mathcal{D}^{\rho-1/2,\chi_1}_{j,l}} = \|\lambda((f,\dot f),\beta_0,\beta_0,\beta_0)\|_{\mathcal{D}^{\rho-1/2,\chi_1}_{j,l}}.$$

Statements i) and ii) of Proposition 4.7 then give that

$$\|\beta_1-\beta_0\|_{\mathcal{D}^{\rho-1/2,\chi_1}_{j,l}} \leq C\big(\|(f,\dot f)\|_{M^\rho_N} + \|\beta_0\|^2_{\mathcal{D}^{0,0}_{L,N}}\big)\|\beta_0\|_{\mathcal{D}^{3/2-\rho,0}_{L,N}},$$

where C depends on j, l, ρ and, L and N depend on j and l. This proves together with (4.129) and the definition of the norm in E^ρ_N that

$$\|\beta_1-\beta_0\|_{\mathcal{D}^{\rho-1/2,\chi_1}_{j,l}} \leq C(1+\|u\|_{E_N})\|u\|_{E_N}\|\alpha\|_{D_N}, \quad j,l\geq 0,$$

where C depends on ρ, j and l and N depend on j and l. This proves statement ii) for $n=0$. To prove statement ii) for $n\geq 1$, we observe that for $g=(f,\dot f)$, it follows from the definition of β_n and from definition (4.91) of λ that

$$\begin{aligned}\beta_{n+1}-\beta_n &= \lambda(g,\beta_n,\beta_n,\beta_n) - \lambda(g,\beta_{n-1},\beta_{n-1},\beta_{n-1})\\ &= \lambda(g,0,0,\beta_n-\beta_{n-1}) + \lambda(0,\beta_n-\beta_{n-1},\beta_n,\beta_n)\\ &\quad + \lambda(0,\beta_{n-1},\beta_n-\beta_{n-1},\beta_n) + \lambda(0,\beta_{n-1},\beta_{n-1},\beta_n-\beta_{n-1}), \quad n\geq 1.\end{aligned} \tag{4.131}$$

It follows from Proposition 4.7 and from (4.131) that

$$\begin{aligned}\|\beta_{n+1}-\beta_n\|_{\mathcal{D}^{d_{n+1},\chi_{n+1}}_{j,l}} \leq C_1\Big(&\|(f,\dot f)\|_{M^\rho_N}\|\beta_n-\beta_{n-1}\|_{\mathcal{D}^{d_n,\chi_n}_{L,N}}\\ &+ \|\beta_n-\beta_{n-1}\|_{\mathcal{D}^{0,\chi_n}_{L,N}}\|\beta_n\|_{\mathcal{D}^{0,0}_{L,N}}\|\beta_n\|_{\mathcal{D}^{d_n,0}_{L,N}}\\ &+ \|\beta_{n-1}\|_{\mathcal{D}^{0,0}_{L,N}}\|\beta_n-\beta_{n-1}\|_{\mathcal{D}^{0,\chi_n}_{L,N}}\|\beta_n\|_{\mathcal{D}^{d_n,0}_{L,N}}\\ &+ \|\beta_{n-1}\|_{\mathcal{D}^{0,0}_{L,N}}\|\beta_{n-1}\|_{\mathcal{D}^{0,0}_{L,N}}\|\beta_n-\beta_{n-1}\|_{\mathcal{D}^{d_n,\chi_n}_{L,N}}\Big), \quad n\geq 1.\end{aligned} \tag{4.132}$$

As a matter of fact, the hypothesis of statements i) and ii) of Proposition 4.7 are satisfied since $d_n+\chi_n > \rho-1/2>0$ for $n\geq 0$. Moreover $d_{n+1}=1-d_n$ and $\chi_{n+1}=1/2-\rho+$

$\chi_n + d_n$ in agreement with statement i) of Proposition 4.7 and χ', given by statement ii) of Proposition 4.7, satisfies $\chi' \geq \chi_{n+1}$. Inequality (4.132) now follows from the topological inclusion

$$\mathcal{D}^{d',\chi'}_{j,l} \subset \mathcal{D}^{d,\chi}_{j,l}, \quad \chi' \geq \chi, d' + \chi' \geq d + \chi, j \geq 0, l \geq 0.$$

Since $d_n \geq 3/2 - \rho$ for $n \geq 0$, inequality (4.132) and this topological inclusion give

$$\begin{aligned} &\|\beta_{n+1} - \beta_n\|_{\mathcal{D}^{d_{n+1},\chi_{n+1}}_{j,l}} \\ &\quad \leq C_2\big(\|(f,\dot f)\|_{M^\rho_N} + \|\beta_n\|^2_{\mathcal{D}^{3/2-\rho,0}_{L,N}} + \|\beta_{n-1}\|^2_{\mathcal{D}^{3/2-\rho,0}_{L,N}}\big)\|\beta_n - \beta_{n-1}\|_{\mathcal{D}^{d_n,\chi_n}_{L,N}}, \end{aligned} \tag{4.133}$$

$j \geq 0$, $l \geq 0$, $n \geq 1$, where C_2 depends on j, l, ρ and, L and N depends on j, l. This inequality and statement i) of the proposition give that

$$\begin{aligned} &\|\beta_{n+1} - \beta_n\|_{\mathcal{D}^{d_{n+1},\chi_{n+1}}_{j,l}} \\ &\quad \leq C'(1 + \|u\|_{E^\rho_{N'}})^{N'}\|u\|_{E^\rho_{N'}}\|\beta_n - \beta_{n-1}\|_{\mathcal{D}^{d_n,\chi_n}_{L,N'}}, \quad n \geq 1, j \geq 0, l \geq 0, \end{aligned} \tag{4.134}$$

where C' depending on j, l, n, ρ and N' depending on j, l, n are sufficiently large. Since we already have proved that statement ii) of the proposition is true for $n = 0$, it follows from (4.134) by induction that it is true for every $n \geq 0$. This proves the proposition.

For $(f, \dot f, \alpha) \in E^{\circ\rho}_\infty$, $1/2 < \rho < 1$ and for the sequence β_n, $n \geq 0$, given by (4.128a) and (4.128b), we introduce (cf. (4.87a))

$$A_{0,\mu}t) = \cos((-\Delta)^{1/2}t)f_\mu + (-\Delta)^{-1/2}\sin((-\Delta)^{1/2})\dot f_\mu, \tag{4.135a}$$
$$A_{n+1,\mu}(t) = A_{0,\mu}(t) + \sum_{\varepsilon=\pm}(G_{\varepsilon,\mu}(\beta^{(\varepsilon)}_n, \beta^{(\varepsilon)}_n))(t), \quad 0 \leq \mu \leq 3, t \geq 0, n \geq 0,$$

where $G_{\varepsilon,\mu}$ is given by (4.1a), and we introduce

$$\phi_n(t) = e^{-i\vartheta(A_n,t)}\sum_{\varepsilon=\pm} e^{i\varepsilon\omega(-i\partial)t}\beta^{(\varepsilon)}_n(t), \quad n \geq 0, t \geq 0, \tag{4.135b}$$

where $(\vartheta(A_n,t))(x) = \vartheta(A_n,(t,x))$, $x \in \mathbb{R}^3$, with ϑ given by (1.19). For n sufficiently large, (A_n, ϕ_n) is an approximate solution of the M-D equations (in the sense that inequality of Theorem 4.11 is satisfied). To prove this, we need decay properties for $A_{n,\mu}(t)$ and $\phi_n(t)$ which are direct consequences of Lemma 4.1 and Proposition 4.8. Before stating the result, we introduce

$$\mathcal{R}^0_{N_0,L} = 0 \tag{4.136a}$$

and

$$\mathcal{R}^l_{N_0,L}(v_1, \ldots, v_l) = \sum_{1\leq j\leq l}\prod_{i\neq j}\|v_i\|_{E^\rho_{N_0}}\|v_j\|_{E^\rho_{N_0+L}}, \tag{4.136b}$$

where $l \geq 1$, $N_0 \geq 0$, $L \geq 0$ and $v_1, \ldots, v_l \in E^\rho_\infty$.

We also introduce $A_{n,Y}(u)$, $\dot{A}_{n,Y}(u)$, $\phi'_{n,Y}(u)$, for $n \geq 0$, $Y \in U(\mathfrak{p})$ and $u \in E^{\circ\rho}_{\infty}$, by

$$(A_{n,Y}(u))(t) = (\xi^M_Y A_n)(t), \quad (\dot{A}_{n,Y}(u))(t) = \frac{d}{dt}(\xi^M_Y A_n)(t), \tag{4.137a}$$
$$(\phi'_{n,Y}(u))(t) = (\xi^D_Y \phi'_n)(t), \quad \phi'_n(t) = e^{i\vartheta(A_n,t)}\phi_n(t),$$

where $A_n(t)$, $\phi_n(t)$ are given as functions of $u = (f, \dot{f}, \alpha) \in E^{\circ\rho}_{\infty}$ by (4.128a), (4.128b), (4.135a) and (4.135b). We note that $A_{n,Y}(u)$, $\dot{A}_{n,Y}(u)$ and $\phi'_{n,Y}(u)$ are polynomials in u and we note for future reference that

$$A_{n+1,\mu}(t) = A_{0,\mu}(t) - \int_t^\infty \frac{\sin(|\nabla|(t-s))}{|\nabla|} \sum_{\varepsilon=\pm}((\phi'^{\varepsilon}_n)^+\gamma^0\gamma_\mu\phi'^{\varepsilon}_n)(s)ds \tag{4.137b}$$

and

$$\phi'^{\varepsilon}_{n+1}(t) = \phi'^{\varepsilon}_0(t) + i\int_t^\infty e^{(t-s)\mathcal{D}}((A_{n,\mu} + B_{n,\mu})\gamma^0\gamma^\mu\phi'^{\varepsilon}_n)(s)ds, \tag{4.137c}$$

where $\phi'_n = \sum_\varepsilon \phi'^{\varepsilon}_n$, $\phi'_0(t) = e^{t\mathcal{D}}\alpha$, $\phi'^{\varepsilon}_0(t) = P_\varepsilon(-i\partial)\phi'_0(t)$,

$$A_{0,\mu}(t) = \cos((-\Delta)^{1/2}t)f_\mu + (-\Delta)^{1/2}\sin((-\Delta)^{1/2}t)\dot{f}_\mu, \quad u = (f, \dot{f}, \alpha),$$

and $B_{n,\mu}(y) = -\partial_\mu\vartheta(A_{n,y})$.

Theorem 4.9. *If $n \geq 0$ and $1/2 < \rho < 1$, then there exists $N_0 \geq 0$ such that*

$$\begin{aligned}
&\sup_{t\geq 0}\|\big(D^l(A_{n,Y}, \dot{A}_{n,Y}, \phi'_{n,Y})(u; v_1, \ldots, v_l)\big)(t)\|_{E^\rho_0} \\
&\quad + \sup_{t\geq 0, x\in\mathbb{R}^3}\Big((1+t+|x|)^{3/2}|\big((D^l\phi'_{n,Y})(u; v_1, \ldots, v_l)\big)(t,x)| \\
&\quad + (1+t+|x|)^{3/2-\rho}|\big((D^l A_{n,Y})(u; v_1, \ldots, v_l)\big)(t,x)| \\
&\quad + (1+t+|x|)(1+\big|t-|x|\big|)^{3/2-\rho}|\big((D^l A_{n,P_\mu Y})(u; v_1, \ldots, v_l)\big)(t,x)|\Big) \\
&\qquad \leq F_{L,l}(\|u\|_{E^\rho_{N_0}})\mathcal{R}^l_{N_0,L}(v_1, \ldots, v_l) + F'_{L,l}(\|u\|_{E^\rho_{N_0}})\|u\|_{E^\rho_{N_0+L}}\|v_1\|_{E^\rho_{N_0}} \cdots \|v_l\|_{E^\rho_{N_0}},
\end{aligned}$$

for every $L \geq 0$, $l \geq 0$, $Y \in \Pi'$, $|Y| \leq L$, $u, v_1, \ldots, v_l \in E^{\circ\rho}_{\infty}$, where $F_{L,l}$ and $F'_{L,l}$ are increasing polynomials on $[0, \infty[$. Moreover $(A_{n,Y}, \dot{A}_{n,Y}, \phi'_{n,Y})$ is a polynomial on $E^{\circ\rho}_{\infty}$ with vanishing constant term.

Proof. This follows from Lemma 4.1, Proposition 4.8, definitions (4.135a), (4.135b) of $A_{n,\mu}(t)$ and $\phi_n(t)$, and from the covariance of the function $u \mapsto (A_{n,\mathbb{I}}(u), \dot{A}_{n,\mathbb{I}}(u), \phi'_{n,\mathbb{I}}(u))$ from E^ρ_∞ to E^ρ_0 under the Lorentz subgroup $SL(2,\mathbb{C})$ of $\mathcal{P}_0$, i.e. $(A_{n,\mathbb{I}}(U^1_g(u))(y) = \Lambda_L A_{n,\mathbb{I}}(\Lambda_L^{-1}(y-a))$, $g = (a, L) \in \mathcal{P}_0$, and similarly for $\phi'_{n,\mathbb{I}}(u)$.

Theorem 4.10. *If $1/2 < \rho < 1$, $0 \leq \rho' \leq 1$, and $n \geq 0$ is such that $\rho' - 1/2 + \chi_n > 0$, where χ_n is given in Proposition 4.8, then there exists N_0 such that*

$$\sup_{t\geq 0}\Big((1+t)^{\rho'-1/2+\chi_n}\|((D^l(A_{n+1,Y}-A_{n,Y},A_{n+1,P_0Y}-A_{n,P_0Y}))(u;v_1,\ldots,v_l))(t)\|_{M^{\rho'}}$$
$$+(1+t)^{\chi_{n+1}}\|\big((D^l(\phi'_{n+1,Y}-\phi'_{n,Y}))(u;v_1,\ldots,v_l)\big)(t)\|_D\Big)$$
$$\leq F_{L,l}(\|u\|_{E^\rho_{N_0}})\mathcal{R}^l_{N_0,L}(v_1,\ldots,v_l)+F'_{L,l}(\|u\|_{E^\rho_{N_0}})\|u\|_{E^\rho_{N_0+L}}\|v_1\|_{E^\rho_{N_0}}\cdots\|v_l\|_{E^\rho_{N_0}},$$

for all $L \geq 0$, $l \geq 0$, $Y \in \Pi'$, $|Y| \leq L$, $u, v_1, \ldots, v_l \in E^{\circ\rho}_\infty$ and there exists N depending on $\alpha \in \mathbb{N}^4$ such that

$$|\big((D^l(A_{n+1,P^\alpha Y}-A_{n,P^\alpha Y}))(u;v_1,\ldots,v_l)\big)(t,x)|(1+t+|x|)^{1+\chi_n+|\alpha|-\varepsilon}(1+t)^\varepsilon$$
$$+|\big((D^l(\phi'_{n+1,Y}-\phi'_{n,Y}))(u;v_1,\ldots,v_l)\big)(t,x)|(1+t+|x|)^{3/2+\chi_{n+1}}$$
$$\leq G_{L,l,\varepsilon}(\|u\|_{E^\rho_N})\mathcal{R}^l_{N,L}(v_1,\ldots,v_l)+G'_{L,l,\varepsilon}(\|u\|_{E^\rho_N})\|u\|_{E^\rho_{N+L}}\|v_1\|_{E^\rho_N}\cdots\|v_l\|_{E^\rho_N},$$

for all $t \geq 0$, $x \in \mathbb{R}^3$, $l \geq 0$, $Y \in \Pi'$, $\alpha \in \mathbb{N}^4$, $|Y|+|\alpha| \leq L$, $\varepsilon > 0$, $u, v_1, \ldots, v_l \in E^{\circ\rho}_\infty$. Here $F_{L,l}$, $F'_{L,l}$, $G_{L,l,\varepsilon}$ and $G'_{L,l,\varepsilon}$ are increasing polynomials on $[0,\infty[$ and $P^\alpha = P^{\alpha_0}P^{\alpha_1}P^{\alpha_2}P^{\alpha_3}$. Moreover $A_{n+1,Y} - A_{n,Y}$ and $\phi'_{n+1,Y} - \phi'_{n,Y}$ are polynomials on $E^{\circ\rho}_\infty$ with constant and linear term vanishing.

Proof. The result follows as in the proof of Theorem 4.9.

We are now ready to prove that $t \mapsto (A_n(t), \phi_n(t))$ where $A_n(t)$ and $\phi_n(t)$ are given by (4.135a) and (4.135b), is an approximate solution of the M-D equations (1.2a) and (1.2b) for n sufficiently large. We introduce for $u = (f, \dot{f}, \alpha) \in E^{\circ\rho}_\infty$:

$$\Delta^M_{n,\mu}(t) = \cos(|\nabla|t)f_\mu + |\nabla|^{-1}\sin(|\nabla|t)\dot{f}_\mu - \int_t^\infty |\nabla|^{-1}\sin(|\nabla|(t-s))\overline{\phi'_n(s)}\gamma_\mu\phi'_n(s)ds - A_{n,\mu}(t) \tag{4.138a}$$

and

$$\Delta^D_n(t) = e^{i\mathcal{D}t}\alpha + i\int_t^\infty e^{i\mathcal{D}(t-s)}\gamma^0\gamma^\mu\big(A_{n,\mu}(s)+B_{n,\mu}(s)\big)\phi'_n(s)ds - \phi'_n(t), \tag{4.138b}$$

where $n \geq 0$, $t \geq 0$, $B_{n,i}(t) = -\partial_i\vartheta(A_n,t)$, $1 \leq i \leq 3$, and $B_{n,0}(t) = -\frac{d}{dt}\vartheta(A_n,t)$. Similarly to (4.137), we introduce $\Delta^M_{n,Y}(u)$, $\Delta^D_{n,Y}(u)$ for $u \in E^{\circ\rho}_\infty$ and $Y \in U(\mathfrak{p})$ by

$$\begin{aligned}(\Delta^M_{n,Y}(u))_\mu(t) &= (\xi^M_Y\Delta_n)_\mu(t),\\ (\Delta^D_{n,Y}(u))(t) &= (\xi^D_Y\Delta^D_n)(t).\end{aligned} \tag{4.139}$$

Theorem 4.11. *If $1/2 < \rho < 1$ and $n \geq n_0$, where n_0 is the integer part of $3+1/(2(1-\rho))$, then there exists N_0 such that*

$$(1+t)^{1-\rho}\|\big((D^l\Delta^M_{n,Y})(u;v_1,\ldots,v_l)\big)(t)\|_{L^2(\mathbb{R}^3,\mathbb{R}^4)}$$

$$
\begin{aligned}
&+ (1+t)^{2-\rho} \| \big((D^l \Delta^M_{n,P_\mu Y})(u; v_1, \ldots, v_l) \big)(t) \|_{L^2(\mathbb{R}^3,\mathbb{R}^4)} \\
&+ (1+t)^{3/2-\rho} \| \big((D^l \Delta^D_{n,Y})(u; v_1, \ldots, v_l) \big)(t) \|_D \\
&+ (1+t+|x|)^{5/2-\rho} | \big((D^l \Delta^M_{n,Y})(u; v_1, \ldots, v_l) \big)(t,x) | \\
&+ (1+t+|x|)^{7/2-\rho} | \big((D^l \Delta^M_{n,P_\mu Y})(u; v_1, \ldots, v_l) \big)(t,x) | \\
&+ (1+t+|x|)^{3-\rho} | \big((D^l \Delta^D_{n,Y})(u; v_1, \ldots, v_l) \big)(t,x) | \\
&\quad \leq F_{L,l}(\|u\|_{E^\rho_{N_0}}) \mathcal{R}^l_{N_0,L}(v_1, \ldots, v_l) + F'_{L,l}(\|u\|_{E^\rho_{N_0}}) \|u\|_{E^\rho_{N_0+L}} \|v_1\|_{E^\rho_{N_0}} \cdots \|v_l\|_{E^\rho_{N_0}},
\end{aligned}
$$

for all $L \geq 0$, $l \geq 0$, $Y \in \Pi'$, $|Y| \leq L$, $u, v_1, \ldots, v_l \in E^{\circ\rho}_\infty$, where $F_{L,l}$ and $F'_{L,l}$ are increasing polynomials on $[0, \infty[$. Moreover $\Delta^M_{n,Y}$ and $\Delta^D_{n,Y}$ are polynomials on $E^{\circ\rho}_\infty$ with constant and linear terms vanishing.

Proof. By the construction of $\Delta^M_{n,\mu}(t)$ and $\Delta^D_n(t)$ and by the definition of $A_{n,\mu}(t)$, $B_{n,\mu}(t)$ and $\phi'_n(t)$ we obtain

$$
\begin{aligned}
\Delta^M_{n,\mu}(t) &= A_{n+1,\mu}(t) - A_{n,\mu}(t) \\
&\quad - \int_t^\infty |\nabla|^{-1} \sin((t-s)|\nabla|) \sum_{\varepsilon=\pm} (e^{-i\varepsilon\omega(-i\partial)s} \beta_n^{(-\varepsilon)}(s))^+ \gamma^0 \gamma_\mu (e^{i\varepsilon\omega(-i\partial)s} \beta_n^{(\varepsilon)}(s)) ds
\end{aligned}
\tag{4.140a}
$$

and

$$
\Delta^D_{n,\mu}(t) = \phi'_{n+1}(t) - \phi'_n(t). \tag{4.140b}
$$

We set $\varepsilon_1 = -\varepsilon$, $\varepsilon_2 = \varepsilon$, $m_1 = m_2 = m$, $f_1(t) = \beta_n^{(-\varepsilon)}(t)$, $f_2(t) = \beta_n^{(\varepsilon)}(t)$ in definitions (4.48a) and (4.48c) of $(H_\mu(f_1, f_2))(t)$, which we denote by $(H_n^{(\varepsilon)}(\phi_n'^{(-\varepsilon)}, \phi_n'^{(\varepsilon)}))_\mu(t)$, where $\phi_n'^{(\varepsilon)}(t) = e^{i\varepsilon\omega(-i\partial)} \beta_n^{(\varepsilon)}(t)$. (4.140a) then gives

$$
\Delta^M_{n,\mu}(t) = \sum_{\varepsilon=\pm} (H_n(\phi_n'^{(-\varepsilon)}, \phi_n'^{(\varepsilon)}))_\mu(t) + A_{n+1,\mu}(t) - A_{n,\mu}(t). \tag{4.141}
$$

The theorem now follows from applications of Lemma 4.3 to each term

$$
H_n(\xi^D_{Y_1} \phi_n'^{(-\varepsilon)}, \xi^D_{Y_2} \phi_n'^{(\varepsilon)}), \quad Y_1, Y_2 \in \Pi', |Y_1| + |Y_2| = |Y|, \tag{4.142}
$$

in the expansion of $\xi^D_Y \Delta^M_n$ and from Theorem 4.9 and Theorem 4.10.

Let $B_{n,0}(t) = -\frac{d}{dt}\vartheta(A_n, t), B_{n,i}(t) = -\partial_i \vartheta(A_n, t)$, $1 \leq i \leq 3$, $n \geq 0$. We introduce for $Y \in U(\mathfrak{p})$ and $u \in E^{\circ\rho}_\infty$,

$$
\begin{aligned}
&\big(\vartheta_{n,Y}(u)\big)(t) = \xi_Y \vartheta\big(A_n, (t,x)\big), \\
&(B_{n,Y}(u))(t) = (\xi^M_Y B_n)(t), \quad (\dot{B}_{n,Y}(u))(t) = \frac{d}{dt}(\xi^M_Y B_n)(t).
\end{aligned}
$$

Corollary 4.12. *If $n \geq 0$ and $1/2 < \rho < 1$, then there exists $N_0 \geq 0$ such that*

$$\begin{aligned}
&\sup_{t\geq 0, x\in\mathbb{R}^3} (1+|x|+t)^{1/2-\rho} \big| \big((D^l \vartheta_{n,Y})(u; v_1, \ldots, v_l) \big)(t,x) \big| \\
&\quad + \sup_{t\geq 0} \| (D^l (B_{n,Y}, \dot{B}_{n,Y})(u; v_1, \ldots, v_l))(t) \|_{M^\rho} \\
&\quad + \sup_{t\geq 0, x\in\mathbb{R}^3} ((1+t+|x|)^{3/2-\rho} \big| \big((D^l B_{n,Y})(u; v_1, \ldots, v_l) \big)(t,x) \big| \\
&\quad + (1+t+|x|)(1+\big|t-|x|\big|)^{3/2-\rho} \big| \big((D^l B_{n,P_\mu Y})(u; v_1, \ldots, v_l) \big)(t,x) \big|) \\
&\qquad \leq F_{L,l}(\|u\|_{E^\rho_{N_0}}) \mathcal{R}^l_{N_0,L}(v_1, \ldots, v_l) + F'_{L,l}(\|u\|_{E^\rho_{N_0}}) \|u\|_{E^\rho_{N_0+L}} \|v_1\|_{E^\rho_{N_0}} \cdots \|v_l\|_{E^\rho_{N_0}},
\end{aligned}$$

for every $L \geq 0$, $l \geq 0$, $Y \in \Pi'$, $|Y| \leq L$, $u, v_1, \ldots, v_l \in E^{\circ\rho}_\infty$, where $F_{L,l}$ and $F'_{L,l}$ are increasing polynomials on $[0,\infty[$. Moreover $(B_{n,Y}, \dot{B}_{n,Y})$ is a polynomial on $E^{\circ\rho}_\infty$ with vanishing constant term.

Proof. This is a direct consequence of Corollary 4.6 and Theorem 4.9.

Corollary 4.13. *If $n \geq 1$ and $1/2 < \rho < 1$, then there exists N_0 such that*

$$\sup_{t\geq 0} (1+t)^{k(\rho)} \big\| \big((A_{n,Y}, \dot{A}_{n,Y}, \phi'_{n,Y}) - (A_{0,Y}, \dot{A}_{0,Y}, \phi'_{0,Y}) \big)(t) \big\|_{E^\rho_0} \leq F_L(\|u\|_{E^\rho_{N_0}}) \|u\|_{E^\rho_{N_0+L}}$$

for every $L \geq 0$, $Y \in \Pi'$, $|Y| \leq L$ and $u \in E^{\circ\rho}_\infty$, where F_L are increasing polynomials on $[0,\infty[$ and $k(\rho) = \min(\rho - 1/2, 2 - 2\rho)$.

Proof. This follows from Theorem 4.10 taking $\rho' = \rho$.

5. Energy estimates and $L^2 - L^\infty$ estimates for the Dirac field.

In this chapter we shall derive various new $L^2 - L^2$ estimates for the inhomogeneous Dirac equation in $\mathbb{R}^+ \times \mathbb{R}^3$,

$$(i\gamma^\mu \partial_\mu + m)h - \gamma^\mu G_\mu h = g \tag{5.1a}$$

for different choices of $g \in C^0(\mathbb{R}^+, D), D = L^2(\mathbb{R}^3, \mathbb{C}^4)$, and of electromagnetic potential G. To use these estimates we shall also derive new $L^2 - L^\infty$ estimates. We shall suppose that the initial data $h(t_0)$ at $t_0 \in \mathbb{R}^+$ satisfies for some $\tau \in \mathbb{Z}$

$$h(t_0) \in (1-\Delta)^{-\tau/2} D, \tag{5.1b}$$

where $(1-\Delta)^{-\tau/2}D$ is the Hilbert space of tempered distributions f such that $(1-\Delta)^{\tau/2} f \in D$. We shall also consider the case $h(t_0) = 0$ where $t_0 = \infty$ or more precisely $\lim\limits_{t\to\infty} \|h(t)\|_\tau \to 0$, where $\|\cdot\|_\tau$ is the norm in $(1-\Delta)^{-\tau/2}D$. We shall suppose that

$$g \in C^0(\mathbb{R}^+, (1-\Delta)^{-\tau/2} D) \tag{5.1c}$$

and that the electromagnetic potential G satisfies

$$(G, 0) \in C^0(\mathbb{R}^+, (1-\Delta)^{-(|\tau|+1)/2} M^1). \tag{5.1d}$$

It follows from inequality (2.61) and from the definition of M^1 that $G \in C^0(\mathbb{R}^+, W_{|\tau|+1,6}$ $(\mathbb{R}^3, \mathbb{R}^4))$ and then that $G \in C^0(\mathbb{R}^+, W_{|\tau|,\infty}(\mathbb{R}^3, \mathbb{R}^4))$ since $W_{|\tau|+1,6}(\mathbb{R}^3) \subset W_{|\tau|,\infty}(\mathbb{R}^3)$. If $B(t) = -i\gamma^0\gamma^\mu G_\mu(t)$, we therefore obtain that $B \in C^0(\mathbb{R}^+, L_b((1-\Delta)^{-n/2}D))$, $n \in \mathbb{N}$, $n \leq |\tau|$, where $L_b(X)$ is the linear space, endowed with the norm topology, of linear continuous operators on a Banach space X. By duality we also have $B \in C^0(\mathbb{R}^+, L_b((1-\Delta)^{n/2}D))$, $n \in \mathbb{N}$, $n \leq |\tau|$. We define the operator $\mathcal{L}(t)$ in $(1-\Delta)^{-\tau/2}D$ with domain $(1-\Delta)^{-(\tau+1)/2}D$ by

$$\mathcal{L}(t) = \mathcal{D} - i\gamma^0\gamma^\mu G_\mu(t), \quad t \geq 0, \tag{5.2}$$

Since $B \in C^0(\mathbb{R}^+, L_b((1-\Delta)^{-\tau/2}D))$ and $\mathcal{D} \in L((1-\Delta)^{-(\tau+1)/2}D, (1-\Delta)^{-\tau/2}D)$ it follows that $\mathcal{L} \in C^0(\mathbb{R}^+, L_b((1-\Delta)^{-(\tau+1)/2}D, (1-\Delta)^{-\tau/2}D))$. Moreover if

$$\widetilde{\Delta}(t) = (1-\Delta)^{1/2}\mathcal{L}(t)(1-\Delta)^{-1/2} - \mathcal{L}(t),$$

then $\widetilde{\Delta} \in C^0(\mathbb{R}^+, L_b((1-\Delta)^{-\tau/2}D))$. In fact using lemma A.3 of [16] it follows for $\tau \geq 0$ that the norm of the operators $\widetilde{\Delta}(t)$ (resp. $\widetilde{\Delta}(t+\varepsilon) - \widetilde{\Delta}(t)$) are bounded by

$$C\|(1-\Delta)^{(\tau+1/2)/2}|\nabla|G(t)\|_{L^2} \quad (\text{resp. } C\|(1-\Delta)^{(\tau+1/2)/2}|\nabla|(G(t+\varepsilon) - G(t))\|_{L^2}).$$

If $\tau \leq 0$, then considering the transposed of $\widetilde{\Delta}(t)$, we obtain the above bounds with $\tau + 1/2$ being replaced by $-\tau - 1/2$. Thus the statement follows from condition (5.1d). It now follows from theorem 1 of [15] that there exists a strongly continuous evolution

operator $w(s,s')$, $s,s'\geq 0$, in $(1-\Delta)^{-\tau/2}D$. Moreover the function $(s,s')\mapsto w(s,s')\alpha\in(1-\Delta)^{-\tau/2}D$ is C^1 for $\alpha\in(1-\Delta)^{-(\tau+1)/2}D$ and

$$w(s,s)=I, \qquad w(s,s')w(s',s'')=w(s,s''), \tag{5.3a}$$

$$\frac{d}{ds}w(s,s')=\mathcal{L}(s)w(s,s'), \qquad \frac{d}{ds'}w(s,s')=-w(s,s')\mathcal{L}(s'), \tag{5.3b}$$

where equalities (5.3a) are defined on $(1-\Delta)^{-\tau/2}D$ and equalities (5.3b) on $(1-\Delta)^{-(\tau+1)/2}D$. Since $\mathcal{L}(t)$ is kew-adjoint on D with domain $(1-\Delta)^{-1/2}D$, it follows that $w(s,s')$ is unitary on D. The unique solution $h\in C^1(\mathbb{R}^+,(1-\Delta)^{-\tau/2}D)$ of equation (5.1a) with electromagnetic potential G satisfying (5.1d) and satisfying, with τ replaced by $\tau+1$, conditions (5.1b) and (5.1c), is given by

$$h(t)=w(t,t_0)h(t_0)+\int_{t_0}^{t}w(t,s)(-i\gamma^0)g(s)ds, \quad t\geq 0. \tag{5.3c}$$

Let $h_n(t_0)\in(1-\Delta)^{-(\tau+1)/2}D$ (resp. $g_n\in C^0(\mathbb{R}^+,(1-\Delta)^{-(\tau+1)/2}D)), n\geq 0$, be a Cauchy sequence in $(1-\Delta)^{-\tau/2}D$ (resp. $C^0(\mathbb{R}^+,(1-\Delta)^{-\tau/2}D)$) converging to $h(t_0)$ (resp. g) and let $h_n\in C^1(\mathbb{R}^+,(1-\Delta)^{-\tau/2}D)$ be the corresponding sequence of solutions of (5.1a) given by (5.3c). It then follows from (5.1a) that

$$\|\frac{d}{dt}(h_{n_1}(t)-h_{n_2}(t))\|_{\tau-1}\leq C(t)\|h_{n_1}(t)-h_{n_2}(t)\|_\tau+\|g_{n_1}(t)-g_{n_2}(t)\|_{\tau-1}$$

and from (5.3c), since $w(s,s')$ is a bounded operator on $(1-\Delta)^{-\tau/2}D$, that

$$\|h_{n_1}(t)-h_{n_2}(t)\|_\tau\leq C'(t)\big(\|h_{n_1}(t_0)-h_{n_2}(t_0)\|_\tau+\sup_{0\leq s\leq t}\|g_{n_1}(s)-g_{n_2}(s)\|_\tau\big),$$

for some constants $C(t)$ and $C'(t)$. This shows that if conditions (5.1b), (5.1c) and (5.1d) are satisfied then equation (5.1a) has a unique solution $h\in C^0(\mathbb{R}^+,(1-\Delta)^{-\tau/2}D)$. This solution is given by (5.3c) and moreover $h\in C^1(\mathbb{R}^+,(1-\Delta)^{-(\tau-1)/2}D)$.

We are mainly interested in the particular case of equation (5.1a) given by

$$(i\gamma^\mu\partial_\mu+m)h-G_\mu\gamma^\mu h=F_\mu\gamma^\mu r, \quad t\geq 0, \tag{5.4a}$$

where r satisfies

$$(i\gamma^\mu\partial_\mu+m)r=q, \quad t\geq 0, \tag{5.4b}$$

and where F,r and q satisfy conditions such that (5.1c) is satisfied with $g=F_\mu\gamma^\mu r$. For example if $\tau=0$ then this is obviously the case if

$$\begin{aligned}&F\in W^{1,\infty}(\mathbb{R}^+\times\mathbb{R}^3,\mathbb{R}^4),\\ &r\in C^1(\mathbb{R}^+,D)\cap C^0(\mathbb{R}^+,(1-\Delta)^{-1/2}D),\\ &q\in C^0(\mathbb{R}^+,D).\end{aligned} \tag{5.5}$$

For this case we can obtain a much better energy estimate than

$$\|h(t)\|_D \leq \|h(t_0)\|_D + \int_{\min(t,t_0)}^{\max(t,t_0)} \|g(s)\|_D \, ds, \tag{5.6}$$

which follows from (5.3c) and the unitarity of $w(t,s)$ in D. We note that, using $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, one obtains

$$\begin{aligned}(m - i\gamma^\mu\partial_\mu + \gamma^\mu G_\mu)F_\nu\gamma^\nu r& \\ = \gamma^\nu F_\nu(m + i\gamma^\mu\partial_\mu - \gamma^\mu G_\mu)r - 2iF^\mu\partial_\mu r - ir\partial_\mu F^\mu& \\ - \frac{i}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)r(\partial_\mu F_\nu - \partial_\nu F_\mu) + 2G_\mu F^\mu r.&\end{aligned} \tag{5.7a}$$

We also note for later reference, that if $(i\gamma^\mu\partial_\mu + m - \gamma^\mu G_\mu)h = \gamma^\mu F_\mu r + g_1$, then

$$\begin{aligned}(i\gamma^\mu\partial_\mu + m - \gamma^\mu G_\mu)(\partial_\nu h + iG_\nu h + iF_\nu r)& \\ = \gamma^\mu h(\partial_\nu G_\mu - \partial_\mu G_\nu) + \gamma^\mu r(\partial_\nu F_\mu - \partial_\mu F_\nu) + \gamma^\mu F_\mu\partial_\nu r& \\ + i\gamma^\mu(G_\nu F_\mu - G_\mu F_\nu)r + iF_\nu(i\gamma^\mu\partial_\mu + m)r + \partial_\nu g_1 + iG_\nu g_1, \quad 0 \leq \nu \leq 3.&\end{aligned} \tag{5.7b}$$

This expression is useful for estimating L^2-norms of derivatives of h, because of the gauge invariance of the electromagnetic fields in the first two terms on the right-hand side. We also have that

$$\begin{aligned}(i\gamma^\mu\partial_\mu + m - \gamma^\mu G_\mu)(h + iF_\nu r)& \\ = \gamma^\mu r(\partial_\nu F_\mu - \partial_\mu F_\nu) + F_\nu(i\gamma^\mu\partial_\mu + m - \gamma^\mu G_\mu)r,&\end{aligned} \tag{5.7b$'$}$$

if $(i\gamma^\mu\partial_\mu + m - \gamma^\mu G_\mu)h = \gamma^\mu(\partial_\nu F_\mu)r$. Finally, we also note that if $y \in \mathbb{R}^+ \times \mathbb{R}^3$ and $\nu \in \{0,1,2,3\}$ is given, then

$$(a + y^\nu)G_\mu\partial^\mu = y^\mu G_\mu\partial^\nu + aG_\mu\partial^\mu + G_\mu(y^\nu\partial^\mu - y^\mu\partial^\nu), \tag{5.7c}$$

where $a \in \mathbb{R}$ and where the summation convention is used for μ. This gives that

$$\begin{aligned}\Big(1 + \sum_\nu |y^\nu|\Big)\big|\sum_\mu G_\mu\partial^\mu f\big|& \\ \leq C\Big(\big|\sum_\mu G_\mu\partial^\mu f\big| + \sum_\nu \big|\sum_\mu y^\mu G_\mu\partial^\nu f + \sum_\mu G_\mu(y^\nu\partial^\mu f - y^\mu\partial^\nu f)\big|\Big),&\end{aligned} \tag{5.7d}$$

where the sums are taken over $0 \leq \nu \leq 3, 0 \leq \mu \leq 3$.

Theorem 5.1. *Let* $Q = \gamma^\mu \partial_\mu + i\gamma^\mu G_\mu$, *(where the summation convention is used for* $0 \leq \mu \leq 3$*), let* $\dot{G}_\mu(t) = \frac{d}{dt} G_\mu(t)$ *and let* $t, t' \geq 0$.

ia) *If* $h \in C^1(\mathbb{R}^+, D) \cap C^0(\mathbb{R}^+, (1-\Delta)^{-1/2} D)$ *and* $(G, 0) \in C^0(\mathbb{R}^+, (1-\Delta)^{-1/2} M^1)$ *then*

$$\|h(t)\|_D^2 = \|h(t')\|_D^2 + \int_{t'}^t 2Re\big(h(s), -i\gamma^0((m+iQ)h)(s)\big)_D \, ds,$$

ib) *If* $h \in C^2(\mathbb{R}^+, D) \cap C^1(\mathbb{R}^+, (1-\Delta)^{-1/2} D) \cap C^0(\mathbb{R}^+, (1-\Delta) D)$ *and* $(G, \dot{G}) \in C^0(\mathbb{R}^+, (1-\Delta)^{1/2} M^1)$ *then*

$$\begin{aligned}\|h(t)\|_D^2 = \|h(t')\|_D^2 + m^{-1} Re\Big(&\big(h(t), ((m+iQ)h)(t)\big)_D \\ &- \big(h(t'), ((m+iQ)h)(t')\big)_D + \int_{t'}^t \big(h(s), -i\gamma^0((m^2+Q^2)h)(s)\big)_D \, ds\Big),\end{aligned}$$

iia) *If* $(G, 0) \in C^0(\mathbb{R}^+, (1-\Delta)^{-1/2} M^1), h(t_0) \in D, g \in C^0(\mathbb{R}^+, D)$ *and if* $h \in C^0(\mathbb{R}^+, D)$ *is the unique solution of equation (5.1a) then*

$$\big|\|h(t)\|_D - \|h(t')\|_D\big| \leq \int_t^{t'} \|g(s)\|_D \, ds, \qquad \text{for } 0 \leq t \leq t',$$

iib) *If* $(G, \dot{G}) \in C^0(\mathbb{R}^+, (1-\Delta)^{1/2} M^1), h(t_0) \in D, g = g^{(1)} + g^{(2)}, \ g^{(1)}, g^{(2)} \in C^0(\mathbb{R}^+, D)$, $\partial_\mu g^{(2)} \in L^1_{\text{loc}}(\mathbb{R}^+, D)$, $0 \leq \mu \leq 3$ *and if* $h \in C^0(\mathbb{R}^+, D)$ *is the unique solution of equation (5.1a) then*

$$\begin{aligned}&\big|\|h(t) - (2m)^{-1} g^{(2)}(t)\|_D - \|h(t') - (2m)^{-1} g^{(2)}(t')\|_D\big| \\ &\quad \leq \int_t^{t'} \|g^{(1)}(s) + (2m)^{-1}((m-iQ)g^{(2)})(s)\|_D \, ds, \qquad \text{for } \ 0 \leq t \leq t'.\end{aligned}$$

Proof. Since the operator $\mathcal{L}(t)$ defined in (5.2) is skew-adjoint on D with domain $(1-\Delta)^{-1/2} D$, when the hypothesis on G in ia) is satisfied, it follows that

$$\begin{aligned}\frac{d}{dt}\|h(t)\|_D^2 &= 2Re\big(h(t), (\frac{d}{dt} - \mathcal{L}(t))h(t)\big)_D \\ &= 2Re\big(h(t), -i\gamma^0((m+iQ)h)(t)\big)_D,\end{aligned}$$

for $h \in C^1(\mathbb{R}^+, D) \cap C^0(\mathbb{R}^+, (1-\Delta)^{-1/2} D)$. Integration of this equality from t' to t proves statement ia).

To prove the second statement introduce:

$$f = (m - iQ)h \quad \text{and} \quad g = (m+iQ)h. \tag{5.8}$$

Then $f, g \in C^1(\mathbb{R}^+, D) \cap C^0(\mathbb{R}^+, (1-\Delta)^{-1/2}D)$, according to the hypothesis of statement ib), since $G \in C^0(\mathbb{R}^+, W_{1,\infty}(\mathbb{R}^3, \mathbb{R}^4))$ which was proved after (5.1d) and since $\dot{G} \in C^0(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathbb{R}^4))$ by Sobolev embedding. It follows from statement ia) that

$$\|f(t)\|_D^2 = \|f(t')\|_D^2 + \int_{t'}^t 2Re\big(f(s), -i\gamma^0((m+iQ)f)(s)\big)_D\, ds. \tag{5.9}$$

Substitution of the expressions

$$f = 2mh - g \quad \text{and} \quad (m+iQ)f = (m-iQ)g, \tag{5.10}$$

which follows from (5.8), into the equality (5.9) gives

$$\begin{aligned} &4m^2\|h(t)\|_D^2 - 4mRe(h(t), g(t))_D + \|g(t)\|_D^2 \\ &\quad = 4m^2\|h(t')\|_D^2 - 4mRe(h(t'), g(t'))_D + \|g(t')\|_D^2 \\ &\qquad + \int_{t'}^t Re\big(4mh(s) - 2g(s), -i\gamma^0((m-iQ)g)(s)\big)_D\, ds. \end{aligned}$$

This equality proves the equality of statement ib) since we obtain that

$$2Re\big(g(s), i\gamma^0((m-iQ)g)(s)\big)_D = \frac{d}{ds}\|g(s)\|_D^2,$$

for $g \in C^1(\mathbb{R}^+, D) \cap C^0(\mathbb{R}^+, (1-\Delta)^{-1/2}D)$, the operator

$$i\gamma^0 m + \sum_{j=1}^3 \gamma^0\gamma^j\partial_j + i\gamma^0\gamma^\mu G_\mu(s)$$

being skew-adjoint on D with domain $(1-\Delta)^{-1/2}D$.

To prove the inequality of statement iia) we note that, according to the introductory remarks of this chapter:

$$h(t) = w(t,t')h(t') + \int_{t'}^t w(t,s)(-i\gamma^0)g(s)ds. \tag{5.11}$$

Since $w(t_1, t_2), t_1, t_2 \geq 0$ is unitary in D it follows that

$$\big|\|h(t)\|_D - \|h(t')\|_D\big| \leq \int_t^{t'} \|g(s)\|_D ds, \quad 0 \leq t \leq t',$$

which proves statement iia).

To prove iib) let $g_n^{(2)} \in S(\mathbb{R}^4, \mathbb{C}^4), n \geq 0$, be a sequence such that

$$\sup_{0\leq s\leq T}\Big(\|g_n^{(2)}(s) - g^{(2)}(s)\|_D + \int_0^T \sum_{0\leq\mu\leq 3} \|(\partial_\mu(g_n^{(2)} - g^{(2)}))(s)\|_D ds\Big) \to 0,$$

when $n \to \infty$ for every $T \geq 0$. Since $\|G(s)\|_{L^\infty(\mathbb{R}^3,\mathbb{R}^4)} \leq C\|(1-\Delta)^{1/2}(G(s),0))\|_{M^1}$ it then follows from the hypothesis of statement iib) that

$$\sup_{0\leq s\leq T}\Big(\|g_n^{(2)}(s)-g^{(2)}(s)\|_D+\int_0^T\|((m-iQ)(g_n^{(2)}-g^{(2)}))(s)\|_D ds\Big)\to 0$$

when $n \to \infty$ for every $T \geq 0$ and that $(m-iQ)g_n^{(2)} \in C^0(\mathbb{R}^+,D)$. Let $h_n^{(2)} \in C^0(\mathbb{R}^+,D)$ be the unique solution of $(m+iQ)h_n^{(2)} = g_n^{(2)}, h_n^{(2)}(t_0)=0$. If $f_n=(m-iQ)h_n^{(2)}$, then $f_n = 2mh_n^{(2)} - g_n^{(2)}$ and $(m+iQ)f_n=(m-iQ)g_n^{(2)}$, according to (5.10), so $(m+iQ)(h_n^{(2)} - (2m)^{-1}g_n^{(2)}) = (2m)^{-1}(m-iQ)g_n^{(2)}$. Let $h_n \in C^0(\mathbb{R}^+,D)$ be the unique solution of equation $(m+iQ)h_n = g^{(1)}+g_n^{(2)}, h_n(t_0)=h(t_0)\in D$. Then it follows from statement iia) and the properties of g_n that

$$\|h_n(t)-h(t)\|_D \leq \Big|\int_{t_0}^t \|g_n^{(2)}(s)-g^{(2)}(s)\|_D ds\Big|,$$

so $\sup_{0\leq s\leq T}\|h_n(t)-h(t)\|_D \to 0$ when $n\to\infty$ for every $T\geq 0$. The definition of h_n gives that

$$\begin{aligned}(m+iQ)(h_n-(2m)^{-1}g_n^{(2)}) &= (m+iQ)((h_n-h_n^{(2)})+(h_n^{(2)}-(2m)^{-1}g_n^{(2)}))\\ &= g^{(1)}+(2m)^{-1}(m-iQ)g_n^{(2)}.\end{aligned}$$

Since $g^{(1)}+(2m)^{-1}(m-iQ)g_n^{(2)} \in C^0(\mathbb{R}^+,D)$ and

$$h_n(t_0)-(2m)^{-1}g_n^{(2)}(t_0) = h(t_0)-(2m)^{-1}g_n^{(2)}(t_0)\in D,$$

it follows from statement iia) that

$$\begin{aligned}&\big|\|h_n(t)-(2m)^{-1}g_n^{(2)}(t)\|_D - \|h_n(t')-(2m)^{-1}g_n^{(2)}(t')\|_D\big| \\ &\quad\leq \int_t^{t'}\|g^{(1)}(s)+(2m)^{-1}((m-iQ)g_n^{(2)})(s)\|_D\,ds, \quad 0\leq t\leq t'.\end{aligned} \tag{5.12}$$

The inequality in statement iib) follows from the convergence properties, proved above, of h_n and $g_n^{(2)}$ and by taking the limit $n\to\infty$ in inequality (5.12). This proves the theorem.

Corollary 5.2. *Let $F^{(l)}, r^{(l)}, q^{(l)}$, $l\geq 0$, be a finite number of functions satisfying conditions (5.4b) and (5.5). Let $Q^{(l)}(t,x) = tF_0^{(l)}(t,x)+\sum_{1\leq i\leq 3}x_iF_i(t,x)$ and let ξ_X^D, $X\in\mathfrak{p}$, be defined by (4.81d) and (4.81e). If $(G,\dot{G})\in C^0(\mathbb{R}^+,(1-\Delta)^{-1}M^1)$, $h(t_0)\in D$, $g_1\in C^0(\mathbb{R}^+,D)$ and $0\leq a_l\leq 1$, then the unique solution $h\in C^1(\mathbb{R}^+,(1-\Delta)^{1/2}D)\cap C^0(\mathbb{R}^+,D)$ of equation (5.4a), with $g=g_1+\sum_l\gamma^\mu F_\mu^{(l)}r^{(l)}$ satisfies:*

$$\begin{aligned}
&\big|\|h(t)-(2m)^{-1}\sum_l \gamma^\mu F^{(l)}_\mu(t)r^{(l)}(t)\|_D-\|h(t')-(2m)^{-1}\sum_l \gamma^\mu F^{(l)}_\mu(t')r^{(l)}(t')\|_D\big|\\
&\quad\le\int_t^{t'}\|g_1(s)\|_D ds+(2m)^{-1}\int_t^{t'}\Big(2\sum_l\big|(1+s)^{a_l-1}\big|Q^{(l)}(s)+sF^{(l)}_0(s)\partial_0 r^{(l)}(s)\\
&\qquad-\sum_{1\le i\le 3}F^{(l)}_i(s)((\xi^D_{M_{0i}}r^{(l)})(s)-\sigma_{0i}r^{(l)}(s))\big|^{1-a_l}\big|F^{(l)}_\mu(s)\partial^\mu r^{(l)}(s)\big|^{a_l}\big|_D\\
&\qquad+\big|\sum_l\Big(r^{(l)}(s)\partial^\mu F^{(l)}_\mu(s)+\frac12 r^{(l)}(s)\gamma^\mu\gamma^\nu(\partial_\mu F^{(l)}_\nu(s)-\partial_\nu F^{(l)}_\mu(s))\\
&\qquad+i\gamma^\mu\gamma^\nu G_\mu(s)F^{(l)}_\nu(s)r^{(l)}(s)+i\gamma^\mu F^{(l)}_\mu(s)q^{(l)}(s)\Big)\big|_D\Big)ds,
\end{aligned}$$

where $0\le t\le t', \partial_0$ is the time derivative and the summation convention is used for repeated upper and lower indices μ and ν.

Proof. Let $g^{(2)}=\sum_l\gamma^\mu F^{(l)}_\mu r^{(l)}$. It follows from (5.5) that the hypotheses of statement iib) of Theorem 5.1 are satisfied, which gives that

$$\begin{aligned}
&\big|\|h(t)-(2m)^{-1}g^{(2)}(t)\|_D-\|h(t')-(2m)^{-1}g^{(2)}(t')\|_D\big|\\
&\quad\le\int_t^{t'}\|g_1(s)\|_D ds+(2m)^{-1}\int_t^{t'}\|((m-iQ)g^{(2)})(s)\|_D ds.
\end{aligned}$$

The explicit expression of $(m-iQ)g^{(2)}$ is given by (5.7a). Let $\gamma^\mu F_\mu r$ be one of the terms $\gamma^\mu F^{(l)}_\mu r^{(l)}$ in $g^{(2)}$. Consider the term $-2iF^\mu\partial_\mu r$ on the right-hand side of (5.7a). Since

$$F_0(t,x)=(1+t)^{-1}\Big(Q(t,x)-\sum_{1\le i\le 3}x_iF_i(t,x)+F_0(t,x)\Big)$$

it follows from definition (4.81b) and (4.81e) of $\xi_{M_{0i}}$ and $\xi^D_{M_{0i}}$ that

$$\begin{aligned}
&F_0(t,x)\frac{\partial}{\partial t}-\sum_{1\le i\le 3}F_i(t,x)\partial_i\\
&\quad=(1+t)^{-1}(Q(t,x)+F_0(t,x))\frac{\partial}{\partial t}-(1+t)^{-1}\sum_{1\le i\le 3}F_i(t,x)(x_i\frac{\partial}{\partial t}+t\partial_i)\\
&\quad=(1+t)^{-1}\Big((Q(t,x)+F_0(t,x))\frac{\partial}{\partial t}-\sum_{1\le i\le 3}F_i(t,x)(\xi^D_{M_{0i}}-\sigma_{0i})\Big).
\end{aligned}$$

Substitution of this expression into the factor $|F^\mu\partial_\mu r|^{1-a}$ in inequality

$$\begin{aligned}
&|(m-iQ)\gamma^\mu F_\mu r|\\
&\quad\le 2|F^\mu\partial_\mu r|^{1-a}|F^\mu\partial_\mu r|^a+|r\partial_\mu F^\mu+\frac12 r\gamma^\mu\gamma^\nu(\partial_\mu F_\nu-\partial_\nu F_\mu)+i\gamma^\mu\gamma^\nu G_\mu F^\nu r+i\gamma^\mu F_\mu q|
\end{aligned}$$

gives the announced result, which proves the corollary.

In order to use Corollary 5.2 with L^∞-norms of the potentials we need supplementary *decay properties of the Dirac field* h in a conic neighbourhood of the light-cone as well as outside the light-cone. This will be proved in the next two theorems. To state the results we introduce the following notation:

$$(L_i(t)f)(x) = x_i((\mathcal{D}f)(x) + V(t,x)f(x)) + t\partial_i f, \quad t \geq 0, 1 \leq i \leq 3, \tag{5.13a}$$

$(l_i(t))(x) = x_i F(t,x)$, where $f \in D_\infty$, $V \in C(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathrm{Mat}(4,\mathbb{C})))$, $F \in C(\mathbb{R}^+, D)$, $\mathrm{Mat}(4,\mathbb{C})$ being the space of 4×4 complex matrices,

$$(R_{ij}f)(x) = x_i \partial_j f(x) - x_j \partial_i f(x), \quad 1 \leq i \leq 3, 1 \leq j \leq 3; \tag{5.13b}$$

$$\begin{aligned} M_{n,t}(f) = \Big(&\|q_t^{n/2} f\|_D^2 + \sum_{1 \leq i \leq 3} \|q_t^{n/2} \partial_i f\|_D^2 + \sum_{1 \leq i < j \leq 3} \|q_t^{n/2} R_{ij} f\|_D^2 \\ &+ \sum_{1 \leq i \leq 3} \|q_t^{n/2}(L_i(t)f + l_i(t))\|_D^2 \Big)^{1/2}, \quad t \geq 0, n \geq 0, \end{aligned} \tag{5.13c}$$

$$q_t(x) = (1+t)(1 + \big|t - |x|\big|)^{-1}.$$

Theorem 5.3. *If $V \in C^0(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathrm{Mat}(4,\mathbb{C}))), F \in C^0(\mathbb{R}^+, D)$ and $n \in \mathbb{N}$ then there exists $C_n > 0$ independent of t, f, V, F such that*

$$\begin{aligned} \|q_t^{(n+1)/2} f\|_D \leq C_n \Big(&\big(1 + \|q_t^{1/2} V(t)\|_{L^\infty} + (1+t)^{-1/2} \|V(t)\|_{L^\infty}\big) M_{n,t}(f) \\ &+ \|q_t^{1+n/2} F(t)\|_D + \|q_t^{n/2} F(t)\|_D \Big), \end{aligned}$$

for $f \in D_\infty$, $t \geq 0$.

Proof. Introduce $h_t(x) = (1 + \delta^2 \big|t - |x|\big|^2)^{-1/4}$, $K_t = \mathcal{D} + V(t)$ and $N_{n,t}(f) = (\|h_t^n \mathcal{D} f\|_D^2 + (1+t)^{-n} M_{n,t}(f)^2)^{1/2}$ and let $\varepsilon > 0, \delta > 0$. Introduce also $\tau_t(V) = (1+\delta t)^{1/2} \|h_t V(t)\|_{L^\infty} + (1+\delta t)^{-1/2} \|V(t)\|_{L^\infty}$, and $\lambda_{n,t}(F) = (1+\delta t)\|h_t^{n+2} F(t)\|_D + \|h_t^n F(t)\|_D$.

If $0 \leq t \leq 1$, then $|q_t(x)| \leq 2$ so the inequality of the theorem is trivially true with $C \geq 2^{1/2}$. For a given $f \in D_\infty$ let now $T_+(f)$ (resp. $T_-(f)$) be the set of real numbers $t \geq 1$ such that

$$\|h_t^n \mathcal{D} f\|_D^2 + \sum_{1 \leq i \leq 3} \|h_t^n (L_i(t) f + l_i(t))\|_D^2 - 2\varepsilon t \|h_t^{n+1} f\|_D^2 \geq 0 \quad (\text{ resp } < 0). \tag{5.14}$$

Let $t \in T_+(f)$. Since $t \geq 1$ it follows from (5.14) that

$$(1+t)\|h_t^{n+1} f\|_D^2 \leq \varepsilon^{-1} N_{n,t}(f)^2, \quad t \in T_+(f). \tag{5.15}$$

Let $g_0 = K_t f + F(t), g_i = \partial_i f, 1 \leq i \leq 3, g = (g_1, g_2, g_3)$ and let $|g| = (\sum_{1 \leq i \leq 3} |g_i|^2)^{1/2}$. Equality

$$(t - |x|)|\mathcal{D} f| = t(|\mathcal{D} f| - |g|) - (|x||\mathcal{D} f| - t|g|)$$

gives

$$\begin{aligned}\big|t-|x|\big||\mathcal{D}f| &\geq t(|\mathcal{D}f|-|g|)-|x\mathcal{D}f+tg|\\ &\geq t(|\mathcal{D}f|-|g|)-|xg_0+tg|-\big|t-|x|\big||V(t)||f|\\ &\quad -t|V(t)||f|-\big|t-|x|\big||F(t)|-t|F(t)|.\end{aligned}$$

It follows that

$$\begin{aligned}&2(1+\delta^2\big|t-|x|\big|^2)^{1/2}|\mathcal{D}f|^2\\ &\quad\geq |\mathcal{D}f|^2+\delta t(|\mathcal{D}f|^2-|\mathcal{D}f||g|)-\delta|\mathcal{D}f||xg_0+tg|\\ &\qquad -\delta\big|t-|x|\big|(|V(t)||f||\mathcal{D}f|+|F(t)||\mathcal{D}f|)-\delta t(|V(t)||f||\mathcal{D}f|+|F(t)||\mathcal{D}f|).\end{aligned}$$

Since $0\leq h_t\leq 1$ and $|\mathcal{D}f|^2-|\mathcal{D}f||g|\geq \frac{1}{2}(|\mathcal{D}f|^2-|g|^2)$ we get

$$\begin{aligned}&2h_t^{2n}|\mathcal{D}f|^2\\ &\quad\geq \frac{1}{2}h_t^{2n+2}(1+\delta t)(|\mathcal{D}f|^2-|g|^2)-\delta h_t^{2n}|\mathcal{D}f||xg_0+tg|\\ &\qquad -h_t^{2n}(|V(t)||f||\mathcal{D}f|+|F(t)||\mathcal{D}f|)-\delta t(h_t^{2n+2}|V(t)||\mathcal{D}f||f|+h_t^{2n+2}|F(t)||\mathcal{D}f|).\end{aligned}\tag{5.16}$$

Integration of this inequality over $\mathbb{R}^3$ and Schwarz inequality give that

$$\begin{aligned}&2\|h_t^n\mathcal{D}f\|_D^2\\ &\quad\geq \frac{1}{2}(1+\delta t)\Big(\|h_t^{n+1}\mathcal{D}f\|_D^2-\sum_{1\leq i\leq 3}\|h_t^{n+1}\partial_i f\|_D^2\Big)\\ &\qquad -\delta\|h_t^n\mathcal{D}f\|_D\Big(\sum_{1\leq i\leq 3}\|h_t^n(L_i(t)f+l_i(t))\|_D^2\Big)^{1/2}\\ &\qquad -(1+\delta t)\Big(\|h_tV(t)\|_{L^\infty}\|h_t^{n+1}f\|_D\|h_t^n\mathcal{D}f\|_D+\|h_t^{n+2}F(t)\|_D\|h_t^n\mathcal{D}f\|_D\Big)\\ &\qquad -\Big(\|V(t)\|_{L^\infty}\|h_t^n\mathcal{D}f\|_D\|h_t^nf\|_D+\|h_t^nF(t)\|_D\|h_t^n\mathcal{D}f\|_D\Big).\end{aligned}\tag{5.17}$$

According to the definitions of $N_{n,t},\tau_t$ and $\lambda_{n,t}$ we obtain, using $ab\leq\frac{1}{2}(a^2+b^2)$,

$$\begin{aligned}&\frac{1}{2}(1+\delta t)\Big(\|h_t^{n+1}\mathcal{D}f\|_D^2-\sum_{1\leq i\leq 3}\|h_t^{n+1}\partial_i f\|_D^2\Big)\\ &\quad -\delta\|h_t^n\mathcal{D}f\|_D\Big(\sum_{1\leq i\leq 3}\|h_t^n(L_i(t)f+l_i(t))\|_D^2\Big)^{1/2}\\ &\qquad\leq \frac{5}{2}N_{n,t}(f)^2+\frac{1}{2}\lambda_{n,t}(F)^2+\tau_t(V)\|h_t^n\mathcal{D}f\|_D(1+\delta t)^{1/2}\|h_t^{n+1}f\|_D.\end{aligned}\tag{5.18}$$

Let $t\in T_-(f)$. It then follows from inequality (5.14) and inequality (5.18) that

$$\begin{aligned}&(1+\delta t)\Big(\|h_t^{n+1}\mathcal{D}f\|_D^2-\sum_{1\leq i\leq 3}\|h_t^{n+1}\partial_i f\|_D^2-2\varepsilon\|h_t^{n+1}f\|_D^2\Big)\\ &\quad\leq 5N_{n,t}(f)^2+\lambda_{n,t}(F)^2+2\tau_t(V)\|h_t^n\mathcal{D}f\|_D(1+\delta t)^{1/2}\|h_t^{n+1}f\|_D.\end{aligned}\tag{5.19}$$

Let $j \in C^\infty(\mathbb{R}^+), 0 \leq j(y) \leq 1, j(y) = 0$ for $0 \leq y \leq 1/4$ and $j(y) = 1$ for $y \geq 1/2$. Let $\varphi_t(x) = h_t^{2n+2}(x) j(|x|/t)$ and $\psi_t(x) = h_t^{2n+2}(x)(1 - j(|x|/t)), t > 0$. Since $0 \leq \psi_t(x) \leq (2)^{1/2}(1+\delta t)^{-1} h^{2n}(x)$, we obtain that

$$
\begin{aligned}
&(1+\delta t)\Big(\|h_t^{n+1}\mathcal{D}f\|_D^2 - \sum_{1\leq i\leq 3} \|h_t^{n+1}\partial_i f\|_D^2\Big) \\
&\quad = (1+\delta t)\Big((\mathcal{D}f, (\varphi_t + \psi_t)\mathcal{D}f)_D - \sum_{1\leq i\leq 3} (\partial_i f, (\varphi_t + \psi_t)\partial_i f)_D\Big) \\
&\quad \geq (1+\delta t)\Big((\mathcal{D}f, \varphi_t \mathcal{D}f)_D - \sum_{1\leq i\leq 3} (\partial_i f, \varphi_t \partial_i f)_D\Big) \\
&\qquad - 2^{1/2}\Big(\|h_t^n \mathcal{D}f\|_D^2 + \sum_{1\leq i\leq 3} \|h_t^n \partial_i f\|_D^2\Big).
\end{aligned}
$$

This inequality and inequality (5.19) give

$$
\begin{aligned}
&(1+\delta t)\Big((\mathcal{D}f, \varphi_t \mathcal{D}f)_D - \sum_{1\leq i\leq 3} (\partial_i f, \varphi_t \partial_i f)_D - 2\varepsilon \|h_t^{n+1} f\|_D^2\Big) \\
&\quad \leq C\Big(N_{n,t}(f)^2 + \lambda_{n,t}(F)^2 + 2\tau_t(V)\|h_t^n \mathcal{D}f\|_D (1+\delta t)^{1/2} \|h_t^{n+1} f\|_D\Big), \quad t \in T_-(f),
\end{aligned}
\tag{5.20}
$$

where C is a constant independent of f, V, F. The function φ_t is C^∞ and bounded together with its derivatives for $t > 0$, so $\varphi_t \mathcal{D}f \in D_\infty$. We add the operator $\mathcal{D}\varphi_t\mathcal{D} = \varphi_t \mathcal{D}^2 + [\mathcal{D}, \varphi_t]\mathcal{D}$ and its adjoint. Since $\mathcal{D}^2 = \Delta - m^2$ and since $\mathcal{D}$ is skew-adjoint we obtain:

$$
\mathcal{D}\varphi_t\mathcal{D} = \frac{1}{2}\big((\Delta - m^2)\varphi_t + \varphi_t(\Delta - m^2) - [\mathcal{D}, [\mathcal{D}, \varphi_t]]\big)
$$

and similarly we obtain that

$$
\sum_{1\leq i\leq 3} \partial_i \varphi_t \partial_i = \frac{1}{2}\Big(\Delta\varphi_t + \varphi_t \Delta - \sum_{1\leq i\leq 3} [\partial_i, [\partial_i, \varphi_t]]\Big).
$$

These two equalities and inequality (5.20) give, when $t \in T_-(f)$:

$$
\begin{aligned}
&(1+\delta t)\big((m^2 - 2\varepsilon)\|h_t^{n+1} f\|_D^2 - m^2 (f, \psi_t f)_D\big) \\
&\quad + (1+\delta t)\frac{1}{2}\big(f, [\mathcal{D}, [\mathcal{D}, \varphi_t]] - \sum_{1\leq i\leq 3} [\partial_i, [\partial_i, \varphi_t]] f\big)_D \\
&\qquad \leq C\Big(N_{n,t}(f)^2 + \lambda_{n,t}(F)^2 + 2\tau_t(V)\|h_t^n \mathcal{D}f\|_D (1+\delta t)^{1/2} \|h_t^{n+1} f\|_D\Big).
\end{aligned}
\tag{5.21}
$$

We shall calculate the commutators in expression (5.21). Since $[\partial_i, \varphi_t] = (\partial_i \varphi_t)$ we get

$$
\sum_{1\leq i\leq 3} [\partial_i, [\partial_i, \varphi_t]] = (\Delta\varphi_t)
$$

and

$$[\mathcal{D},[\mathcal{D},\varphi_t]] = \sum_{\substack{1\le i\le 3\\ 1\le j\le 3}} [\gamma^0\gamma^i\partial_i, \gamma^0\gamma^j(\partial_j\varphi_t)] - [i\gamma^0 m, \sum_{1\le j\le 3}\gamma^0\gamma^j(\partial_j\varphi_t)].$$

The last equalities and equalities $[\gamma^0,\gamma^0\gamma^j] = 2\gamma^j, 1\le j\le 3$,

$$\sum_{i,j}[\gamma^0\gamma^i\partial_i,\gamma^0\gamma^j(\partial_j\varphi_t)] = (\Delta\varphi_t) + \sum_{i,j}\gamma^i\gamma^j((\partial_i\varphi_t)\partial_j - (\partial_j\varphi_t)\partial_i)$$

and

$$\partial_i\varphi_t = x_i|x|^{-2}\sum_{1\le j\le 3} x_j\partial_j\varphi_t,$$

which follows from the spherical symmetry of φ_t, give that

$$\begin{aligned}\Big([\mathcal{D},[\mathcal{D},\varphi_t]]f &- \sum_{1\le i\le 3}[\partial_i,[\partial_i,\varphi_t]]f\Big)(x) \\ &= -2im\nu_t(x)\sum_{1\le j\le 3}\gamma^j|x|^{-1}x_jf(x) + 2|x|^{-1}\nu_t(x)\sum_{1\le i<j\le 3}\gamma^i\gamma^j(R_{ij}f)(x), \quad t>0,\end{aligned} \tag{5.22}$$

where $\nu_t(x) = \sum_{1\le l\le 3}|x|^{-1}x_l(\partial_l\varphi_t)(x)$. We observe that $\nu_t(x)=0$ for $4|x|\ge t>0$ and $2|x|\le t$, according to the definition of φ_t. Since $|\sum\gamma^jx_j| = |x|$ we obtain from (5.22) that

$$\begin{aligned}&\Big|\frac{1}{2}\Big(f,[\mathcal{D},[\mathcal{D},\varphi_t]]f - \sum_{1\le i\le 3}[\partial_i,[\partial_i,\varphi_t]]f\Big)_D\Big| \\ &\quad\le m\|h_t^{-2-2n}\nu_t\|_{L^\infty}\|h_t^{n+1}f\|_D^2 \\ &\qquad + 3^{1/2}4t^{-1}\|h_t^{-2n}\nu_t\|_{L^\infty}\|h_t^nf\|_D\Big(\sum_{1\le i<j\le 3}\|h_t^nR_{ij}f\|_D^2\Big)^{1/2}, \quad t>0.\end{aligned}$$

A direct calculation gives

$$\nu_t(x) = t^{-1}h_t^{2+2n}(x)j'(|x|/t) + (1+n)\delta^2(t-|x|)h_t^{6+2n}(x)j(|x|/t),$$

where j' is the derivative of j. It follows that $|h_t^{-2n}\nu_t(x)|$ and $|h_t^{-2-2n}\nu_t(x)|$ are bounded by $(1+n)\delta + t^{-1}\|j'\|_{L^\infty}$. This gives

$$\begin{aligned}\frac{1}{2}\Big|\Big(f,[\mathcal{D},[\mathcal{D},\varphi_t]]f &- \sum_{1\le i\le 3}[\partial_i,[\partial_i,\varphi_t]]f\Big)_D\Big| \\ &\le m(1+n)\delta\|h_t^{1+n}f\|_D^2 + t^{-1}CN_{n,t}(f)^2, \quad t>0,\end{aligned} \tag{5.23}$$

where the constant C depends only on the function j and on n. Since $(1+\delta t)(f,\psi_tf)_D \le 2^{3/2}\|h_t^nf\|_D^2$ it follows from inequalities (5.21) and (5.23) that, with $t\in T_-(f)$,

$$\begin{aligned}(1+\delta t)(m^2 - 2\varepsilon - m(1+n)\delta)&\|h_t^{n+1}f\|_D^2 \\ &\le C^2(N_{n,t}(f)^2 + \lambda_{n,t}(F)^2) + 2CN_{n,t}(f)\tau_t(V)(1+\delta t)^{1/2}\|h_t^{n+1}f\|_D,\end{aligned} \tag{5.24}$$

where C is a new constant. Let $\varepsilon = m^2/8$ and $(1+n)\delta = m/4$. It then follows from inequality (5.24) that (with a new constant C):

$$\begin{aligned}(1+\delta t)&\|h_t^{n+1} f\|_D^2 \\ &\leq C^2(N_{n,t}(f)^2 + \lambda_{n,t}(F)^2) + 2C\tau_t(V)N_{n,t}(f)(1+\delta t)^{1/2}\|h_t^{n+1} f\|_D, \quad t \in T_-(f),\end{aligned}$$

which gives

$$(1+\delta t)^{1/2}\|h_t^{n+1} f\|_D \leq 2C\big((1+\tau_t(V))N_{n,t}(f) + \lambda_{n,t}(F)\big), \tag{5.25}$$

with $(1+n)\delta = m/4$ and $\varepsilon = m^2/8$.

Since $T_+(f) \cup T_-(f) = [1,\infty[$ it follows from inequality (5.15) with $\varepsilon = m^2/8$, from inequality (5.25) and from the definition of τ_t and $\lambda_{n,t}$ that

$$\begin{aligned}\|q_t^{(n+1)/2} f\|_D &\leq C'_n\big((1+\tau_t(V))(1+t)^{n/2}N_{n,t}(f) + (1+t)^{n/2}\lambda_{n,t}(F)\big) \\ &\leq C''_n\big((1 + \|q_t^{1/2}V(t)\|_{L^\infty} + (1+t)^{-1/2}\|V(t)\|_{L^\infty})M_{n,t}(f) \\ &\qquad + \|q_t^{1+n/2}F(t)\|_D + \|q_t^{n/2}F(t)\|_D\big),\end{aligned}$$

for some constants C'_n, C''_n. We have here used the fact that $(1+t)^{n/2}N_{n,t}(f) \leq C_n M_{n,t}(f)$ for some C_n. This proves the theorem.

In order to state the next theorem we introduce the following notation:

$$\begin{aligned}M_t^{(n)}(f) = \Big(\|r_t^{n/2} f\|_D^2 &+ \sum_{1\leq i\leq 3}\|r_t^{n/2}\partial_i f\|_D^2 + \sum_{1\leq i<j\leq 3}\|r_t^{n/2}R_{ij} f\|_D^2 \\ &+ \sum_{1\leq i\leq 3}\|r_t^{n/2}(L_i(t)f + l_i(t))\|_D^2\Big)^{1/2}, \quad n \geq 0, t \geq 0,\end{aligned} \tag{5.26}$$

where $L_i(t), l_i(t)$ and R_{ij} are defined in (5.13a) and (5.13b) and where $r_t(x) = 0$ if $|x| < t$ and $r_t(x) = 1 + |x|$ if $|x| \geq t$.

Theorem 5.4. *If $V \in C^0\big(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathrm{Mat}(4,\mathbb{C}))\big)$, $F \in C^0(\mathbb{R}^+, D)$ and $\tau_t(V) = \sup_{x\in\mathbb{R}^3}((1+|x|+t)^{1/2}|V(t,x)|) < \infty$, then there exist constants C_n independent of t, V, f, F, such that*

$$\begin{aligned}\|r_t^{(n+1)/2}&f\|_D \\ &\leq C_n(1+\tau_t(V))\big(M_{n,t}(f) + M_t^{(n)}(f) + \|q_t^{1+n/2}F(t)\|_D + \|r_t^{1+n/2}F(t)\|_D\big)\end{aligned}$$

for $n \geq 0$ and $f \in D_\infty$.

Proof. We first consider the case where $F = 0$. Let $\psi_t \in C^\infty(\mathbb{R}^3), t \geq 0$, be a cut-off function defined by $\psi_t(x) = u(|x| - t)$, where $u \in C^\infty(\mathbb{R}^+), 0 \leq u(y) \leq 1$ for $y \in \mathbb{R}^+, u(y) = 0$ for $0 \leq y \leq 1$ and $u(y) = 1$ for $2 \leq y$. Let $K_t = \mathcal{D} + V(t), g_0 = K_t f, g_i = \partial_i f, 1 \leq i \leq 3, g = (g_1, g_2, g_3)$, where $f \in D_\infty$.

Since in the support of $r_t^{(n+1)/2}(1-\psi_t)$ we have $0 \le t \le |x| \le 2+t$, it follows that

$$\begin{aligned}|r_t(x)^{(n+1)/2}(1-\psi_t(x))| &\le C_n'((1+t)(1+\big|t-|x|\big|)^{-1})^{(n+1)/2} \\ &= C_n' q_t(x)^{(n+1)/2},\end{aligned}$$

where q_t is defined in (5.13c). This gives for some constant C_n that

$$\|r_t^{(n+1)/2}(1-\psi_t)f\|_D \le C_n \|q_t^{(n+1)/2} f\|_D, \quad t \ge 0. \tag{5.27}$$

Since $|x|-t \ge 1$ and $r_t(x) = 1+|x|$ in the support of ψ_t it follows that

$$\begin{aligned}r_t^n \psi_t^2(|\mathcal{D}f|^2 + |xg_0+tg|^2) &\ge 2r_t^n\psi_t^2|\mathcal{D}f||xg_0+tg| \\ &\ge 2r_t^n\psi_t^2|\mathcal{D}f|\big(|x|(|\mathcal{D}f|-|g|) + (|x|-t)|g| - |x||V(t)||f|\big) \\ &\ge r_t^n\psi_t^2|x|\big(|\mathcal{D}f|^2 - |g|^2 - 2|x||V(t)||\mathcal{D}f||f|\big) \\ &\ge \frac{1}{2} r_t^{n+1}\psi_t^2(|\mathcal{D}f|^2-|g|^2) - 2r_t^{n+1}\psi_t^2|V(t)||f||\mathcal{D}f|.\end{aligned}$$

Integration of this inequality over $\mathbb{R}^3$ gives

$$\begin{aligned}&\|r_t^{(n+1)/2}\psi_t\mathcal{D}f\|_D^2 - \sum_{1\le i\le 3}\|r_t^{(n+1)/2}\psi_t\partial_i f\|_D^2 \\ &\quad\le C^2 M_t^{(n)}(f)^2 + 2C\|r_t^{1/2}V(t)\|_{L^\infty}\|r_t^{n/2}\mathcal{D}f\|_D\|r_t^{(n+1)/2}\psi_t f\|_D \\ &\quad\le C_n^2 M_t^{(n)}(f)^2 + 2C_n\|r_t^{1/2}V(t)\|_{L^\infty} M_t^{(n)}(f)\|r_t^{(n+1)/2}\psi_t f\|_D,\end{aligned} \tag{5.28}$$

where C and C_n are constants depending only on the mass m.

Let $\varphi_t = \psi_t^2 r_t^{n+1}$. Following the proof of Theorem 5.3 from (5.20) to (5.22) we obtain, since φ_t is C^∞,

$$\begin{aligned}&\|r_t^{(n+1)/2}\psi_t\mathcal{D}f\|_D^2 - \sum_{1\le i\le 3}\|r_t^{(n+1)/2}\psi_t\partial_i f\|_D^2 \\ &\quad\ge m^2\|r_t^{(n+1)/2}\psi_t f\|_D^2 - m(f,\nu_t f)_D - \sum_{1\le i<j\le 3}|(f,|x|^{-1}\nu_t R_{ij} f)_D|,\end{aligned}$$

where $\nu_t(x) = 0$ for $|x| \le 1+t$ and $\nu_t(x) = \sum_{1\le l\le 3}|x|^{-1}x_l(\partial_l\varphi_t)(x)$ otherwise. It follows from the definition of φ_t that

$$\nu_t(x) = (n+1)r_t^n(x)\psi_t^2(x) + r_t^{n+1}(x)\psi_t(x)u'(|x|-t),$$

where u' is the derivative of u. Because supp $u' \subset [1,2]$, we have that

$$\begin{aligned}|r_t^{n+1}(x)\psi_t(x)u'(|x|-t)| &\le C_n(1+t)^{n+1}(1+\big|t-|x|\big|)^{-n-1} \\ &= C_n q_t(x)^{n+1}\end{aligned}$$

for some constant C_n, so $|\nu_t| \leq (n+1)r_t^n\psi_t^2 + C_n q_t^{n+1}$. We also have $|x|^{-1}|\nu_t(x)| \leq C_n r_t^n$ for some constant C_n. This gives

$$\begin{aligned}&\|r_t^{(n+1)/2}\psi_t\mathcal{D}f\|_D^2 - \sum_{1\leq i\leq 3}\|r_t^{(n+1)/2}\psi_t\partial_i f\|_D^2\\&\qquad\geq m^2\|r_t^{(n+1)/2}\psi_t f\|_D^2 - C_n'\|q_t^{(n+1)/2}f\|_D^2 - C_n'\|r_t^{n/2}f\|_D\Big(\sum_{1\leq i<j\leq 3}\|r_t^{n/2}R_{ij}f\|_D^2\Big)^{1/2}.\end{aligned}$$

Applying the definition of $M_t^{(n)}$ to the last term on the right-hand side of this inequality, we obtain

$$\begin{aligned}&\|r_t^{(n+1)/2}\psi_t\mathcal{D}f\|_D^2 - \sum_{1\leq i\leq 3}\|r_t^{(n+1)/2}\psi_t\partial_i f\|_D^2\\&\qquad\geq m^2\|r_t^{(n+1)/2}\psi_t f\|_D^2 - C_n^2(\|q_t^{(n+1)/2}f\|_D^2 + M_t^{(n)}(f)^2), \quad t\geq 0,\end{aligned} \tag{5.29}$$

for some constant C_n. It follows from inequalities (5.28) and (5.29) that (with a new constant C_n)

$$\begin{aligned}&\|r_t^{(n+1)/2}\psi_t f\|_D^2\\&\qquad\leq C_n^2(M_t^{(n)}(f)^2 + \|q_t^{(n+1)/2}f\|_D^2) + 2C_n\|r_t^{1/2}V(t)\|_{L^\infty}M_t^{(n)}(f)\|r_t^{(n+1)/2}\psi_t f\|_D.\end{aligned}$$

This inequality shows that (with a new constant C_n)

$$\begin{aligned}\|r_t^{(n+1)/2}\psi_t f\|_D &\leq C_n\big(M_t^{(n)}(f) + \|q_t^{(n+1)/2}f\|_D + \|r_t^{1/2}V(t)\|_{L^\infty}M_t^{(n)}(f)\big)\\&= C_n\big((1+\|r_t^{1/2}V(t)\|_{L^\infty})M_t^{(n)}(f) + \|q_t^{(n+1)/2}f\|_D\big).\end{aligned} \tag{5.30}$$

It follows from inequalities (5.27) and (5.30) that, for $t\geq 0$ and $F=0$,

$$\|r_t^{(n+1)/2}f\|_D \leq C_n\big((1+\|r_t^{1/2}V(t)\|_{L^\infty})M_t^{(n)}(f) + \|q_t^{(n+1)/2}f\|_D\big). \tag{5.31}$$

To study the case where $F\neq 0$ we stress the dependence of $M_t^{(n)}(f)$ on F by the notation $M_t^{(n)}(f,F)$. It follows from definition (5.26) that

$$M_t^{(n)}(f,0) \leq C\big(M_t^{(n)}(f,F) + \|r_t^{1+n/2}F(t)\|_D\big),$$

where C is independent of n, t, f, F. This inequality and inequality (5.31) give (with new C_n):

$$\begin{aligned}\|r_t^{(n+1)/2}f\|_D \leq C_n\Big(&(1+\|r_t^{1/2}V(t)\|_{L^\infty})M_t^{(n)}(f,F)\\&+\|q_t^{(n+1)/2}f\|_D + (1+\|r_t^{1/2}V(t)\|_{L^\infty})\|r_t^{1+n/2}F(t)\|_D\Big).\end{aligned} \tag{5.32}$$

It follows from this inequality, Theorem 5.3 and from the definitions of q_t and r_t, that

$$\|r_t^{(n+1)/2} f\|_D \leq C_n(1+\sup_{x\in\mathbb{R}^3}((1+t+|x|)^{1/2})|V(t,x)|)$$
$$\big(M_{n,t}(f,F)+M_t^{(n)}(f,F)+\|q_t^{1+n/2}F(t)\|_D+\|r_t^{1+n/2}F(t)\|_D\big).$$

This proves the theorem.

Before applying Theorem 5.3 and Theorem 5.4 to solutions of equation (5.1a), we note that

$$\frac{1+t+|x|}{1+\big|t-|x|\big|} \leq 2(1+q_t(x)) \leq 4\frac{1+t+|x|}{1+\big|t-|x|\big|}, \quad t\geq 0, x\in\mathbb{R}^3, \tag{5.33}$$

and we introduce the following notation:

$$h_Y=\xi_Y^D h, \quad g_Y=\xi_Y^D g, \quad G_{Y\mu}=(\xi_Y^M G)_\mu, \tag{5.34a}$$

where $Y\in U(\mathfrak{p}), 0\leq\mu\leq 3$ and where h, g and G are the functions in equation (5.1a).

We now introduce the summation symbol $\sum_{Y_1,\ldots,Y_p}^{Y}, Y\in U(\mathfrak{p})$. Let $\mathcal{V}$ be a real vector space and $f:(U(\mathfrak{p})^p)\to\mathcal{V}$. We define inductively $\sum_{Y_1,\ldots,Y_p}^{Y} f(Y_1,\ldots,Y_p)$, for $Y\in\Pi'$, by

$$\sum_{Y_1,\ldots,Y_p}^{\mathbb{I}} f(Y_1,\ldots,Y_p)=f(\mathbb{I},\ldots,\mathbb{I}) \tag{5.34b}$$

$$\sum_{Y_1,\ldots,Y_p}^{XY} f(Y_1,\ldots,Y_p) \tag{5.34c}$$

$$=\sum_{1\leq l\leq p}\sum_{Z_1,\ldots,Z_p}^{Y} f(Z_1,\ldots,Z_{l-1},XZ_l,Z_{l+1},\ldots,Z_p), X\in\Pi, XY\in\Pi'$$

If $Y\in U(\mathfrak{p})$, we extend this definition by linearity in Y. If $\mathcal{E}\subset(\Pi')^p$ and if in $\sum_{Y_1,\ldots,Y_p}^{Y}$ $f(Y_1,\ldots,Y_p)$ we add only the elements $f(Y_1,\ldots,Y_p)$ for which $(Y_1,\ldots,Y_p)\in\mathcal{E}$, the element of $\mathcal{V}$ so obtained is denoted by $\sum_{(Y_1,\ldots,Y_p)\in\mathcal{E}}^{Y} f(Y_1,\ldots,Y_p)$.

Theorem 5.5. *Let $n\geq 0$, $k\geq 1, 0\leq L\leq n+k-1$ be integers, let $G_Y\in C^0(\mathbb{R}^+,L^\infty(\mathbb{R}^3,\mathbb{R}^4))$ for $0\leq|Y|\leq L$, $Y\in\Pi'$ and let $G_Y\in C^0(\mathbb{R}^+,L^2_{\text{loc}}(\mathbb{R}^3,\mathbb{R}^4))$ for $|Y|\leq n+k-1$, $Y\in\Pi'$. Let*

$$\tau_0^{(l)}(t)=\sum_{\substack{|Y|\leq l\\ Y\in\Pi'}}(1+t)^{1/2}\|G_Y(t)\|_{L^\infty}, \quad l\geq 0,$$
$$\tau_1^{(l)}(t)=\sum_{\substack{|Y|\leq l\\ Y\in\Pi'}}\sup_{x\in\mathbb{R}^3}((1+t+|x|)^{1/2}|G_Y(t,x)|), \quad l\geq 0,$$

and let

$$\tau_{i,j}(t)=\sum_{1\leq p\leq j}\sum_{l_1+\cdots+l_p=j}\prod_{1\leq q\leq p}\tau_i^{(l_q)}(t), \quad i=0,1, j\geq 0.$$

For $h \in C^0(\mathbb{R}^+, D)$*, let* $g = (i\gamma^\mu \partial_\mu + m - \gamma^\mu G_\mu)h$*, and let*

$$(\lambda_0(t))(x) = q_t(x), (\lambda_1(t))(x) = q_t(x) + r_t(x), \quad t \geq 0, x \in \mathbb{R}^3.$$

If $h_Y \in C^0(\mathbb{R}^+, D)$ *for* $0 \leq |Y| \leq n+k, Y \in \Pi'$*, and if* $G_{Y_1 \mu} h_{Y_2} \in C^0(\mathbb{R}^+, D)$ *for* $L+1 \leq |Y_1| \leq n+k-1$*,* $0 \leq |Y_2| \leq n+k-2-L, Y_1, Y_2 \in \Pi'$*, then*

$$\begin{aligned}
&\wp_n^D\big((1+\lambda_i(t))^{k/2} h(t)\big) \\
&\quad \leq C'_k\Big(\wp^D_{n+k}(h(t))^2 + \sum_{0 \leq j \leq k-1} \Big(\wp^D_{n+j}\big((1+\lambda_i(t))^{(k+1-j)/2} g(t)\big)^2 \\
&\qquad + \sum_{\substack{Y \in \Pi' \\ |Y| \geq L+1 \\ |Y| = n+j}} \|(1+\lambda_i(t))^{(k+1-j)/2} \gamma^\mu G_{Y\mu}(t) h_{\mathbb{I}}(t)\|_D^2\Big)\Big)^{1/2} \\
&\qquad + C'_{n+k}\Big(\wp^D_{n+k-1}(h(t)) + \sum_{\substack{0 \leq j \leq k-1 \\ Z_1, Z_2 \in \Pi' \\ |Z_1|+|Z_2| = n+j \\ 1 \leq |Z_2| \leq n+j-L-1}} \|(1+\lambda_i(t))^{(k+1-j)/2} \gamma^\mu G_{Z_1\mu}(t) h_{Z_2}(t)\|_D\Big) \\
&\qquad + C'_{n+k} \sum_{1 \leq l \leq L} (1+\tau_{i,l}(t))\Big(\wp^D_{n+k-l}(h(t)) + \sum_{\substack{0 \leq j \leq k-1 \\ n+j-l \geq 0}} \wp^D_{n+j-l}\big((1+\lambda_i(t))^{(k+1-j)/2} g(t)\big) \\
&\qquad + \sum_{\substack{0 \leq j \leq k-1 \\ Z_1, Z_2 \in \Pi' \\ |Z_2| \leq n+j-L-1 \\ |Z_1|+|Z_2| \leq n+j-l}} \|(1+\lambda_i(t))^{(k+1-j)/2} \gamma^\mu G_{Z_1\mu}(t) h_{Z_2}(t)\|_D\Big),
\end{aligned}$$

$i = 0, 1, t \geq 0$*. The constants* $C'_N, N \geq 1$*, depend only on* $\tau_{i,0}(t)$*.*

Proof. According to the definition of g, we obtain that

$$g_Y = (i\gamma^\mu \partial_\mu + m)h_Y - \sum_{Y_1, Y_2}^{Y} \gamma^\mu G_{Y_1\mu} h_{Y_2}, \quad Y \in \Pi', \tag{5.35}$$

$|Y| \leq n+k-1$. Since $\|(i\gamma^\mu\partial_\mu + m)h_Y(t)\|_D \leq C\wp^D_{n+k}(h(t))$, $\|G_{Y_1\mu}(t) h_{Y_2}(t)\|_D \leq \|G_{Y_1\mu}(t)\|_{L^\infty} \|h_{Y_2}(t)\|_D$ for $0 \leq |Y_1| \leq L$, $0 \leq |Y_2| \leq n+k-1$ and since $G_{Y_1\mu} h_{Y_2} \in C^0(\mathbb{R}^+, D)$ for $L+1 \leq |Y_1| \leq n+k-1$, $0 \leq |Y_2| \leq n+k-2-L$ according to the hypothesis it follows that $g_Y \in C^0(\mathbb{R}^+, D)$ for $|Y| \leq n+k-1, Y \in \Pi'$.

For given $Y \in \Pi', l \geq 1, |Y| + l \leq n+k$, let

$$F = -i \sum_{\substack{Y_1, Y_2 \\ |Y_2| \leq |Y|-1}}^{Y} \gamma^0 \gamma^\mu G_{Y_1\mu} h_{Y_2} - i\gamma^0 g_Y \quad \text{and} \quad V = -i\gamma^0\gamma^\mu G_\mu. \tag{5.36}$$

It follows as in the case of g_Y that $F \in C^0(\mathbb{R}^+, D)$. Moreover $V \in C^0(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathrm{Mat}(4, \mathbb{C})))$, where $\mathrm{Mat}(4, \mathbb{C})$ is the linear space of 4×4 complex matrices. According to the definition of $\xi^D_{M_{0i}}$ it follows from (5.34a) and (5.36) that

$$\xi^D_{M_{0i}} h_Y = t\partial_i h_Y + x_i V h_Y + x_i F + \sigma_{0i} h_Y, \quad 1 \leq i \leq 3. \tag{5.37}$$

It follows from (5.37), Theorem 5.3 and Theorem 5.4, that

$$\begin{aligned}&\|q_t^{l/2}h_Y(t)\|_D &(5.38a)\\ &\quad\leq C_l\Big((1+\tau_{0,0}(t))\wp_1^D(q_t^{(l-1)/2}h_Y(t))+\|q_t^{(l+1)/2}F(t)\|_D+\|q^{(l-1)/2}F(t)\|_D\Big)\end{aligned}$$

and

$$\begin{aligned}&\|r_t^{l/2}h_Y(t)\|_D &(5.38b)\\ &\quad\leq C_l(1+\tau_{1,0}(t))\Big(\wp_1^D(q^{(l-1)/2}h_{\cdot Y}(t))+\wp_1^D(r_t^{(l-1)/2}h_{\cdot Y}(t))\\ &\qquad+\|r_t^{(l+1)/2}F(t)\|_D+\|r_t^{(l-1)/2}F(t)\|_D\Big),\end{aligned}$$

where $\wp_1^D$ is applied to the linear functions $Z\mapsto q_t^{(l-1)/2}h_{ZY}(t)$ and $Z\mapsto r_t^{(l-1)/2}h_{ZY}(t)$, $Z\in\Pi'$. Using that $(1+q_t)^{l/2}\leq C_l(1+q_t^{l/2})$ and that $(1+q_t+r_t)^{l/2}\leq C_l(1+q_t^{l/2}+r_t^{l/2})$ for some constant C_l it follows from (5.38a), (5.38b) and the expression (5.36) of F that

$$\begin{aligned}&\|(1+\lambda_i(t))^{l/2}h_Y(t)\|_D &(5.39)\\ &\quad\leq C_l(1+\tau_{i,0}(t))\Big(\wp_1^D\big((1+\lambda_i(t))^{(l-1)/2}h_{\cdot Y}(t)\big)\\ &\qquad+\sum_{\substack{Y_1,Y_2\\|Y_2|\leq|Y|-1}}^{Y}\|(1+\lambda_i(t))^{(l+1)/2}\gamma^\mu G_{Y_1\mu}(t)h_{Y_2}(t)\|_D\\ &\qquad+\|(1+\lambda_i(t))^{(l+1)/2}g_Y(t)\|_D\Big),\quad i=0,1,t\geq 0,l\geq 1,\end{aligned}$$

$Y\in\Pi',|Y|+l\leq n+k$ for some constants C_l depending only on l.

Let $l=1$ and $Y=\mathbb{I}$, then it follows from (5.39) that

$$\begin{aligned}\|(1+\lambda_i(t))^{1/2}h_{\mathbb{I}}(t)\|_D&=\wp_0^D\big((1+\lambda_i(t))^{1/2}h(t)\big) &(5.40)\\ &\leq C_1(1+\tau_{i,0}(t))\big(\wp_1^D(h(t))+\wp_0^D((1+\lambda_i(t))g(t))\big),\quad i=0,1,t\geq 0,\end{aligned}$$

which shows that the inequality of the theorem is true for $n=0,k=1$. Suppose that it is true for $0\leq n\leq N$ and $k=1$, for some $N\geq 0$. Let $Y\in\Pi',|Y|=N+1$, and let $0\leq L\leq N+1$. It follows from (5.39) that

$$\begin{aligned}&\|(1+\lambda_i(t))^{1/2}h_Y(t)\|_D &(5.41)\\ &\quad\leq C_1(1+\tau_{i,0}(t))\Big(\wp_1^D(h_{\cdot Y}(t))+\sum_{\substack{Y_1,Y_2\\|Y_1|\leq L\\|Y_2|\leq|Y|-1}}^{Y}8\tau_{i,|Y_1|}(t)\wp_{|Y_2|}^D\big((1+\lambda_i(t))^{1/2}h(t)\big)\\ &\qquad+\sum_{\substack{Y_1,Y_2\\|Y_2|\leq N-L}}^{Y}\|(1+\lambda_i(t))\gamma^\mu G_{Y_1\mu}(t)h_{Y_2}(t)\|_D+\|(1+\lambda_i(t))g_Y(t)\|_D\Big),\end{aligned}$$

where we have used that $1+\lambda_0(t)\leq 2(1+t)$ and $1+(\lambda_1(t))(x)\leq 3(1+t+|x|)$ and where we have used that $|Y_1|\geq L+1$ if and only if $|Y_2|\leq N-L$ and that $N-L\leq |Y|-1$. Let

$$I_Y(t)=C_1(1+\tau_{i,0}(t))\sum^{Y}_{\substack{Y_1,Y_2\\|Y_1|\leq L\\|Y_2|\leq|Y|-1}}8\tau_{i,|Y_1|}(t)\wp^D_{|Y_2|}\big((1+\lambda_i(t))^{1/2}h(t)\big), \tag{5.42}$$

be the second term on the right-hand side of inequality (5.41). Since $|Y|=N+1$, we obtain that

$$I_Y(t)\leq C''_N\sum_{1\leq j\leq L}\tau_{i,j}(t)\wp^D_{N+1-j}\big((1+\lambda_i(t))^{1/2}h(t)\big), \tag{5.43}$$

where C''_N is a constant depending only on $\tau_{i,0}(t)$. Inequality (5.41) and the definition of $I_Y(t)$ give that

$$\begin{aligned}
&\|(1+\lambda_i(t))^{1/2}h_Y(t)\|^2_D\\
&\quad\leq C^2C_1^2(1+\tau_{i,0}(t))^2\Big(\wp^D_1(h_{\cdot Y}(t))^2\\
&\qquad+\|(1+\lambda_i(t))\gamma^\mu G_{Y\mu}(t)h_{\mathbb{I}}(t)\|^2_D+\|(1+\lambda_i(t))g_Y(t)\|^2_D\Big)\\
&\qquad+C^2\Big(C_1(1+\tau_{i,0}(t))\sum^{Y}_{\substack{Y_1,Y_2\\1\leq|Y_2|\leq N-L}}\|(1+\lambda_i(t))\gamma^\mu G_{Y_1\mu}(t)h_{Y_2}(t)\|_D+I_Y(t)\Big)^2,
\end{aligned}$$

for some constant C (independent of all the variables in the inequality). Summation over $Y\in\Pi', |Y|=N+1$ and the fact that

$$\sum_{\substack{|Y|=N+1\\Y\in\Pi'}}\wp^D_1(h_{\cdot Y}(t))^2\leq 2\wp^D_{N+2}(h(t))^2+C_N\wp^D_{N+1}(h(t))^2,$$

for some constant C_N, give that

$$\begin{aligned}
&\sum_{\substack{Y\in\Pi'\\|Y|=N+1}}\|(1+\lambda_i(t))^{1/2}h_Y(t)\|^2_D \qquad\qquad (5.44)\\
&\leq C_1'^2\Big(\wp^D_{N+2}(h(t))^2+\sum_{\substack{Y\in\Pi'\\|Y|=N+1\\|Y|\geq L+1}}\|(1+\lambda_i(t))\gamma^\mu G_{Y\mu}(t)h_{\mathbb{I}}(t)\|^2_D+\wp^D_{N+1}\big((1+\lambda_i(t))g(t)\big)^2\Big)\\
&+C_N'^2\Big(\wp^D_{N+1}(h(t))+\sum_{\substack{Y\in\Pi'\\|Y|=N+1}}I_Y(t)+\sum_{\substack{Y_1,Y_2\in\Pi'\\|Y_1|+|Y_2|=N+1\\1\leq|Y_2|\leq N-L}}\|(1+\lambda_i(t))\gamma^\mu G_{Y_1\mu}(t)h_{Y_2}(t)\|_D\Big)^2,
\end{aligned}$$

where C_1' and C_N' are constants depending only on $\tau_{i,0}(t)$.

According to the induction hypothesis, we obtain from the inequality of the theorem that

$$\begin{aligned}
&\wp_n^D\big((1+\lambda_i(t))^{1/2}h(t)\big) && (5.45)\\
&\quad\le C'_n \sum_{0\le l\le K}(1+\tau_{i,l}(t))\Big(\wp^D_{n+1-l}(h(t))+\wp^D_{n-l}\big((1+\lambda_i(t))g(t)\big)\\
&\qquad+\sum_{\substack{Z_1,Z_2\in\Pi'\\|Z_1|+|Z_2|\le n-l\\|Z_2|\le n-K-1}}\|(1+\lambda_i(t))\gamma^\mu G_{Z_1\mu}(t)h_{Z_2}(t)\|_D\Big),
\end{aligned}$$

for $0\le n\le N$, $0\le K\le n$, where C'_n is a constant depending only on $\tau_{i,0}(t)$. Since $\tau_{i,l}(t)\le\tau_{i,l+1}(t)$, it follows from (5.45) that

$$\begin{aligned}
&\wp_N^D\big((1+\lambda_i(t))^{1/2}h(t)\big) && (5.46a)\\
&\quad\le C'_N \sum_{1\le l\le L}(1+\tau_{i,l}(t))\Big(\wp^D_{N+2-l}(h(t))+\wp^D_{N+1-l}\big((1+\lambda_i(t)g(t)\big)\\
&\qquad+\sum_{\substack{|Z_1|+|Z_2|\le N+1-l\\|Z_2|\le N+1-L-1}}\|(1+\lambda_i(t))\gamma^\mu G_{Z_1\mu}(t)h_{Z_2}(t)\|_D\Big),
\end{aligned}$$

if $1\le L\le N+1$. According to the induction hypothesis the inequality of the theorem (with $L=0$) gives that

$$\begin{aligned}
&\wp_N^D\big((1+\lambda_i(t))^{1/2}h(t)\big) && (5.46b)\\
&\quad\le C'_1\Big(\wp^D_{N+1}(h(t))^2+\wp^D_N\big((1+\lambda_i(t))g(t)\big)^2\Big)^{1/2}\\
&\qquad+C'_N\sum_{\substack{Z_1,Z_2\in\Pi'\\|Z_1|+|Z_2|\le N\\|Z_2|\le N-1}}\|(1+\lambda_i(t))\gamma^\mu G_{Z_1\mu}(t)h_{Z_2}(t)\|_D,
\end{aligned}$$

where C'_1 and C'_N are constants depending only on $\tau_{i,0}(t)$.

Adding $\big(\wp^D_N((1+\lambda_i(t))^{1/2}h(t)\big)^2$ to both sides of inequality (5.44) and using (5.46b) for $L=0$ and (5.46a) for $1\le L\le N+1$ we obtain that

$$\begin{aligned}
&\wp_{N+1}^D\big((1+\lambda_i(t))^{1/2}h(t)\big) && (5.47)\\
&\le C'_1\Big(\wp^D_{N+2}(h(t))^2+\wp^D_{N+1}\big((1+\lambda_i(t))g(t)\big)^2+\sum_{\substack{Y\in\Pi'\\|Y|=N+1\\|Y|\ge L+1}}\|(1+\lambda_i(t))\gamma^\mu G_{Y\mu}(t)h_{\mathbb{I}}(t)\|_D^2\Big)^{1/2}\\
&\quad+C'_N\Big(\wp^D_{N+1}(h(t))+\sum_{\substack{Z_1,Z_2\in\Pi'\\|Z_1|+|Z_2|=N+1\\1\le|Z_2|\le N-L}}\|(1+\lambda_i(t))\gamma^\mu G_{Z_1\mu}(t)h_{Z_2}(t)\|_D
\end{aligned}$$

$$+ \sum_{\substack{Y\in\Pi' \\ |Y|=N+1}} I_Y(t) + \sum_{1\le l\le L} (1+\tau_{i,l}(t))\big(\wp^D_{N+2-l}(h(t)) + \wp^D_{N+1-l}((1+\lambda_i(t))g(t))$$

$$+ \sum_{\substack{Z_1,Z_2\in\Pi' \\ |Z_2|\le N-L \\ |Z_1|+|Z_2|\le N+1-l}} \|(1+\lambda_i(t))\gamma^\mu G_{Z_1\mu}(t)h_{Z_2}(t)\|_D\Big)\Big),$$

where C_1' and C_N' are constants depending only on $\tau_{i,0}(t)$.

Let $n = N+1-j$ and $K = L-j$ for $1 \le j \le L$, in inequality (5.45). Then $0 \le n \le N$ and $0 \le K \le n$, so it follows from inequalities (5.43) and (5.45) that

$$I_Y(t) \le C_N' \sum_{1\le j\le L} \sum_{0\le l\le L-j} \tau_{i,l+j}(t)\Big(\wp^D_{N+2-l-j}(h(t)) + \wp^D_{N+1-l-j}\big((1+\lambda_i(t))g(t)\big)$$

$$+ \sum_{\substack{Z_1,Z_2\in\Pi' \\ |Z_2|\le N-L \\ |Z_1|+|Z_2|\le N+1-l-j}} \|(1+\lambda_i(t))\gamma^\mu G_{Z_1\mu}(t)h_{Z_2}(t)\|_D\Big),$$

where C_N' is a constant depending only on $\tau_{i,0}(t)$. We have here used the fact that $(1+\tau_{i,l}(t))\tau_{i,j}(t) \le C_{l,j}\tau_{i,l+j}(t)$ for some constant $C_{l,j}$. Substituting $j' = j, l' = j+l$, we obtain that

$$I_Y(t) \le C_N' \sum_{1\le l'\le L} \tau_{i,l'}(t)\Big(\wp^D_{N+2-l'}(h(t)) + \wp^D_{N+1-l'}\big((1+\lambda_i(t))g(t)\big) \tag{5.48}$$

$$+ \sum_{\substack{Z_1,Z_2\in\Pi' \\ |Z_2|\le N-L \\ |Z_1|+|Z_2|\le N+l-l'}} \|(1+\lambda_i(t))\gamma^\mu G_{Z_1\mu}(t)h_{Z_2}(t)\|_D\Big),$$

where C_N' is a constant depending only on $\tau_{i,0}(t)$. It follows from inequalities (5.47) and (5.48) that $\wp^D_{N+1}\big((1+\lambda_i(t))^{1/2}h(t)\big)$ is bounded by the left-hand side of inequality (5.47) without the term $I_Y(t)$ and with a new constant C_N' depending only on $\tau_{i,0}(t)$. This proves the inequality of the theorem with $0 \le n, 0 \le L \le n$ and $k=1$, by induction.

For $n \ge 0, k \ge 1, 0 \le L \le n+k-1$, let

$$Q^{(k)}_{n,L}(t) \tag{5.49}$$

$$= C_k'\Big(\wp^D_{n+k}(h(t))^2 + \sum_{0\le j\le k-1}\Big(\wp^D_{n+j}\big((1+\lambda_i(t))^{(k+1-j)/2}g(t)\big)^2$$

$$+ \sum_{\substack{Y\in\Pi' \\ |Y|=n+j \\ |Y|\ge L+1}} \|(1+\lambda_i(t))^{(k+1-j)/2}\gamma^\mu G_{Y\mu}(t)h_{\mathbb{I}}(t)\|^2_D\Big)\Big)^{1/2}$$

$$+ C_{n+k}'\Big(\wp^D_{n+k-1}(h(t)) + \sum_{\substack{0\le j\le k-1 \\ Z_1,Z_2\in\Pi' \\ |Z_1|+|Z_2|=n+j \\ 1\le|Z_2|\le n+j-L-1}} \|(1+\lambda_i(t))^{(k+1-j)/2}\gamma^\mu G_{Z_1\mu}(t)h_{Z_2}(t)\|_D\Big)$$

$$+ C'_{n+k} \sum_{1\le l\le L} (1+\tau_{i,l}(t))\Big(\wp^D_{n+k-l}(h(t)) + \sum_{\substack{0\le j\le k-1\\ n+j-l\ge 0}} \wp^D_{n+j-l}\big((1+\lambda_i(t))^{(k+1-j)/2} g(t)\big)$$

$$+ \sum_{\substack{0\le j\le k-1\\ Z_1,Z_2\in\Pi'\\ |Z_1|+|Z_2|=n+j-l\\ |Z_2|\le n+j-L-1}} \|(1+\lambda_i(t))^{(k+1-j)/2}\gamma^\mu G_{Z_1\mu}(t) h_{Z_2}(t)\|_D\Big), \quad t\ge 0,$$

where $C'_N, N\ge 1$, are constants depending only on $\tau_{i,0}(t)$. We note that with an appropriate choice of the constants $C'_N, N\ge 1$,

$$Q^{(k)}_{n,L}(t) \le Q^{(k)}_{n+1,L}(t) \quad \text{for } n\ge 0, k\ge 1, 0\le L\le n+k-1, \tag{5.50a}$$

$$Q^{(k)}_{n+1,L}(t) \le Q^{(k+1)}_{n,L}(t) \quad \text{for } n\ge 0, k\ge 1, 0\le L\le n+k, \tag{5.50b}$$

$$Q^{(k)}_{n,L}(t) \le Q^{(k)}_{n+1,L+1}(t) \quad \text{for } n\ge 0, k\ge 1, 0\le L\le n+k-1, \tag{5.50c}$$

and moreover that the inequality of the theorem reads

$$\wp^D_n\big((1+\lambda_i(t))^{k/2} h(t)\big) \le Q^{(k)}_{n,L}(t), \tag{5.50d}$$

where $n\ge 0, k\ge 1, 0\le L\le n+k-1, t\ge 0$.

We have proved that inequality (5.50d) is true for $n\ge 0, k=1, 0\le L\le n$ and we make the induction hypothesis H_K that it is true for $n\ge 0, 1\le k\le K, 0\le L\le n+k-1$, where $K\ge 1$.

It follows from inequality (5.39) that

$$\begin{aligned} &\|(1+\lambda_i(t))^{(k+1)/2} h_{\mathbb{I}}(t)\|_D \\ &\quad\le C'_{K+1}\Big(\wp^D_1\big((1+\lambda_i(t))^{K/2} h(t)\big) + \wp^D_0\big((1+\lambda_i(t))^{(K+2)/2} g(t)\big)\Big), \end{aligned} \tag{5.51}$$

where C'_{K+1} is a constant depending only on $\tau_{i,0}(t)$. Using the induction hypothesis H_K for the first term on the right-hand side of inequality (5.51) and the definition of $Q^{(K+1)}_{0,L}(t)$ for the second term, we obtain, choosing C'_{K+1} appropriately,

$$\wp^D_0\big((1+\lambda_i(t))^{(K+1)/2} h(t)\big) \le Q^{(K)}_{1,L}(t) + Q^{(K+1)}_{0,L}(t).$$

Inequality (5.50b) then gives that

$$\wp^D_0\big((1+\lambda_i(t))^{(K+1)/2} h(t)\big) \le Q^{(K+1)}_{0,L}(t), \quad 0\le L\le K, \tag{5.52}$$

after a redefinition of the constants $C'_N, N\ge 1$. This shows that inequality (5.50d) is true for $n=0$, $k=K+1$, $0\le L\le K$, if the hypothesis H_K is true. We now make the induction hypothesis $H_{K+1,N}$, where $K\ge 1, N\ge 0$, that (5.50d) is true for $0\le n\le N$, $1\le k\le K+1, 0\le L\le n+k-1$. Thus $H_{K+1,0}$ is true if H_K is true. Let $0\le L\le N+K+1$

and let $Y \in \Pi', |Y| = N+1$. It then follows from inequality (5.39), in a similar way as (5.41) was obtained, that

$$\begin{aligned} &\|(1+\lambda_i(t))^{(K+1)/2} h_Y(t)\|_D \\ &\quad \leq C_{K+1}(1+\tau_{i,0}(t))\Big(\wp_1^D\big((1+\lambda_i(t))^{K/2}h_Y(t)\big) + \|(1+\lambda_i(t))^{(K+2)/2} g_Y(t)\|_D \\ &\qquad + \sum^{Y}_{\substack{Y_1,Y_2 \\ |Y_1|\leq L \\ |Y_2|\leq |Y|-1}} \tau_{i,|Y_1|}(t)\wp^D_{|Y_2|}\big((1+\lambda_i(t))^{(K+1)/2}h(t)\big) \\ &\qquad + \sum^{Y}_{\substack{Y_1,Y_2 \\ |Y_2|\leq N-L}} \|(1+\lambda_i(t))^{(K+2)/2}\gamma^\mu G_{Y_1\mu}(t) h_{Y_2}(t)\|_D\Big), \end{aligned} \tag{5.53}$$

where we have redefined the constant C_{K+1}. Defining

$$\begin{aligned} &I_Y^{(K+1)}(t) \\ &\quad = C_{K+1}(1+\tau_{i,0}(t)) \sum_{\substack{Y_1,Y_2 \\ |Y_1|\leq L \\ |Y_2|\leq |Y|-1}} \tau_{i,|Y_1|}\wp^D_{|Y_2|}\big((1+\lambda_i(t))^{(K+1)/2}h(t)\big), \end{aligned} \tag{5.54}$$

where $|Y| = N+1, Y \in \Pi', 0 \leq L \leq N+K+1$, we obtain from inequality (5.53) in the same way as we obtained inequality (5.44)

$$\begin{aligned} &\sum_{\substack{Y\in\Pi' \\ |Y|=N+1}} \|(1+\lambda_i(t))^{(K+1)/2} h_Y(t)\|_D^2 \\ &\quad \leq C''^2_{K+1}\Big(\wp^D_{N+2}\big((1+\lambda_i(t))^{K/2}h(t)\big)^2 \\ &\qquad + \sum_{\substack{Y\in\Pi' \\ |Y|=N+1 \\ |Y|\geq L+1}} \|(1+\lambda_i(t))^{(K+2)/2} G_{Y\mu}(t) h_{\mathbb{I}}(t)\|^2_D + \wp^D_{N+1}\big((1+\lambda_i(t))^{(K+2)/2}g(t)\big)^2\Big) \\ &\qquad + (C''_{N+K+2})^2\Big(\wp^D_{N+1}\big((1+\lambda_i(t))^{K/2}h(t)\big) + \sum_{\substack{Y\in\Pi' \\ |Y|=N+1}} I_Y^{(K+1)}(t) \\ &\qquad + \sum_{\substack{Y_1,Y_2\in\Pi' \\ |Y_1|+|Y_2|=N+1 \\ 1\leq |Y_2|\leq N-L}} \|(1+\lambda_i(t))^{(K+2)/2}\gamma^\mu G_{Y_1\mu}(t) h_{Y_2}(t)\|_D\Big)^2, \end{aligned} \tag{5.55}$$

where C''_{K+1} and C''_{N+K+2} are constants depending only on $\tau_{i,0}(t)$. Adding $\big(\wp^D_N((1+\lambda_i(t))^{(K+1)/2}h(t))\big)^2$ to both sides of inequality (5.55), redefining the constants C''_{K+1} and C''_{N+K+2}, using that $\wp^D_N\big((1+\lambda_i(t))^{(K+1)/2}h(t)\big) \leq Q^{(K+1)}_{N,L'}$, for $0 \leq L' \leq N+K$, according to the hypothesis $H_{K+1,N}$ and using that $\wp^D_{N+2}\big((1+\lambda_i(t))^{K/2}h(t)\big) \leq Q^{(K)}_{N+2,L}(t)$ for

$0 \le L \le N+K+1$ according to H_K, we obtain that

$$\begin{aligned}
&\wp^D_{N+1}\big((1+\lambda_i(t))^{(K+1)/2}h(t)\big)^2 \qquad (5.56)\\
&\quad \le Q^{(K+1)}_{N,L'}(t)^2 + (C''_{K+1})^2\Big(Q^{(K)}_{N+2,L}(t)^2\\
&\qquad + \sum_{\substack{Y\in\Pi'\\ |Y|=N+1\\ |Y|\ge L+1}} \|(1+\lambda_i(t))^{(K+2)/2}G_{Y\mu}(t)h_{\mathbb{I}}(t)\|^2_D + \wp^D_{N+1}\big((1+\lambda_i(t))^{(K+2)/2}g(t)\big)^2\Big)\\
&\qquad + (C''_{N+K+2})^2\Big(Q^{(K)}_{N+1,L'}(t) + \sum_{\substack{Y\in\Pi'\\ |Y|=N+1}} I^{(K+1)}_Y(t)\\
&\qquad + \sum_{\substack{Y_1,Y_2\in\Pi'\\ |Y_1|+|Y_2|=N+1\\ 1\le|Y_2|\le N-L}} \|(1+\lambda_i(t))^{(K+2)/2}\gamma^\mu G_{Y_1\mu}(t)h_{Y_2}(t)\|_D\Big)^2,
\end{aligned}$$

for $0 \le L \le N+K+1$ and $0 \le L' \le N+K$.

Using hypothesis $H_{K+1,N}$ it follows from definition (5.54) of $I^{(K+1)}_Y(t)$, that

$$\sum_{\substack{Y\in\Pi'\\ |Y|=N+1}} I^{(K+1)}_Y(t) \le C''_{K+1}\sum_{1\le l\le L}\tau_{i,l}(t)Q^{(K+1)}_{N+1-l,L-l}(t),$$

$0 \le L \le N+K+1$. Since $\tau_{i,l}(t)(1+\tau_{i,j}(t)) \le \tau_{i,l+j}(t)$ it follows from expression (5.49) of $Q^{(k)}_{n,L}(t)$ that

$$\begin{aligned}
&\sum_{\substack{Y\in\Pi'\\ |Y|=N+1}} I^{(K+1)}_Y(t) \qquad (5.57)\\
&\quad \le C''_{N+K+2}\sum_{1\le l\le L}(1+\tau_{i,l}(t))\Big(\wp^D_{N+K+2-l}(h(t))\\
&\qquad + \sum_{\substack{0\le j\le K\\ N+1+j-l\ge 0}} \wp^D_{N+1+j-l}\big((1+\lambda_i(t))^{(K+2-j)/2}g(t)\big)\\
&\qquad + \sum_{\substack{0\le j\le K\\ Z_1,Z_2\in\Pi'\\ |Z_1|+|Z_2|\le N+1+j-l\\ |Z_2|\le N+1+j-L-1}} \|(1+\lambda_i(t))^{(K+2-j)/2}\gamma^\mu G_{Z_1\mu}(t)h_{Z_2}(t)\|_D\Big),
\end{aligned}$$

where C''_{N+K+2} is a constant depending only on $\tau_{i,0}(t)$. If $0 \le L \le N+K$, then it follows from inequality (5.56) with $L = L'$, inequality (5.57) and expression (5.49) of $Q^{(K+1)}_{N+1,L}$, that

$$\wp^D_{N+1}\big((1+\lambda_i(t))^{(K+1)/2}h(t)\big) \le Q^{(K+1)}_{N+1,L}(t), \quad 0 \le L \le N+K, \qquad (5.58)$$

after a suitable definition of C'_{K+1} and C'_{K+N+2}. If $L = N + k + 1$, then it follows using (5.50c) for $Q^{(K+1)}_{N,L-1}$ and inequalities (5.56) and (5.57), that

$$\wp^D_{N+1}\big((1+\lambda_i(t))^{(K+1)/2}h(t)\big) \leq Q^{(K+1)}_{N+1,L}(t), \quad L = N+k+1, \tag{5.59}$$

after a suitable definition of C'_{K+1} and C'_{K+N+2}. Inequalities (5.58) and (5.59) prove that $H_{K+1,N+1}$ is true if $H_{K+1,N}$ and H_K are true. Since $H_{K+1,0}$ is true if H_K is true, it follows by induction that $H_{K+1,N}$ is true for all $N \geq 0$, i.e. H_{K+1} is true if H_K is true. Since H_1 is true it now follows by induction that H_K is true for all $K \geq 1$. This proves the theorem.

In order to eliminate L^∞-norms coming from the right-hand side of the inequality of Theorem 5.5 and in later energy estimates we shall *derive appropriate $L^2 - L^\infty$ estimates for the solution of the inhomogeneous Dirac equation and wave equation.* Let u, as in Proposition 2.15, be the solution of the wave equation $\Box u = 0$ with initial data $(f, \dot{f}) \in M^\rho_\infty$. Since the evolution operator in M^ρ_0 defined by the wave equation is unitary in M^ρ_0 it follows that

$$\|(f,\dot{f})\|^2_{M_n} = \sum_{\substack{Y\in\Pi' \\ |Y|\leq n}} \|T^{M1}_{Y(t)}(u(t),\dot{u}(t))\|^2_{M_0}, \quad n \geq 0, \tag{5.60}$$

where $Y(t)$ is defined by (1.11). If $(g, \dot{g}) \in M^\rho_\infty$, is an initial data for the wave equation at time t, if follows from Proposition 2.15 and (5.60) that

$$\begin{aligned}(1+|x|+t)^{3/2-\rho}|g(x)| \\ + (1+|x|+t) \sum_{0\leq|\nu|\leq n-1} (1+\big|t-|x|\big|)^{3/2-\rho+|\nu|}(|\partial^\nu\nabla g(x)| + |\partial^\nu \dot{g}(x)|) \\ \leq C_{n,\rho}\Big(\sum_{\substack{Y\in\Pi' \\ |Y|\leq n+2}} \|T^{M1}_{Y(t)}(g,\dot{g})\|^2_{M_0}\Big)^{1/2}, \quad n\geq 1, t\geq 0, 1/2<\rho<1.\end{aligned} \tag{5.61}$$

Similarly, using the $L^1 - L^\infty$ estimate in [21], we obtain

$$(1+|x|+t)^{3/2}|\alpha(x)| \leq C\Big(\sum_{\substack{Y\in\Pi' \\ |Y|\leq 3}} \|T^{D1}_{Y(t)}\alpha\|^2_D\Big)^{1/2}, \quad t\geq 0, \tag{5.62}$$

$\alpha \in D_\infty$. We note that estimates (5.61) and (5.62) are relations expressed directly on the space of initial data and that the bound is given by the canonical seminorms (in the space of differentiable vectors) composed by the time evolution defined in (1.11) of the enveloping algebra of the Poincaré Lie algebra. We shall generalize this to a certain extent to the nonlinear case by first expressing the derivatives ∂_i and the second derivatives $\partial_i\partial_j, 1 \leq i \leq j$, in terms of the nonlinear generators for the representation of $\mathfrak{sl}(2,\mathbb{C})$ and then use the Sobolev inequalities for weighted L^p spaces developed in [14].

In the case of the wave equation we introduce, for $F, \dot{F} \in C^\infty(\mathbb{R}^3), t \in \mathbb{R}, x \in \mathbb{R}^3, i, j \in \{1,2,3\}$,

$$F_{M_{0i}}(x) = x_i\dot{F}(x), \quad F_{M_{ij}}(x) = (x_i\partial_j - x_j\partial_i)F(x), \quad F_{P_i} = \partial_i F, \quad F_{P_0} = \dot{F}, \tag{5.63}$$

where $M_{\mu\nu}$ are for $0 \leq \mu < \nu \leq 3$ the generators of $\mathfrak{sl}(2,\mathbb{C})$ and $M_{\mu\nu} = -M_{\nu\mu}$. Moreover for $f, \dot{f} \in C^\infty(\mathbb{R}^3)$ let $K_Y(t, f, \dot{f}, F, \dot{F})$ and $\dot{K}_Y(t, f, \dot{f}, F, \dot{F})$, where $Y \in U(\mathfrak{sl}(2,\mathbb{C}))$ and the degree of Y is at most two, be defined by

$$(K_{\mathbb{I}}(t), \dot{K}_{\mathbb{I}}(t)) = (f, \dot{f}), \tag{5.64a}$$

$$(K_{M_{ij}}(t), \dot{K}_{M_{ij}}(t))(x) = (x_i\partial_j - x_j\partial_i)(f(x), \dot{f}(x)), \tag{5.64b}$$

$$\begin{aligned}(K_{M_{0i}}(t), \dot{K}_{M_{0i}}(t))(x) = (x_i\dot{f}(x) + t\partial_i f(x), x_i\Delta f(x)\\ + \partial_i f(x) + t\partial_i\dot{f}(x) + x_i F(x)),\end{aligned} \tag{5.64c}$$

$$(K_{M_{ij}M_{\mu\nu}}(t), \dot{K}_{M_{ij}M_{\mu\nu}}(t))(x) = (x_i\partial_j - x_j\partial_i)(K_{M_{\mu\nu}}(t), \dot{K}_{M_{\mu\nu}}(t))(x), \tag{5.64d}$$

$$(K_{M_{0i}M_{0j}}(t))(x) = x_i(\dot{K}_{M_{0j}}(t))(x) + t\partial_i(K_{M_{0j}}(t))(x), \tag{5.64e}$$

$$\begin{aligned}(\dot{K}_{M_{0i}M_{0j}}(t))(x) = x_i\Delta(K_{M_{0j}}(t))(x) + (\partial_i K_{M_{0j}}(t))(x)\\ + (t\partial_i\dot{K}_{M_{0j}}(t))(x) + x_i F_{M_{0j}+tP_j}(x),\end{aligned} \tag{5.64f}$$

where $\mathbb{I}$ is the unit element in $U(\mathfrak{sl}(2,\mathbb{C}))$, $\mu, \nu \in \{0,1,2,3\}$ and $i, j \in \{1,2,3\}$.

When there is no risk of confusion we omit the arguments $f, \dot{f}, F, \dot{F}$ in $(K, \dot{K})$ and write $(K_Y(t), \dot{K}_Y(t))$. If u is a solution of the inhomogeneous wave equation $\square u = G$, then it follows from the definition of $\xi_Y, Y \in U(\mathfrak{p})$, that

$$\big((\xi_Y u)(t), \frac{d}{dt}(\xi_Y u)(t)\big) = (K_Y, \dot{K}_Y)\big(t, u(t), \frac{d}{dt}u(t), G(t), \frac{d}{dt}G(t)\big), \tag{5.65}$$

for $Y \in U(\mathfrak{sl}(2,\mathbb{C}))$ and Y being of degree not greater than 2.

K and $\dot{K}$ defined by (5.64a)–(5.64f) satisfy

$$\begin{aligned}&\partial_i(t\dot{f}(x) + \sum_{1\leq j\leq 3} x_j\partial_j f(x))\\ &\quad = (\dot{K}_{M_{0i}}(t))(x) - \sum_{1\leq j\leq 3}(\partial_j K_{M_{ij}}(t))(x) - 2\partial_i f(x) - x_i F(x)\end{aligned} \tag{5.66a}$$

and

$$\begin{aligned}&\partial_i\Big(t(\dot{K}_{M_{\mu\nu}}(t))(x) + \sum_{1\leq j\leq 3} x_j(\partial_j K_{M_{\mu\nu}}(t))(x)\Big)\\ &\quad = \Big(\dot{K}_{M_{0i}M_{\mu\nu}}(t) - \sum_{1\leq j\leq 3}\partial_j K_{M_{ij}M_{\mu\nu}}(t) - 2\partial_i K_{M_{\mu\nu}}(t)\Big)(x) - x_i(F_{M_{\mu\nu}(t)}(t))(x),\end{aligned} \tag{5.66b}$$

where $\mu, \nu \in \{0,1,2,3\}, 1 \leq i \leq 3$ and $M_{ij}(t) = M_{ij}$ for $1 \leq i < j \leq 3$ and $M_{0i}(t) = M_{0i} + tP_i$. These two formulas express the dilatation generator $y^\mu\partial_\mu$ in terms of the action of $\mathfrak{sl}(2,\mathbb{C})$.

In 4-dimensional conventional notations with contravariant coordinates $y^\mu, 0 \leq \mu \leq 3$, we have, using the summation convention,

$$y^\nu y_\nu\partial_\mu = y_\mu y^\nu\partial_\nu + y^\nu(y_\nu\partial_\mu - y_\mu\partial_\nu),$$

which reads

$$\lambda\partial_\mu = y_\mu \overline{D} + y^\nu \overline{M}_{\nu\mu}, \quad 0 \le \mu \le 3, \tag{5.67}$$

with $\overline{D} = y^\nu \partial_\nu, \lambda = y^\nu y_\nu$ and $\overline{M}_{\mu\nu} = y_\mu \partial_\nu - y_\nu \partial_\mu$. It follows from (5.67) that

$$\begin{aligned} \partial_\mu \partial_\nu = {} & \lambda^{-1}(y_\mu y_\nu \Box + g_{\mu\nu}\overline{D}) \\ & + \lambda^{-2}\Big(y^\alpha y^\beta(\overline{M}_{\alpha\mu}\overline{M}_{\beta\nu} + \overline{M}_{\alpha\nu}\overline{M}_{\beta\mu}) + y_\mu y^\alpha \overline{D}\,\overline{M}_{\alpha\nu} + y_\nu y^\alpha \overline{D}\,\overline{M}_{\alpha\mu} \\ & - y_\mu y^\alpha \overline{M}_{\alpha\nu} - y_\nu y^\alpha \overline{M}_{\alpha\mu} - y_\mu y_\nu (\frac{1}{2}\overline{M}_{\alpha\beta}\overline{M}^{\alpha\beta} + 4\overline{D})\Big), \end{aligned} \tag{5.68}$$

for $0 \le \mu \le 3, 0 \le \nu \le 3$. Formulas (5.66a), (5.66b), (5.67) and (5.68) give $\partial_i, \partial_i\partial_j$ expressed in terms of $K_Y, \dot{K}_Y$, $F, \dot{F}$ since, on initial conditions, $\Box$ can be replaced by F.

For later reference we shall state particular L^∞-estimates for the electromagnetic potential.

Proposition 5.6. *Let $t \ge 0, (f, \dot{f}) \in M^0_\infty, (F, \dot{F}) \in M^0_\infty$ and suppose that the function $x \mapsto x_i(F(x), \dot{F}(x))$ is an element of M^0_∞, $1 \le i \le 3$.*
i) *If $a \in \mathbb{R}$, then*

$$(1+|x|)^{1+a}|f(x)| \le C\Big(\sum_Y \|(1+|\cdot|^2)^{a/2}K_Y(t)\|_{L^2} + \|(1-\Delta)f\|_{L^2}\Big), \quad x \in \mathbb{R}^3,$$

where the sum is taken over $Y \in \Pi' \cap U(\mathfrak{su}(2)), |Y| \le 2$.
ii) *If $0 < \delta < 1$, then*

$$\begin{aligned} (1+t)^{3/2}|f(x)| \le C_\delta\Big(&\sum_Y \|(K_Y(t), \dot{K}_Y(t))\|_{M^0} + \|(1-\Delta)f\|_{L^2} \\ &+ \sum_Z (1+t)\|F_{Z(t)}(t)\|_{L^{6/5}}\Big), \quad 0 \le |x| \le \delta t, \end{aligned}$$

where $Z(t)$ is given by (1.11) and the sums are taken over $Y, Z \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C})), |Y| \le 2$ and $|Z| \le 2$.
iii) *If $0 < \delta_1 < 1 < \delta_2$, then*

$$\begin{aligned} (1+t)(1+|t-|x||)^{1/2}|f(x)| \le C_{\delta_1,\delta_2}\Big(&\sum_Y \|(K_Y(t), \dot{K}_Y(t))\|_{M^0} + \|(1-\Delta)f\|_{L^2} \\ &+ \sum_Z (1+t)\|F_{Z(t)}(t)\|_{L^{6/5}}\Big), \quad \delta_1 t \le |x| \le \delta_2 t, \end{aligned}$$

where the domains of summation are as in ii).
iv) *If $0 < \delta < 1$ and $a \in \mathbb{R}$ then*

$$\begin{aligned} (1+|x|)^{3/2+a}|f(x)| \le C_{\delta,a}\Big(&\sum_Y \|(1+|\cdot|^2)^{a/2}(K_Y(t), \dot{K}_Y(t))\|_{M^0} + \|(1-\Delta)f\|_{L^2} \\ &+ \sum_Z \|(1+|\cdot|)^{1+a}F_{Z(t)}(t)\|_{L^{6/5}}\Big), \quad 0 \le t \le \delta|x|, \end{aligned}$$

where domains of summation are as in ii).

Proof. Statement i) of the proposition follows from [[14], Theorem 3.1]. To prove statement ii) let, for given $0 < \delta < 1, \delta'$ be such that $0 < \delta < \delta' < 1$ and let $\varphi \in C^\infty(\mathbb{R}^3, \mathbb{R})$ be a positive function with supp $\varphi \subset \{x \big| |x| \leq \delta'\}, \varphi(x) = 1$ for $|x| \leq \delta$. For $t \geq 1$, let $\psi_t(x) = \varphi(x/t)$. It follows from (5.66a)–(5.68) that

$$\begin{aligned} &\|\psi_t f\|_{L^2} + t \sum_{1 \leq i \leq 3} \|\partial_i \psi_t f\|_{L^2} + t^2 \sum_{\substack{1 \leq i \leq 3 \\ 1 \leq j \leq 3}} \|\partial_i \partial_j \psi_t f\|_{L^2} \\ &\quad \leq C_{\delta, \delta'} \Big(\sum_Y \|(K_Y(t), \dot{K}_Y(t))\|_{M^0} + \sum_Z (1+t) \||\nabla|^{-1} F_{Z(t)}(t)\|_{L^2} \Big), \quad t \geq 1, \end{aligned} \tag{5.69}$$

where $Y, Z \in \Pi' \cap U(\mathfrak{sl}(2, \mathbb{C}))$ and $|Y| \leq 2, |Z| \leq 2$. Let $g_t(y) = \psi_t(ty) f(ty), t \geq 1$. Since $\||\nabla|^{-1} F_{Z(t)}(t)\|_{L^2} \leq C \|F_{Z(t)}(t)\|_{L^{6/5}}$ and

$$\begin{aligned} \sup_{|x| \leq \delta t} t^{3/2} |f(x)| &\leq t^{3/2} \|g_t\|_{L^\infty} \\ &\leq C t^{3/2} \|(1 - \Delta) g_t\|_{L^2} \\ &\leq C \Big(\|\psi_t f\|_{L^2} + t \sum_{1 \leq i \leq 3} \|\partial_i \psi_t f\|_{L^2} + t^2 \sum_{\substack{1 \leq i \leq 3 \\ 1 \leq j \leq 3}} \|\partial_i \partial_j \psi_t f\|_{L^2} \Big), \quad t \geq 1, \end{aligned} \tag{5.70}$$

it follows from (5.69) that statement ii) is true for $t \geq 1$. For $0 \leq t \leq 1$ it follows from $\|f\|_{L^\infty} \leq C \|(1 - \Delta) f\|_{L^2}$.

To prove statement iii) we introduce the metric $ds^2 = (1 + (R - t)^2)^{-1} dR^2 + (1 + R^2)^{-1} ds'^2$ in $\mathbb{R}^3 - \{x \big| |x| \leq \frac{1}{2}\}$ where $R = |x|$ and where ds'^2 is the Euclidean metric on the unit sphere. It then follows from (5.66a)–(5.68) and from [[14], Proposition 2.1] that statement iii) is true. Statement iv) follows similarly. This proves the proposition.

In the case of the Dirac equation, let h be a solution in a time interval containing t of

$$(i\gamma^\mu \partial_\mu + m) h = g, \quad g \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^3). \tag{5.71}$$

Let

$$\overline{\mathcal{M}}_{\mu\nu} = \overline{M}_{\mu\nu} + \frac{1}{4}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu). \tag{5.72}$$

It follows from (5.67) that

$$\lambda \partial_\mu = y_\mu \overline{D} - \frac{1}{4} y^\alpha (\gamma_\alpha \gamma_\mu - \gamma_\mu \gamma_\alpha) + y^\alpha \overline{\mathcal{M}}_{\alpha\mu}, \tag{5.73}$$

and it follows from the Dirac equation (5.71) that

$$\overline{D}(e^{-im\gamma^\nu y_\nu} h) = -e^{-im\gamma^\nu y_\nu}(-3/2h + \lambda^{-1} \gamma^\mu y_\mu y^\alpha \gamma^\beta \overline{\mathcal{M}}_{\alpha\beta} h + i\gamma^\alpha y_\alpha g). \tag{5.74}$$

We observe that

$$[\overline{\mathcal{M}}_{\mu\nu}, \gamma^\alpha y_\alpha] = 0, \quad (\gamma^\nu y_\nu)^2 = \lambda, \tag{5.75}$$

which we get by a direct calculation.

Theorem 5.7. *Let $k \in \mathbb{N}, t \in \mathbb{R}^+, x \in \mathbb{R}^3$ and let the function $x \mapsto q_t(x)$ (resp. $x \mapsto r_t(x)$) be defined as in (5.13c) (resp. (5.26)). If h is a solution of equation (5.71), then*

$$
\begin{aligned}
&(1+t+|x|)^{3/2}(1+q_t(x)+r_t(x))^{k/2}|h(t,x)| \\
&\leq C_k\Big(\wp^D_{k+8}(h(t)) + \sum_{0\leq j\leq k+5} \wp^D_{2+j}\big((1+q_t+r_t)^{(k+7-j)/2} g(t)\big) + t\wp^D_1\big((1+q_t)^{(k+4)/2} g(t)\big)\Big),
\end{aligned}
$$

for some constants C_k independent of t, x, h, g.

Proof. Let $y_0 \geq 1+|\vec{y}|, \vec{y} = (y_1,y_2,y_3) \in \mathbb{R}^3$. Then $(\gamma^\mu y_\nu)^2 = y_\nu y^\nu I$ and the operator $\exp(-im\gamma^\nu y_\nu)$ is unitary on $\mathbb{C}^4$. It follows from (5.73), (5.74) and (5.75) that

$$
\begin{aligned}
&y_0|\partial_j e^{-im\gamma^\nu y_\nu} h(y)| \\
&\quad\leq C\Big(q_{y_0}(\vec{y})|h(y)| + q_{y_0}(\vec{y})^{3/2}\sum_{\alpha,\beta}|\overline{\mathcal{M}}_{\alpha\beta} h(y)| + q_{y_0}(\vec{y})^{1/2} y_0 |g(y)|\Big), \quad 1 \leq j \leq 3.
\end{aligned} \tag{5.76}
$$

Let $\varphi \in C^\infty(\mathbb{R})$ be a positive function such that $\varphi(s) = 0$ for $s \leq 1$ and $\varphi(s) = 0$ for $s \geq 2$ and let $\psi_{y_0}(\vec{y}) = \varphi(y_0 - |\vec{y}|)$. Since $|\partial_j \psi_{y_0}(\vec{y})| \leq C(1+y_0)^{-1} q_{y_0}(\vec{y})$, it follows from (5.76), with $h_1(y) = e^{-im\gamma^\nu y_\nu} h(y)$, that

$$
\begin{aligned}
&y_0\|\partial_j q^{k/2}_{y_0}\psi_{y_0} h_1(y_0,\cdot)\|_{L^p(\mathbb{R}^3)} \\
&\quad\leq C_k\Big(\|q^{k/2+1}_{y_0} h(y_0,\cdot)\|_{L^p(Q_{y_0})} + \sum_{\alpha,\beta}\|q^{k/2+3/2}_{y_0}(\overline{\mathcal{M}}_{\alpha\beta} h)(y_0,\cdot)\|_{L^p(Q_{y_0})} \\
&\qquad + y_0\|q^{1/2+k/2}_{y_0} g(y_0,\cdot)\|_{L^p(Q_{y_0})}\Big), \quad y_0 \geq 1, 1 \leq p \leq \infty, 1 \leq j \leq 3, k \geq 0,
\end{aligned} \tag{5.77}
$$

where $Q_{y_0} = \{\vec{y} \in \mathbb{R}^3 \big| y_0 \geq 1 + |\vec{y}|\}$.

It now follows by a substitution of variable as in the proof of statement ii) of Proposition 5.6, using the Sobolev inequality $\|\cdot\|_{L^\infty} \leq C(\|\cdot\|_{L^p} + \sum_i \|\partial_i \cdot\|_{L^p}), p > 3$, that

$$
\begin{aligned}
&y_0^{3/p} q^{k/2}_{y_0}(\vec{y})|h(y)| \\
&\quad\leq C_{k,p}\Big(\|q^{k/2+1}_{y_0} h(y_0,\cdot)\|_{L^p(Q_{y_0})} \\
&\qquad + \sum_{\alpha,\beta}\|q^{k/2+3/2}_{y_0}(\overline{\mathcal{M}}_{\alpha,\beta} h)(y_0,\cdot)\|_{L^p(Q_{y_0})} + y_0\|q^{k/2+1/2}_{y_0} g(y_0,\cdot)\|_{L^p(Q_{y_0})}\Big),
\end{aligned} \tag{5.78}
$$

where $y_0 \geq 1, p > 3, 1 \leq j \leq 3, k \geq 0$ and $|\vec{y}| + 2 \leq y_0$. Similarly, as we obtained inequality (5.77), we get that (with a new function φ)

$$
\begin{aligned}
&y_0\Big(\|\partial_j q^{k/2+1}_{y_0} h_1(y_0,\cdot)\|_{L^p(Q_{y_0})} + \sum_{\alpha,\beta}\|\partial_j q^{k/2+3/2}_{y_0}(\overline{\mathcal{M}}_{\alpha\beta} h_1)(y_0,\cdot)\|_{L^p(Q_{y_0})}\Big) \\
&\leq C_k\Big(\|q^{k/2+2}_{y_0} h(y_0,\cdot)\|_{L^p(Q'_{y_0})} + \sum_{\alpha,\beta}\|q^{k/2+5/2}_{y_0}(\overline{\mathcal{M}}_{\alpha\beta} h)(y_0,\cdot)\|_{L^p(Q'_{y_0})} \\
&\quad + \sum_{\alpha,\beta,\mu,\nu}\|q^{k/2+3}_{y_0}(\overline{\mathcal{M}}_{\alpha\beta}\overline{\mathcal{M}}_{\mu\nu} h)(y_0,\cdot)\|_{L^p(Q'_{y_0})} + y_0\|q^{k/2+3/2}_{y_0} g(y_0,\cdot)\|_{L^p(Q'_{y_0})} \\
&\quad + y_0\sum_{\alpha,\beta}\|q^{k/2+2}_{y_0}(\overline{\mathcal{M}}_{\alpha\beta} g)(y_0,\cdot)\|_{L^p(Q'_{y_0})}\Big), \quad y_0 \geq 1, 1 \leq p \leq \infty, 1 \leq j \leq 3, k \geq 0,
\end{aligned} \tag{5.79}
$$

where $Q'_{y_0} = \{\vec{y} \in \mathbb{R}^3 \big| y_0 \geq 1/2 + |\vec{y}|\}$. Sobolev inequality $\|\cdot\|_{L^p} \leq C(\|\cdot\|_{L^2} + \sum_i \|\partial_i \cdot\|_{L^2})$, $2 \leq p \leq 6$, and inequality (5.79) give, as in the case of (5.78), after a change of variable

$$
\begin{aligned}
&y_0^{3/2-3/p}\Big(\|q_{y_0}^{k/2+1}h(y_0,\cdot)\|_{L^p(Q_{y_0})} + \sum_{\alpha,\beta}\|q_{y_0}^{k/2+3/2}(\overline{\mathcal{M}}_{\alpha\beta}h)(y_0,\cdot)\|_{L^p(Q_{y_0})}\Big) \qquad (5.80)\\
&\leq C_{k,p}\Big(\|q_{y_0}^{k/2+2}h(y_0,\cdot)\|_{L^2(Q'_{y_0})} + \sum_{\alpha,\beta}\|q_{y_0}^{k/2+5/2}(\overline{\mathcal{M}}_{\alpha\beta}h)(y_0,\cdot)\|_{L^2(Q'_{y_0})}\\
&\quad + \sum_{\alpha,\beta,\mu,\nu}\|q_{y_0}^{k/2+3}(\overline{\mathcal{M}}_{\alpha\beta}\overline{\mathcal{M}}_{\mu\nu}h)(y_0,\cdot)\|_{L^2(Q'_{y_0})} + y_0\|q_{y_0}^{k/2+3/2}g(y_0,\cdot)\|_{L^2(Q'_{y_0})}\\
&\quad + y_0\sum_{\alpha,\beta}\|q_{y_0}^{k/2+2}(\overline{\mathcal{M}}_{\alpha\beta}g)(y_0,\cdot)\|_{L^2(Q'_{y_0})}\Big),
\end{aligned}
$$

where $2 \leq p \leq 6, y_0 \geq 1, 1 \leq j \leq 3$ and $k \geq 0$. It follows now from inequalities (5.78) and (5.80) with $p = 6$, that

$$
\begin{aligned}
&y_0^{3/2}q_{y_0}^{k/2}(\vec{y})|h(y)| \qquad (5.81)\\
&\leq C_k\Big(\|q_{y_0}^{k/2+2}h(y_0,\cdot)\|_{L^2(Q'_{y_0})} + \sum_{\alpha,\beta}\|q_{y_0}^{k/2+5/2}(\overline{\mathcal{M}}_{\alpha\beta}h)(y_0,\cdot)\|_{L^2(Q'_{y_0})}\\
&\quad + \sum_{\alpha,\beta,\mu,\nu}\|q_{y_0}^{k/2+3}(\overline{\mathcal{M}}_{\alpha\beta}\overline{\mathcal{M}}_{\mu\nu}h)(y_0,\cdot)\|_{L^2(Q'_{y_0})} + y_0\|q_{y_0}^{k/2+3/2}g(y_0,\cdot)\|_{L^2(Q'_{y_0})}\\
&\quad + y_0\|q_{y_0}^{k/2+2}(\overline{\mathcal{M}}_{\alpha\beta}g)(y_0,\cdot)\|_{L^2(Q'_{y_0})} + y_0\|q_{y_0}^{k/2+1/2}g(y_0,\cdot)\|_{L^6(Q_{y_0})}\Big),\ y_0 \geq 1 + |\vec{y}|, k \geq 0.
\end{aligned}
$$

By considering a time-translation in equation (5.71), by using Sobolev embedding for the L^6-term we obtain from (5.81), after changing the notation,

$$
\begin{aligned}
&(1+t)^{3/2}q_t^{k/2}(x)|h(t,x)| \qquad (5.82)\\
&\leq C_k\Big(\sum_{\substack{Y\in\Pi'\\|Y|\leq 2}}\|q_t^{k/2+3}(\xi_Y^D h)(t)\|_{L^2(Q''_t)} + t\sum_{\substack{Y\in\Pi'\\|Y|\leq 1}}\|q_t^{k/2+2}(\xi_Y^D g)(t)\|_{L^2(Q''_t)}\Big),
\end{aligned}
$$

$t \geq 0, |x| \leq t+2, k \geq 0$, where $Q''_t = \{x \in \mathbb{R}^3 \big| |x| \leq t+3\}$.

We next consider the decrease properties of h outside the light-one. According to the definition of $r_t, t \geq 0$, we have, with $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$,

$$
\sum_{|\alpha|\leq 2}(1+|x|)^{|\alpha|}|\partial^\alpha h(t,x)| = \sum_{|\alpha|\leq 2} r_t(x)^{|\alpha|}|\partial^\alpha h(t,x)|,
$$

for $|x| \geq t$. Using a cut-off function and [[14] Proposition 2.1] we obtain

$$
(1+|x|)^{3/2+l/2}|h(t,x)| \leq C\sum_{|\alpha|\leq 2}\|r_t^{|\alpha|+l/2}\partial^\alpha h(t)\|_{L^2(O_t)}, \qquad (5.83)
$$

$|x| \geq t+1, l \geq 0$, where $O_t = \{x \in \mathbb{R}^3 | |x| \geq t\}$.

Inequality (5.82) and Theorem 5.5, with $G=0, i=0, n=2, L=0$ and k replaced by $k+6$, give that

$$(1+t)^{3/2}(1+q_t(x))^{k/2}|h(t,x)| \leq C_k\Big(\wp^D_{k+8}(h(t)) + \sum_{0\leq j\leq k+5} \wp^D_{2+j}\big((1+q_t)^{(k+7-j)/2}g(t)\big) + t\wp^D_1\big((1+q_t)^{(k+4)/2}g(t)\big)\Big), \tag{5.84}$$

$t \geq 0, |x| \leq t+2, k \geq 0$.

Inequality (5.83) and Theorem 5.5 with $G=0$, $i=1, n=2, L=0$ and k replaced by $l+4$, give that

$$(1+|x|)^{(3+l)/2}|h(t,x)| \leq C\wp^D_2\big((1+q_t+r_t)^{(l+4)/2}h(t)\big) \leq C_l\Big(\wp^D_{l+6}(h(t)) + \sum_{0\leq j\leq l+3} \wp^D_{2+j}\big((1+q_t+r_t)^{(l+5-j)/2}g(t)\big)\Big), \tag{5.85}$$

$|x| \geq t+1, t \geq 0, l \geq 0$.

The inequality of the theorem now follows from inequalities (5.84) and (5.85), since $1+q_t(x)+r_t(x) \leq C(1+q_t(x))$ for $0 \leq |x| \leq t+1$ and $1+q_t(x)+r_t(x) \leq C(1+r_t(x))$ for $|x| \geq t \geq 0$. This proves the theorem.

To illustrate the use of the last theorem in our context, we shall apply it to the equation

$$(i\gamma^\mu\partial_\mu + m - \gamma^\mu G_\mu)h = g. \tag{5.86}$$

In order to state the result we introduce certain notations. Let $0 \leq a^{(0)} \leq \cdots \leq a^{(k)} \leq \cdots$ be a sequence of real numbers and let

$$b_n = \sum_{1\leq p\leq n} \sum_{\substack{n_1+\cdots+n_p=n \\ n_i\geq 1}} \prod_{1\leq j\leq p} a^{(n_j)}, \quad n \geq 1, b_0 = a^{(0)}. \tag{5.87}$$

Let

$$T^{\infty(n)}(t) = \sum_{\substack{Y\in\Pi' \\ |Y|\leq n}} \Big(\|\delta(t)^{1/2}G_Y(t)\|_{L^\infty} + (1+t)\|\xi_Y\partial_\mu G^\mu(t)\|_{L^\infty} + (1+t)\|F_Y(t)\|_{L^\infty} + \|Q_Y(t)\|_{L^\infty}\Big), \quad t \geq 0, n \geq 0 \tag{5.88a}$$

where $(\delta(t))(x) = \delta(t,x) = 1+t+|x|, G_{Y\mu} = (\xi^M_Y G)_\mu$ for $Y \in \Pi'$, $F_{Y\mu\nu} = \partial_\mu G_{Y\nu} - \partial_\nu G_{Y\mu}$, $Q_Y(y) = y^\mu G_{Y\mu}(y)$ and where ξ_Y is given by (4.81). Let

$$T^{2(n)}(t) = \wp^{M^1}_n(G(t), \dot{G}(t)) + \sum_{\substack{Y\in\Pi' \\ |Y|\leq n}} \Big(\|\delta(t)^{-3/2}Q_Y(t)\|_{L^2} + \|\delta(t)^{-1/2}F_Y(t)\|_{L^2} + \|\delta(t)^{-1/2}\xi_Y\partial_\mu G^\mu(t)\|_{L^2}\Big), \tag{5.88b}$$

$t \geq 0, n \geq 0$. We introduce the notation $T_n^\infty(t)$ and $T_{N,n}^2(t), N, n \geq 0$, by

$$T_n^\infty(t) = b_n, \quad (\text{resp. } T_{N,n}^2(t) = b_n), \tag{5.89a}$$

where b_n is given by formula (5.87) for

$$a^{(n)} = T^{\infty(n)}(t), \quad (\text{resp. } a^{(n)} = T^{2(N+n)}(t)). \tag{5.89b}$$

We define

$$T_n^2 = T_{0,n}^2 \tag{5.89c}$$

and we note that it follows from (5.87) that $b_{n_1} b_{n_2} \leq b_{n_1+n_2}$, $n_1, n_2 \geq 1$, which gives that

$$T_{n_1}^\infty(t) T_{n_2}^\infty(t) \leq T_{n_1+n_2}^\infty(t), T_{N,n_1}^2 T_{N,n_2}^2(t) \leq T_{N,n_1+n_2}^2(t), \quad n_1, n_2 \geq 1. \tag{5.89d}$$

Moreover

$$T_{n+N}^2(t) \leq C T_{N,n}^2(t), \tag{5.89e}$$

where C is a function of $T_N^2(t)$.

Theorem 5.8. *Let $N \geq 0, h_Y \in C^0(\mathbb{R}^+, D)$ for $|Y| \leq N+8$, $(G_Y, \dot{G}_Y) \in C^0(\mathbb{R}^+, M^1)$ for $|Y| \leq N+10$, where $Y \in \Pi'$, $h_Y = \xi_Y^D h, G_Y = \xi_Y^M G$ and $\dot{G}_Y(t) = \frac{d}{dt} G_Y(t)$. Let*

$$H_n(t) = \sum_{\substack{Y \in \Pi' \\ |Y|+k \leq n}} \|\delta(t)^{3/2}(1 + q_t + r_t)^{k/2} h_Y(t)\|_{L^\infty}, \quad n \geq 0,$$

where q_t and r_t are as in (5.13c) and (5.26). Let

$$R_n^2(t) = \sum_{l+k \leq n} \wp_l^D\big((1 + q_t + r_t)^{k/2} g(t)\big),$$
$$R_n^\infty(t) = \sum_{\substack{Y \in \Pi' \\ |Y|+k \leq n}} \|\delta(t)^{3/2}(1 + q_t + r_t)^{k/2} g_Y(t)\|_{L^\infty}$$

and

$$R_n'(t) = \sum_{l+k \leq n} \wp_l^D\big(\delta(t)(1 + q_t + r_t)^{k/2} g'(t)\big), \quad n \geq 0,$$

where $g' = (2m)^{-1}(m - i\gamma^\mu \partial_\mu + \gamma^\mu G_\mu) g, g_Y = \xi_Y^D g, g_Y' = \xi_Y^D g'$. If g is defined by formula (5.86) and if $R_{N+9}^2(t) < \infty$, $R_{N+7}'(t) < \infty, T_9^\infty(t) < \infty, T_{9,N}^2(t) < \infty$ then there is a constant a_N depending only on $T_9^\infty(t)$, such that

$$H_N(t) \leq a_N \Big(R_{N+7}'(t) + R_{N+9}^2(t) + R_N^\infty(t) + \wp_{N+8}^D(h(t))$$
$$+ \sum_{\substack{n_1+n_2 \leq N \\ n_2 \leq N-1}} T_{9,n_1}^2(t) \big(R_{n_2+7}'(t) + R_{n_2+9}^2(t) + R_{n_2}^\infty(t) + \wp_{n_2+8}^D(h(t)) \big) \Big).$$

Proof. We prove the theorem by induction in N. Suppose that the inequality of the theorem, with n instead of N, is true for $0 \leq n \leq N-1$ and suppose that the hypotheses of the theorem are true for N.

We first estimate $\wp^D_n\big((1+\lambda_i(t))^{k/2}h(t)\big), i=0$ or $i=1$, for $n+k \leq N+8$, where λ_i is as in Theorem 5.5. According to the hypotheses, $h_Y \in C^0(\mathbb{R}^+, D)$ for $|Y| \leq N+8, G_Y \in C^0(\mathbb{R}^+, L^2_{loc}(\mathbb{R}^3,\mathbb{R}^4))$ (since $\|G_Y(t)\|_{L^6} \leq C\|(G_Y(t),0)\|_{M^1}$ and $L^6(\mathbb{R}^3) \subset L^2_{\text{loc}}(\mathbb{R}^3)$) for $|Y| \leq N+10$ and $G_Y \in C^0(\mathbb{R}^+, L^\infty(\mathbb{R}^3,\mathbb{R}^4))$ since $\|G_Y(t)\|_{L^\infty} \leq C\|(1-\Delta)^{1/2}(G_Y(t),0)\|_{M^1})$ for $|Y| \leq N+9, Y \in \Pi'$. Moreover

$$\|G_{Y_1\mu}(t)h_{Y_2}(t)\|_D \leq \|G_{Y_1}(t)\|_{L^6}\|h_{Y_2}(t)\|_{L^3} \leq C\wp^{M^1}_{|Y_1|}(G(t),0)\wp^D_{|Y_2|+1}(h(t)),$$

by Sobolev embedding. This shows that $G_{Y_1\mu}(t)h_{Y_2}(t) \in C^0(\mathbb{R}^+, D)$ for $Y_1, Y_2 \in \Pi'$ such that $|Y_1| \leq N+7$ and $|Y_2| \leq N+6-L$, where $0 \leq L \leq N+7$. Thus, the hypotheses of Theorem 5.5 are satisfied for $n+k \leq N+8$. Using that

$$\begin{aligned}
&\|(1+\lambda_i(t))^{(k+1-j)/2}\gamma^\mu G_{Z_1\mu}(t)h_{Z_2}(t)\|_D \\
&\quad\leq \|G_{Z_1\mu}(t)\|_{L^6}\|(1+\lambda_i(t))^{(k+1-j)/2}h_{Z_2}(t)\|_{L^3} \\
&\quad\leq C\|(G_{Z_1}(t),0)\|_{M^1}\|\delta(t)^{-3/2}\|_{L^3}\|\delta(t)^{3/2}(1+\lambda_i(t))^{(k+1-j)/2}h_{Z_2}(t)\|_{L^\infty},
\end{aligned}$$

for some constant C and using that $\|\delta(t)^{-3/2}\|_{L^3} \leq C'(1+t)^{-1/2} \leq C'$ for some constant C', we obtain using Theorem 5.5 that

$$\begin{aligned}
&\wp^D_n\big((1+\lambda_i(t))^{k/2}h(t)\big) \\
&\leq C'_{n+k}\sum_{0\leq l\leq L}(1+\tau_{i,l}(t))\sum_{0\leq j\leq k-1}\wp_{n+j-l}\big((1+\lambda_i(t))^{(k+1-j)/2}g(t)\big) \\
&\quad + C'_{n+k}\sum_{0\leq l\leq L}(1+\tau_{i,l}(t))\wp^D_{n+k-l}(h(t)) \\
&\quad + C'_{n+k}\sum_{0\leq l\leq L}(1+\tau_{i,l}(t))\sum_{\substack{0\leq j\leq k-1\\ Z_2\in\Pi'\\ n_1+|Z_2|\leq n+j-l\\ |Z_2|\leq n+j-L-1}} T^{2(n_1)}(t)\|\delta(t)^{3/2}(1+\lambda_i(t))^{(k+1-j)/2}h_{Z_2}(t)\|_{L^\infty},
\end{aligned}$$

where $n+k \leq N+8, 0 \leq L \leq n+k-1$ and C'_{n+k} is a constant depending only on $\tau_{i,0}(t)$. Using the definitions of $R_n(t)$ and $H_n(t)$, we obtain that

$$\begin{aligned}
&\wp^D_n\big((1+\lambda_i(t))^{k/2}h(t)\big) \qquad (5.90)\\
&\quad\leq C'_{n+k}\sum_{\substack{n_1\leq L\\ n_1+n_2=n+k+1}}(1+\tau_{i,n_1}(t))R^2_{n_2}(t) + C'_{n+k}\sum_{\substack{n_1\leq L\\ n_1+n_2=n+k}}(1+\tau_{i,n_1}(t))\wp^D_{n_2}(h(t)) \\
&\qquad + C'_{n+k}\sum_{\substack{n_1+n_2+n_3=n+k+1\\ n_1\leq L, n_3\leq n+k-L}}(1+\tau_{i,n_1}(t))T^{2(n_2)}(t)H_{n_3}(t), \quad 0\leq L\leq n+k-1.
\end{aligned}$$

Let

$$h^{(1)} = h-(2m)^{-1}(G_\mu\gamma^\mu h+g). \qquad (5.91)$$

Then it follows that

$$\begin{aligned}\wp_n^D(h^{(1)}(t)) \leq \wp_n^D(h(t)) + (2m)^{-1}\wp_n^D(g(t))\\ + C_n \sum_{\substack{Z\in\Pi' \\ |Z|+n_2\leq n \\ |Z|\leq L}} \|G_Z(t)\|_{L^\infty}\wp_{n_2}^D(h(t))\\ + C_n \sum_{\substack{Z_1,Z_2\in\Pi' \\ |Z_1|+|Z_2|\leq n \\ |Z_2|\leq n-L-1}} \|G_{Z_1}(t)\|_{L^6}\|h_{Z_2}(t)\|_{L^3}.\end{aligned}$$

Since $\|h_{Z_2}(t)\|_{L^3} \leq C(1+t)^{-1/2}\|\delta(t)^{3/2}h_{Z_2}(t)\|_{L^\infty} \leq C\|\delta(t)^{3/2}h_{Z_2}(t)\|_{L^\infty}$, we obtain that

$$\begin{aligned}\wp_n^D(h^{(1)}(t)) \leq \wp_n^D(h(t)) + (2m)^{-1}\wp_n^D(g(t))\\ + C_n\Big(\sum_{\substack{n_1+n_2=n \\ n_1\leq L}} T_{n_1}^\infty(t)\wp_{n_2}^D(h(t)) + \sum_{\substack{n_1+n_2=n \\ n_2\leq n-L-1}} T_{n_1}^2(t)H_{n_2}(t)\Big),\end{aligned} \tag{5.92}$$

where $0 \leq L \leq n$.

Let

$$\begin{aligned}g^{(1)} = (2m)^{-1}\gamma^\mu\gamma^\nu G_\mu G_\nu h - im^{-1}G^\mu\partial_\mu h\\ - i(2m)^{-1}h\partial_\mu G^\mu - i(8m)^{-1}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)hF_{\mu\nu},\end{aligned} \tag{5.93a}$$

where

$$F_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu, \quad 0 \leq \mu \leq 3, 0 \leq \nu \leq 3. \tag{5.93b}$$

It follows from (5.93a) and (5.93b) that

$$\begin{aligned}\xi_Y^D g^{(1)} = (2m)^{-1}\gamma^\mu\gamma^\nu \sum_{Y_1,Y_2,Y_3}^{Y} G_{Y_1\mu}G_{Y_2\nu}h_{Y_3}\\ - im^{-1}\sum_{Y_1,Y_2}^{Y}\Big(G_{Y_1}^\mu\partial_\mu h_{Y_2} + \frac{1}{2}h_{Y_2}\partial_\mu G_{Y_1}^\mu + \frac{1}{8}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)h_{Y_2}F_{Y_1\mu\nu}\Big),\ Y \in U(\mathfrak{p}),\end{aligned} \tag{5.94a}$$

where

$$F_{Y\mu\nu} = \partial_\mu G_{Y\nu} - \partial_\nu G_{Y\mu}, \quad G_{Y\mu} = (\xi_Y^M G)_\mu. \tag{5.94b}$$

According to inequality (5.7d), we have

$$\begin{aligned}&|\sum_\mu G_{Y_1}^\mu\partial_\mu h_{Y_2}|\delta\\ &\leq C\Big(\sum_\mu |G_{Y_1}^\mu||h_{P_\mu Y_2}| + |Q_{Y_1}|\sum_\nu |h_{P_\nu Y_2}| + \sum_{\mu,\nu}|G_{Y_1}^\mu|(|h_{M_{\mu\nu}Y_2}| + |h_{Y_2}|)\Big),\end{aligned} \tag{5.95}$$

where the sums are taken over $0 \leq \mu \leq 3, 0 \leq \nu \leq 3$. Equality (5.94a) and inequality (5.95) give

$$
\begin{aligned}
|\xi^D_Y g^{(1)}| \leq C\Big(& \sum^{Y}_{Y_1,Y_2,Y_3} |G_{Y_1}||G_{Y_2}||h_{Y_3}| \\
& + \sum^{Y}_{Y_1,Y_2} \Big(\delta^{-1}\big((|G_{Y_1}|+|Q_{Y_1}|)\sum_{\mu}|h_{P_\mu Y_2}| + |G_{Y_1}|\big(|h_{Y_2}| + \sum_{\mu,\nu}|h_{M_{\mu\nu}Y_2}|\big)\big) \\
& + \big|\sum_{\mu}\partial_\mu G^\mu_{Y_1}\big||h_{Y_2}| + |F_{Y_1}||h_{Y_2}|\Big)\Big), \quad Y \in U(\mathfrak{p}).
\end{aligned} \tag{5.96}
$$

Using that

$$
\begin{aligned}
\|G_{Y_1\mu}(t)G_{Y_2\nu}(t)h_{Y_3}(t)\|_D &\leq \|G_{Y_1}(t)\|_{L^6}\|G_{Y_2}(t)\|_{L^6}\|h_{Y_3}(t)\|_{L^6} \\
&\leq C\|(G_{Y_1}(t),0)\|_{M^1}\|(G_{Y_2}(t),0)\|_{M^1}\|h_{Y_3}(t)\|_{L^6},
\end{aligned}
$$

by Hölder inequality and Sobolev embedding, using the argument in the derivation of (5.92) and using definitions (5.88a) and (5.88b) of $T^{\infty(n)}$ and $T^{2(n)}$, we obtain from (5.96)

$$
\begin{aligned}
& \wp^D_n\big(\delta(t)(1+\lambda_1(t))^{k/2}g^{(1)}(t)\big) \\
& \leq C_{n+k} \sum_{\substack{n_1+n_2+n_3\leq n \\ n_1+n_2\leq L}} T^\infty_{n_1}(t)T^\infty_{n_2}(t)\wp^D_{n_3}\big((1+\lambda_1(t))^{k/2}h(t)\big) \\
& \quad + C_{n+k} \sum_{\substack{Z\in\Pi' \\ n_1+n_2+|Z|\leq n \\ n_1+n_2\geq L+1}} T^2_{n_1}(t)T^2_{n_2}(t)\|\delta(t)(1+\lambda_1(t))^{k/2}h_Z(t)\|_{L^6} \\
& \quad + C_{n+k} \sum_{\substack{n_1+n_2\leq n \\ n_1\leq L}} T^\infty_{n_1}(t)\wp^D_{n_2+1}\big((1+\lambda_1(t))^{k/2}h(t)\big) \\
& \quad + C_{n+k} \sum_{\substack{Z\in\Pi', |Z|\geq 1 \\ n_1+|Z|\leq n+1 \\ n_1\geq L+1}} T^2_{n_1}(t)\|\delta(t)^{3/2}(1+\lambda_1(t))^{k/2}h_Z(t)\|_{L^\infty},
\end{aligned}
$$

where $0 \leq L \leq n$. Using inequality (5.89d) we obtain that

$$
\begin{aligned}
& \wp^D_n\big(\delta(t)(1+\lambda_1(t))^{k/2}g^{(1)}(t)\big) \\
& \leq C_{n+k}\Big(\sum_{\substack{n_1+n_2\leq n \\ n_1\leq L}} T^\infty_{n_1}(t)\wp^D_{n_2+1}\big((1+\lambda_1(t))^{k/2}h(t)\big) \\
& \quad + \sum_{\substack{n_1\leq n, Z\in\Pi' \\ n_1+|Z|\leq n+1 \\ n_1\geq L+1}} T^2_{n_1}(t)\|\delta(t)^{3/2}(1+\lambda_1(t))^{k/2}h_Z(t)\|_{L^\infty} \\
& \quad + \sum_{\substack{n_1+|Z|\leq n \\ n_1\geq L+1}} T^2_{n_1}(t)\|\delta(t)(1+\lambda_1(t))^{k/2}h_Z(t)\|_{L^6}\Big),
\end{aligned}
$$

where $0 \leq L \leq n$. Since

$$\|\delta(t)f\|_{L^6} \leq \|f\|_{L^2}^{1/3}\big(\|(\delta(t))^{3/2}f\|_{L^\infty}\big)^{2/3} \leq C\big(\|f\|_{L^2} + \|(\delta(t))^{3/2}f\|_{L^\infty}\big),$$

it follows that

$$\begin{aligned}
&\wp_n^D(\delta(t)(1+\lambda_1(t))^{k/2}g^{(1)}(t)) \\
&\quad \leq C_{n+k}\Big(\sum_{\substack{n_1+n_2\leq n\\ n_1\leq L}} T_{n_1}^\infty(t)\wp_{n_2+1}^D\big((1+\lambda_1(t))^{k/2}h(t)\big) \\
&\qquad + \sum_{\substack{n_1+n_2\leq n\\ n_1\geq L+1}} T_{n_1}^2(t)\wp_{n_2}^D\big((1+\lambda_1(t))^{k/2}h(t)\big) \\
&\qquad + \sum_{\substack{n_1+|Z|\leq n+1\\ n_1\leq n, Z\in\Pi'\\ n_1\geq L+1}} T_{n_1}^2(t)\|\delta(t)^{3/2}(1+\lambda_1(t))^{k/2}h_Z(t)\|_{L^\infty}\Big),
\end{aligned}$$

where $0 \leq L$. This inequality and the definition of H_n, give that

$$\begin{aligned}
&\sum_{n+k\leq M} \wp_n^D\big(\delta(t)(1+\lambda_1(t))^{k/2}g^{(1)}(t)\big) \\
&\quad \leq C_M\Big(\sum_{\substack{n_1+n_2+k\leq M\\ n_1\leq L}} T_{n_1}^\infty(t)\wp_{n_2+1}^D\big((1+\lambda_1(t))^{k/2}h(t)\big) \\
&\qquad + \sum_{\substack{n_1+n_2+k\leq M\\ n_1\geq L+1}} T_{n_1}^2(t)\wp_{n_2}^D\big((1+\lambda_1(t))^{k/2}h(t)\big) \\
&\qquad + \sum_{\substack{n_1+n_2\leq M\\ n_1\geq L+1}} T_{n_1}^2(t)H_{n_2+1}(t)\Big),
\end{aligned} \tag{5.97}$$

where $L \geq 0$ is an integer. With $L = L_0$ and $i = 1$ in inequality (5.90), we obtain, using that $\tau_{1,n}(t) \leq T_n^\infty(t)$,

$$\begin{aligned}
&\wp_n^D\big((1+\lambda_1(t))^{k/2}h(t)\big) \\
&\quad \leq C''_{n+k}\Big(R_{n+k+1}^2(t) + \wp_{n+k}^D(h(t)) + \sum_{\substack{n_1+n_2\leq n+k+1\\ n_2\leq n+k-L_0}} T_{n_1}^2(t)H_{n_2}(t)\Big),
\end{aligned} \tag{5.98}$$

where C''_{n+k} is a constant depending only on $T_{L_0}^\infty(t)$. It follows from inequalities (5.97) and (5.98) that

$$\sum_{\substack{n_1+n_2+k\leq M\\ n_1\leq L}} T^\infty_{n_1}(t)\wp^D_{n_2+1}\big((1+\lambda_1(t))^{k/2}h(t)\big) \tag{5.99}$$

$$\leq a_{M,L_0}\Big(\sum_{\substack{n_1+n_2\leq M+2\\ n_1\leq L}} T^\infty_{n_1}(t)R^2_{n_2}(t)+\sum_{\substack{n_1+n_2\leq M+1\\ n_1\leq L}} T^\infty_{n_1}(t)\wp^D_{n_2}(h(t))$$

$$+\sum_{\substack{n_1+n_2+n_3\leq M+2\\ n_1\leq L, n_3\leq M+1-L_0}} T^\infty_{n_1}(t)T^2_{n_2}(t)H_{n_3}(t)\Big),\quad L\geq 0, L_0\geq 0, M\geq 0,$$

where a_{M,L_0} is a constant depending only on $T^\infty_{L_0}(t)$. Similarly we obtain that

$$\sum_{\substack{n_1+n_2+k\leq M\\ n_1\geq L+1}} T^2_{n_1}(t)\wp^D_{n_2}\big((1+\lambda_1(t))^{k/2}h(t)\big) \tag{5.100}$$

$$\leq a_{M,L_0}\Big(\sum_{\substack{n_1\geq L+1\\ n_1+n_2\leq M+2}} T^2_{n_1}(t)R^2_{n_2}(t)+\sum_{\substack{n_1+n_2\leq M+1\\ n_1\geq L+1}} T^2_{n_1}(t)\wp^D_{n_2}(h(t))$$

$$+\sum_{\substack{n_1+n_2\leq M+2\\ n_1\geq L+1\\ n_2\leq M-L-L_0}} T^2_{n_1}(t)H_{n_2}(t)\Big),\quad L\geq 0, L_0\geq 0, M\geq 0,$$

where a_{M,L_0} is a constant depending only on $T^\infty_{L_0}(t)$. It follows from inequalities (5.97) and (5.99), and from inequality (5.100) with N_0 instead of L_0, that

$$\sum_{n+k\leq M} \wp^D_n\big(\delta(t)(1+\lambda_1(t))^{k/2}g^{(1)}(t)\big) \tag{5.101}$$

$$\leq a_{M,L_0,N_0}\Big(\sum_{\substack{n_1+n_2\leq M+2\\ n_1\leq L}} T^\infty_{n_1}(t)R^2_{n_2}(t)+\sum_{\substack{n_1+n_2\leq M+2\\ n_1\geq L+1}} T^2_{n_1}(t)R^2_{n_2}(t)$$

$$+\sum_{\substack{n_1+n_2\leq M+1\\ n_1\leq L}} T^\infty_{n_1}(t)\wp^D_{n_2}(h(t))+\sum_{\substack{n_1+n_2\leq M+1\\ n_1\geq L+1}} T^2_{n_1}(t)\wp^D_{n_2}(h(t))$$

$$+\sum_{\substack{n_1+n_2+n_3\leq M+2\\ n_1\leq L\\ n_3\leq M+1-L_0}} T^\infty_{n_1}(t)T^2_{n_2}(t)H_{n_3}(t)+\sum_{\substack{n_1+n_2\leq M+2\\ n_1\geq L+1\\ n_2\leq M-L-N_0}} T^2_{n_1}(t)H_{n_2}(t)\Big),$$

where M, L, L_0, N_0 are nonnegative integers and a_{M,L_0,N_0} is a constant depending only on $T^\infty_{L_0}(t)$ and $T^\infty_{N_0}(t)$.

Since h and g satisfy equation (5.86) it follows, as in the proof of Theorem 5.1 (cf. (5.8) and (5.9)), that

$$(i\gamma^\mu\partial_\mu+m)h^{(1)}=g'+g^{(1)}, \tag{5.102a}$$

where $h^{(1)}$ is given by (5.91), $g^{(1)}$ by (5.93a) and $g' = (2m)^{-1}(m - i\gamma^\mu\partial_\mu + \gamma^\mu G_\mu)g$. It follows from (5.102) that

$$(i\gamma^\mu\partial_\mu + m)h_Y^{(1)} = g'_Y + g_Y^{(1)}, \quad Y \in U(\mathfrak{p}). \tag{5.102b}$$

It follows from Theorem 5.7 and equation (5.102b) that

$$\begin{aligned}&\sum_{\substack{Y\in\Pi' \\ |Y|+k\leq N}} (\delta(t,x))^{3/2}(1+\lambda_1(t,x))^{k/2}|h^{(1)}(t,x)| \\ &\quad\leq C_N\Big(\wp^D_{N+8}(h^{(1)}(t)) + R'_{N+7}(t) + \sum_{n+k\leq N+7}\wp^D_n\big(\delta(t)(1+\lambda_1(t))^{k/2}g^{(1)}(t)\big)\Big),\end{aligned} \tag{5.103}$$

for some constant C_N. Let $n = N+8, L = 8$ in (5.92) and let $M = N+7, L = 9, L_0 = 9, N_0 = 0$ in (5.101). It then follows from inequalities (5.92), (5.101) and (5.103) that

$$\sum_{\substack{Y\in\Pi' \\ |Y|+k\leq N}} (\delta(t,x))^{3/2}(1+\lambda_1(t,x))^{k/2}|h_Y^{(1)}(t,x)| \leq b_N(t), \tag{5.104}$$

where

$$\begin{aligned}b_N(t) = a_N\Big(&R'_{N+7}(t) + R^2_{N+9}(t) + \wp^D_{N+8}(h(t)) + \sum_{\substack{n_1+n_2\leq N+9 \\ n_2\leq N-1}} T^2_{n_1}(t)R^2_{n_2}(t) \\ &+ \sum_{\substack{n_1+n_2\leq N+8 \\ n_2\leq N-2}} T^2_{n_1}(t)\wp^D_{n_2}(h(t)) + \sum_{\substack{n_1+n_2\leq N+9 \\ n_2\leq N-1}} T^2_{n_1}(t)H_{n_2}(t)\Big),\end{aligned} \tag{5.105a}$$

for some constant a_N depending only on $T_9^\infty(t)$. According to definition (5.89a) of $T^2_{N,n}$ and according to inequality (5.89e), it follows from (5.104a) that

$$\begin{aligned}b_N(t) \leq a_N\Big(&R'_{N+7}(t) + R^2_{N+9}(t) + \wp^D_{N+8}(h(t)) \\ &+ \sum_{\substack{n_1+n_2\leq N \\ n_2\leq N-1}} T^2_{9,n_1}(t)\big(R^2_{n_2}(t) + \wp^D_{n_2}(h(t)) + H_{n_2}(t)\big)\Big).\end{aligned} \tag{5.105b}$$

Definition (5.91) of $h^{(1)}$ gives that

$$h_Y^{(1)} = h_Y - (2m)^{-1}G_{\mathbb{I}\mu}\gamma^\mu h_Y - (2m)^{-1}g_Y - (2m)^{-1}\sum^{Y}_{\substack{Y_1,Y_2 \\ |Y_2|\leq|Y|-1}} G_{Y_1\mu}\gamma^\mu h_{Y_2},$$

$Y \in \Pi'$, which together with inequality (5.104) shows that

$$\begin{aligned}&\sum_{\substack{Y\in\Pi' \\ |Y|+k\leq N}} \delta(t,x)^{3/2}(1+\lambda_1(t,x))^{k/2}\big(|h_Y(t,x)| - (2m)^{-1}|G_{\mathbb{I}\mu}(t,x)||h_Y(t,x)|\big) \\ &\leq 2m^{-1}R^\infty_N(t) + (2m)^{-1}\sum_{\substack{Y_1,Y_2\in\Pi' \\ |Y|+k\leq N \\ |Y_1|+|Y_2|=|Y| \\ |Y_2|\leq|Y|-1}} \|G_{Y_1}(t)\|_{L^\infty}\|\delta(t)^{3/2}(1+\lambda_1(t))^{k/2}h_{Y_2}(t)\|_{L^\infty} + b_N(t).\end{aligned}$$

Using that $\|G_{Y_1}(t)\|_{L^\infty} \leq \|(1-\Delta)^{1/2}\nabla G_{Y_1}(t)\|_{L^2} \leq \wp^{M^1}_{|Y_1|+1}\big((G(t),0)\big)$, that

$$\begin{aligned}&\delta(t,x)^{3/2}(1+\lambda_1(t,x))^{k/2}|G_\mu(t,x)||h_Y(t,x)|\\&\qquad\leq |\delta(t,x)^{1/2}G(t,x)|\big(\delta(t,x)^{3/2}(1+\lambda_1(t,x))^{k/2}|h_Y(t,x)|\big)^{\varepsilon_k}|h_Y(t,x)|^{1-\varepsilon_k}\end{aligned}$$

where $\varepsilon_k=(2+k)/(3+k)$ and that $|h_Y(t,x)|\leq C\wp^D_{|Y|+2}(h(t))$, we obtain that

$$\begin{aligned}&\delta(t,x)^{3/2}(1+\lambda_1(t,x))^{k/2}|h_Y(t,x)|\\&\quad-(2m)^{-1}T_0^\infty(t)\big(\delta(t,x)^{3/2}(1+\lambda_1(t,x))^{k/2}|h_Y(t,x)|\big)^{\varepsilon_k}(\wp^D_{|Y|+2}(h(t))^{1-\varepsilon_k}\\&\qquad\leq(2m)^{-1}R_N^\infty(t)+(2m)^{-1}\sum_{\substack{n_1+n_2\leq N+1\\ n_2\leq N-1}}T^2_{n_1}(t)H_{n_2}(t)+b_N(t).\end{aligned}\tag{5.106}$$

For fixed t,x,k and Y, let $a=\delta(t,x)^{3/2}(1+\lambda_1(t,x))^{k/2}|h_Y(t,x)|$ and let b'_N be the right-hand side of inequality (5.106). Then

$$a-(2m)^{-1}T_0^\infty(t)\big(\wp^D_{|Y|+2}(h(t))\big)^{1-\varepsilon_k}a^{\varepsilon_k}\leq b'_N.$$

Since $0\leq\varepsilon_k<1$, it follows that

$$a\leq C_1\wp^D_{|Y|+2}(h(t))+C_2b'_N,$$

where C_1 and C_2 are constants depending on ε_k and $(2m)^{-1}T_0^\infty(t)$. Thus, it follows from (5.106) that

$$\begin{aligned}&\delta(t,x)(1+\lambda_1(t,x))^{k/2}|h_Y(t,x)|\\&\qquad\leq C_{k,|Y|}\Big(R_N^\infty(t)+\sum_{\substack{n_1+n_2\leq N\\ n_2\leq N-1}}T^2_{1,n_1}(t)H_{n_2}(t)+b_N(t)\Big),\end{aligned}$$

$Y\in\Pi', |Y|+k\leq N$, which together with inequalities (5.104) and (5.105b) give that

$$\begin{aligned}H_N(t)\leq a_N\Big(&R'_{N+7}(t)+R^2_{N+9}(t)+R^\infty_N(t)+\wp^D_{N+8}(h(t))\\&+\sum_{n_1+n_2\leq N, n_2\leq N-1}T^2_{9,n_1}(t)\big(R^2_{n_2}(t)+\wp^D_{n_2}(h(t))+H_{n_2}(t)\big)\Big),\end{aligned}\tag{5.107}$$

where a_N is a constant depending only on $T_9^\infty(t)$.

The inequality of the theorem for $N=0$ follows by taking $N=0$ in (5.107). For $N\geq 1$ it follows from inequality (5.107), the induction hypothesis and the inequality of the theorem, that

$$\begin{aligned}H_N(t)\leq a_N\Big(&R'_{N+7}(t)+R^2_{N+9}(t)+R^\infty_N(t)+\wp^D_{N+8}(h(t))\\&+\sum_{\substack{n_1+n_2\leq N\\ n_2\leq N-1}}T^2_{9,n_1}(t)\big(R^2_{n_2}(t)+\wp^D_{n_2}(h(t))\big)\\&+\sum_{\substack{n_1+n_2\leq N\\ n_2\leq N-1}}T^2_{9,n_1}(t)a_{n_2}\Big(R'_{n_2+7}(t)+R^2_{n_2+9}(t)+R^\infty_{n_2}(t)+\wp^D_{n_2+8}(h(t))\\&+\sum_{\substack{n_3+n_4\leq n_2\\ n_4\leq n_2-1}}T^2_{9,n_3}(t)\big(R'_{n_4+7}(t)+R^2_{n_4+9}(t)+R^\infty_{n_4}(t)+\wp^D_{n_4+8}(h(t))\big)\Big)\Big),\end{aligned}$$

$N \geq 1$, where a_N depends only on $T_9^\infty(t)$ and where we have chosen $a_0 \leq a_1 \leq \cdots \leq a_N$.

This inequality and the fact that

$$a_N \sum_{\substack{n_1+n_2 \leq N \\ n_2 \leq N-1}} \sum_{\substack{n_3+n_4 \leq n_2 \\ n_4 \leq n_2-1}} T^2_{9,n_1}(t) a_{n_2} \big(R'_{n_4+7}(t) + R^2_{n_4+9}(t) + R^\infty_{n_4}(t) + \wp^D_{n_4+8}(h(t))\big)$$
$$\leq a_N a_{N-1} \sum_{\substack{n_1+n_2 \leq N \\ n_2 \leq N-1}} T^2_{9,n_1}(t) \big(R'_{n_2+7}(t) + R^2_{n_2+9}(t) + R^\infty_{n_2}(t) + \wp^D_{n_2+8}(h(t))\big),$$

obtained by using inequality (5.89d), prove the theorem, after redefinition of the constant a_N.

Theorem 5.5 and Theorem 5.8 give an estimate of $\wp^D_n\big((1+q_t+r_t)^{k/2}(h(t)\big)$ not containing L^∞-norms of $h_Y(t), Y \in \Pi'$. To state the result we adapt the notation of these theorems.

Corollary 5.9. *Let $n \geq 0$, $k \geq 1$, $L \geq 3$, let $h_Y \in C^0(\mathbb{R}^+, D)$ for $Y \in \Pi'$, $|Y| \leq \max(n+k, n+k+8-L)$, let $G_Y \in C^0(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathbb{R}^4))$ for $0 \leq |Y| \leq L$, $Y \in \Pi'$ and let $\delta^{-1} G_Y \in C^0(\mathbb{R}^+, L^2(\mathbb{R}^3, \mathbb{R}^4))$, for $|Y| \leq n+k-1$, $Y \in \Pi'$, if $L \leq n+k-2$, where $(\delta(t))(x) = (1+t+|x|)$. Let*

$$R^i_{p,q}(t) = \Big(\sum_{\substack{0 \leq j \leq q-1 \\ p+j \geq 0}} \big(\wp^D_{p+j}\big((1+\lambda_i(t))^{(q+1-j)/2} g(t)\big)\big)^2\Big)^{1/2}, \quad i = 0, 1,$$

for integers $p \geq -q+1, q \geq 1$ and $R^i_{p,q} = 0$ otherwise, let

$$\Gamma_p(t) = \Big(\sum_{\substack{Y \in \Pi' \\ |Y| \leq p}} \|\delta(t)^{-1} G_Y(t)\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^4)}\Big)^{1/2}, \quad p \geq 0$$

and let $\chi_+(s) = 0$ for $s < 0$ and $\chi_+(s) = 1$ for $s \geq 0, s \in \mathbb{R}$. Let $(G_Y, \dot{G}_Y) \in C^0(\mathbb{R}^+, M^1)$ for $Y \in \Pi', |Y| \leq n+k-L+10$. If $R^2_{N+9}(t) + R'_{N+7}(t) + T^2_{9,N}(t) + T^\infty_9(t) < \infty$ for $N = \max(1, n+k-L)$, then

$$\begin{aligned}
&\wp^D_n\big((1+\lambda_i(t))^{k/2} h(t)\big) \\
&\quad \leq C'_k\big(\wp^D_{n+k}(h(t)) + R^i_{n,k}(t)\big) \\
&\qquad + C'_{n+k} \sum_{\substack{n_1+n_2=n+k \\ 1 \leq n_1 \leq L}} (1+\tau_{1,n_1}(t))\big(\wp^D_{n_2}(h(t)) + R^i_{n_2-k,k}(t)\big) \\
&\qquad + C''_k \chi_+(n+k-L-2)\Gamma_{n+k-1}(t)\big(1+T^2_{9,0}(t)\big)\big(R'_8(t) + R^2_{10}(t) + R^\infty_1(t) + \wp^D_9(h(t))\big) \\
&\qquad + C''_{n+k} \sum_{\substack{n_1+n_2+n_3+n_4=n+k \\ n_1 \leq L, n_2 \leq n+k-2 \\ n_3+n_4 \leq n+k-L-2}} (1+\tau_{1,n_1}(t))\Gamma_{n_2}(t)(1+T^2_{9,n_3}(t)) \\
&\qquad\qquad \big(R'_{n_4+7}(t) + R^2_{n_4+9}(t) + R^\infty_{n_4}(t) + \wp^D_{n_4+8}(h(t))\big),
\end{aligned}$$

where the constants C'_q depend only on $\tau_{1,0}(t)$ and the constants C''_q on $T^\infty_9(t)$.

Proof. The hypotheses of Theorem 5.8 are satisfied for $N = \max(1, n+k-L)$, so $H_N(t)$ is finite. Therefore and according to the hypotheses of the corollary, the hypotheses of Theorem 5.5 are satisfied. It follows from theorem 5.5 that

$$
\begin{aligned}
&\wp_n^D\big((1+\lambda_i(t))^{k/2}h(t)\big) \\
&\quad \le C'_k\Big(\wp_{n+k}^D(h(t)) + R_{n,k}^i(t) + \sum_{\substack{n_1+n_2=n+k \\ n_1\le n+k-1 \\ n_2\le n+k-L-2}} \Gamma_{n_1}(t)H_{n_2}(t)\Big) \\
&\qquad + C'_{n+k} \sum_{\substack{n_1+n_2=n+k \\ n_1\le n+k-2 \\ n_2\le n+k-L-1}} \Gamma_{n_1}(t)H_{n_2}(t) \\
&\qquad + C'_{n+k} \sum_{1\le l\le L} (1+\tau_{1,l}(t))\Big(\wp_{n+k-l}^D(h(t)) + R_{n-l,k}^i(t) + \sum_{\substack{n_1+n_2=n+k-l \\ n_1\le n+k-l-1 \\ n_2\le n+k-L-1}} \Gamma_{n_1}(t)H_{n_2}(t)\Big),
\end{aligned}
$$

where C'_k and C'_{n+k} are constants depending only on $\tau_{1,0}(t)$. This inequality gives

$$
\begin{aligned}
&\wp_n^D\big((1+\lambda_i(t))^{k/2}h(t)\big) \\
&\quad \le C'_k\Big(\wp_{n+k}^D(h(t)) + R_{n,k}^i(t) + \chi_+(n+k-L-2)\Gamma_{n+k-1}(t)H_1(t)\Big) \\
&\qquad + C'_{n+k} \sum_{1\le l\le L} (1+\tau_{1,l}(t))\big(\wp_{n+k-l}^D(h(t)) + R_{n-l,k}^i(t)\big) \\
&\qquad + C'_{n+k} \sum_{0\le l\le L} (1+\tau_{1,l}(t)) \sum_{\substack{n_1+n_2=n+k-l \\ n_1\le n+k-2 \\ n_2\le n+k-L-2}} \Gamma_{n_1}(t)H_{n_2}(t),
\end{aligned} \tag{5.108}
$$

where C'_k and C'_{n+k} are constants depending only on $\tau_{1,0}(t)$.

It follows from Theorem 5.8 that

$$
\begin{aligned}
&\sum_{0\le l\le L} (1+\tau_{1,l}(t)) \sum_{\substack{n_1+n_2=n+k-l \\ n_1\le n+k-2 \\ n_2\le n+k-L-2}} \Gamma_{n_1}(t)H_{n_2}(t) \\
&\quad = \sum_{\substack{n_1+n_2+n_3=n+k \\ n_1\le L, n_2\le n+k-2 \\ n_3\le n+k-L-2}} (1+\tau_{1,n_1}(t))\Gamma_{n_2}(t)H_{n_3}(t) \\
&\quad \le a_{n+k}\Big(\sum_{\substack{n_1+n_2+n_3+n_4=n+k+1 \\ n_1\le L, n_2\le n+k-2 \\ n_3+n_4\le n+k-L-2}} (1+\tau_{1,n_1}(t))\Gamma_{n_2}(t) \\
&\qquad\qquad (1+T_{9,n_3}^2(t))\big(R'_{n_4+7}(t) + R_{n_4+9}^2(t) + R_{n_4}^\infty(t) + \wp_{n_4+8}^D(h(t))\big)\Big),
\end{aligned} \tag{5.109}
$$

where a_{n+k} is a constant depending only on $T_9^\infty(t)$. Theorem 5.8 also gives that

$$
H_1(t) \le a_1(1+T_{9,0}^2(t))\big(R'_8(t) + R_{10}^2(t) + R_1^\infty(t) + \wp_9^D(h(t))\big), \tag{5.110}
$$

where a_1 depends only on $T_9^\infty(t)$. The inequality of the corollary follows from inequalities (5.108), (5.109) and (5.110), and by suitably defining the constants C''_k and C''_{n+k}. This proves the corollary.

The preceding results of this chapter permit to establish L^2-estimates of $\xi^D_Y h$, where h is a solution of equation (5.1) and $Y \in \Pi'$. Suppose that $\xi^D_Y g \in C^0(\mathbb{R}^+, D), Y \in \Pi'$, and that $(\xi^M_Y G, \xi^M_{P_0 Y} G) \in C^0(\mathbb{R}^+, M^\rho), Y \in \Pi'$, for some $1/2 < \rho \leq 1$. We recall that

$$h_Y = \xi^D_Y h, \quad g_Y = \xi^D_Y g, \quad G_{Y\mu} = (\xi^M_Y G)_\mu, \quad Y \in U(\mathfrak{p}), \tag{5.111a}$$

and introduce

$$f_Y(t) = \int_{t_0}^t w(t,s)(-i\gamma^0) g_Y(s) ds, \quad Y \in U(\mathfrak{p}). \tag{5.111b}$$

We also introduce the subset $\boldsymbol{\sigma}^n, n \geq 0$, of Π' defined by

$$\boldsymbol{\sigma}^n = I_n \cap \Pi', \tag{5.112}$$

where I_n is the ideal in $U(\mathfrak{p})$ generated by the elements of order n in $U(\mathbb{R}^4)$.

According to the definition of $\Pi', \boldsymbol{\sigma}^n$ is a basis of I_n. We note that $\Pi' = \boldsymbol{\sigma}^0 \supset \boldsymbol{\sigma}^1 \supset \cdots \supset \boldsymbol{\sigma}^n \supset \boldsymbol{\sigma}^{n+1} \supset \cdots$, and that $\boldsymbol{\sigma}^1 \cap U(\mathfrak{sl}(2,\mathbb{C})) = \{0\}$. When Π is given the standard ordering, $P_0 < P_1 < P_2 < P_3 < M_{23} < M_{13} < M_{12} < M_{01} < M_{02} < M_{03}$, we note that, if $Y \in \boldsymbol{\sigma}^n$, then $Y = XZ$, where $X \in \Pi' \cap U(\mathbb{R}^4)$, $|X| = n, Z \in \Pi'$, and that $\Pi' = \boldsymbol{\sigma}^1 \cup (U(\mathfrak{sl}(2,\mathbb{C})) \cap \Pi')$. When not specified, Π will always be given the standard ordering.

Proposition 5.10. *Let $Y, Y_2, Y_3 \in \Pi'$, let $0 \leq a(Y_1, Y_2) \leq 1$, let $h_Y, G_{Y\mu}, g_Y, f_Y$ be given by (5.111a) and (5.111b), let $\theta_Z = \xi^D_Z(\gamma^\mu G_\mu h) + g_Z, Z \in \Pi'$, let h be given by (5.3c), let $x^\mu G_\mu = 0$ and let $t, t_0 \geq 0$.*

i) *If $h_Z(t_0) \in D$ for $Z \in \Pi', |Z| \leq |Y|$ and if* $h'_Y = h_Y - (2m)^{-1} \sum^Y_{\substack{Y_1, Y_2 \\ |Y_2| \leq |Y|-1 \\ Y_1 \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))}} \gamma^\mu G_{Y_1 \mu} H_{Y_2}$,

then

$$\begin{aligned}
&\big| \|h'_Y(t)\|_D - \|h'_Y(t_0)\|_D \big| - \|f_Y(t)\|_D \\
&\leq \sum^Y_{\substack{Y_1, Y_2 \\ |Y_2| \leq |Y|-1 \\ Y_1 \in \boldsymbol{\sigma}^1}} \int_{\min(t,t_0)}^{\max(t,t_0)} \|G_{Y_1\mu}(s)\gamma^\mu h_{Y_2}(s)\|_D ds \\
&+ 2m^{-1} \sum^Y_{\substack{Y_1, Y_2 \\ |Y_2| \leq |Y|-1 \\ Y_1 \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))}} \int_{\min(t,t_0)}^{\max(t,t_0)} \Big(\Big((1+s)^{-1} \big(\|G_{Y_1 0}(s) h_{P_0 Y_2}(s)\|_D \\
&+ \sum_{1 \leq i \leq 3} \big(\|G_{Y_1 i}(s) h_{M_{0i} Y_2}(s)\|_D + \|G_{Y_1 i}(s) h_{Y_2}(s)\|_D \big) \big) \Big)^{1-a(Y_1,Y_2)} \|G^\mu_{Y_1}(s) \partial_\mu h_{Y_2}(s)\|_D^{a(Y_1,Y_2)} \\
&+ \|(\partial_\mu G^\mu_{Y_1}(s)) h_{Y_2}(s)\|_D + \frac{1}{2} \|\gamma^\mu \gamma^\nu \big(\partial_\mu G_{Y_1\nu}(s) - \partial_\nu G_{Y_1\mu}(s) \big) h_{Y_2}(s)\|_D \\
&+ \|\gamma^\mu \gamma^\nu G_\mu(s) G_{Y_1\nu}(s) h_{Y_2}(s)\|_D + \|\gamma^\mu G_{Y_1\mu}(s) \theta_{Y_2}(s)\|_D \Big) ds,
\end{aligned}$$

ii) *If* $h_Z(t_0) \in D, \partial_\nu h_Z(t_0) \in D$ *for* $Z \in \Pi', 0 \le \nu \le 3, |Z| \le |Y|$ *and if* $h_Z^{(\nu)} = \partial_\nu h_Z + i \sum_{Z_1,Z_2}^{Z} G_{Z_1\nu} h_{Z_2}$,

$$g_{1Z}^{(\nu)} = -\gamma^\mu \sum_{Z_1,Z_2}^{Z} (\partial_\mu G_{Z_1\nu} - \partial_\nu G_{Z_1\mu}) h_{Z_2} + \gamma^\mu \sum_{\substack{Z_1,Z_2 \\ |Z_2|\le|Z|-1 \\ Z_1\in\boldsymbol{\sigma}^1}}^{Z} G_{Z_1\mu}(\partial_\nu h_{Z_2} + iG_\nu h_{Z_2})$$
$$+ i \sum_{\substack{Z_1,Z_2 \\ |Z_2|\le|Z|-1}}^{Z} G_{Z_1\nu}(i\gamma^\mu\partial_\mu + m - \gamma^\mu G_\mu) h_{Z_2}$$
$$+ \sum_{\substack{Z_1,Z_2 \\ |Z_2|\le|Z|-1 \\ Z_1\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))}}^{Z} i\gamma^\mu G_{Z_1\mu} G_\nu h_{Z_2} + iG_\nu g_Z,$$

$$g_{2Z}^{(\nu)} = \gamma^\mu \sum_{\substack{Z_1,Z_2 \\ |Z_2|\le|Z|-1 \\ Z_1\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))}}^{Z} G_{Z_1\mu}\partial_\nu h_{Z_2},$$

then

$$\Big| \|h_Y^{(\nu)}(t) - (2m)^{-1} g_{2Y}^{(\nu)}(t)\|_D - \|h_Y^\nu(t_0) - (2m)^{-1} g_{2Y}^{(\nu)}(t_0)\|_D \Big|$$
$$\le \|f_{P_\nu Y}(t)\|_D + \int_{\min(t,t_0)}^{\max(t,t_0)} \|g_{1Y}^{(\nu)}(s)\|_D ds$$
$$+ (2m)^{-1} \sum_{\substack{Y_1,Y_2 \\ |Y_2|\le|Y|-1 \\ Y_1\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))}}^{Y} \int_{\min(t,t_0)}^{\max(t,t_0)} \Big(\Big((1+s)^{-1}\big(\|G_{Y_1 0}(s) h_{P_0 P_\nu Y_2}(s)\|_D$$
$$+ \sum_{1\le i\le 3} \big(\|G_{Y_1 i}(s) h_{M_{0i} P_\nu Y_2}(s)\|_D$$
$$+ \|G_{Y_1 i}(s) h_{P_\nu Y_2}(s)\|_D\big)\big)\Big)^{1-a(Y_1,Y_2)} \|G_{Y_1}^\mu(s)\partial_\mu h_{P_\nu Y_2}(s)\|_D^{a(Y_1,Y_2)}$$
$$+ \|(\partial_\mu G_{Y_1}^\mu(s)) h_{P_\nu Y_2}(s)\|_D + \frac{1}{2}\|\gamma^\alpha\gamma^\beta\big(\partial_\alpha G_{Y_1\beta}(s) - \partial_\beta G_{Y_1\alpha}(s)\big) h_{P_\nu Y_2}(s)\|_D$$
$$+ \|\gamma^\alpha\gamma^\beta G_\alpha(s) G_{Y_1\beta}(s) h_{P_\nu Y_2}(s)\|_D + \|\gamma^\mu G_{Y_1\mu}(s)\theta_{P_\nu Y_2}(s)\|_D\Big) ds.$$

Proof. Since

$$\xi_Y^D(\gamma^\mu G_\mu h) = \sum_{Y_1,Y_2}^{Y} \gamma^\mu G_{Y_1} h_{Y_2},$$

it follows from equation (5.1) and definition (5.111b) of f_Y that

$$(i\gamma^\mu\partial_\mu + m - \gamma^\mu G_\mu)(h_Y - f_Y) = g_{1Y} + g_{2Y}, \tag{5.113a}$$

where

$$g_{1Y} = \sum^{Y}_{\substack{Y_1,Y_2 \\ |Y_2|\leq|Y|-1 \\ Y_1\in\boldsymbol{\sigma}^1}} \gamma^\mu G_{Y_1\mu} h_{Y_2}, g_{2Y}, \quad g_{2Y} = \sum^{Y}_{\substack{Y_1,Y_2 \\ |Y_2|\leq|Y|-1 \\ Y_1\in\Pi'\cap U(sl(2,\mathbb{C}))}} \gamma^\mu G_{Y_1\mu} h_{Y_2}. \tag{5.113b}$$

We have here used that $\Pi' = \boldsymbol{\sigma}^1 \cup \big(\Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))\big)$ and that $\boldsymbol{\sigma}^1 \cap \big(\Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))\big) = \emptyset$. We note that $f_Y(t_0) = 0$ and that $h'_Y = h_Y - (2m)^{-1} g_{2Y}$, which gives that

$$\begin{aligned}&\big|\|h'_Y(t)\|_D - \|h'_Y(t_0)\|_D\big| \leq \|f_Y(t)\|_D\big| \\ &\quad + \big|\|h_Y(t) - f_Y(t) - (2m)^{-1} g_{2Y}(t)\|_D - \|h_Y(t_0) - f_Y(t_0) - (2m)^{-1} g_{2Y}(t_0)\|_D\big|.\end{aligned}$$

The inequality in statement i) of the proposition now follows from Corollary 5.2 whith h, g, g_1 replaced by $h_Y - f_Y, g_{1Y} + g_{2Y}, g_{1Y}$, respectively, and from the fact that $x_\mu G^\mu_{Y_1}(x) = 0$ for $Y_1 \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$, which in its turn follows from the $\mathfrak{sl}(2,\mathbb{C})$ covariance of the condition $x_\mu G^\mu_{\mathbb{I}}(x) = 0$.

The definitions in statement ii) of the proposition give

$$(i\gamma^\mu \partial_\mu + m - \gamma^\mu G_\mu)(h^{(\nu)}_Y - f_{P_\nu Y}) = g^{(\nu)}_{1Y} + g^{(\nu)}_{2Y}.$$

The inequality in statement ii) follows, similarly as in the case of the inequality in statement i), from this equation and from Corollary 5.2 with h, g, g_1 replaced by $h^{(\nu)}_Y - f_{P_\nu Y}$, $g^{(\nu)}_{1Y} + g^{(\nu)}_{2Y}, g^{(\nu)}_{1Y}$, respectively. This proves the proposition.

In the sequel of this chapter we shall suppose that

$$G_\mu(y) = A_\mu(y) - \partial_\mu \vartheta(A, y), \quad 0 \leq \mu \leq 3, \tag{5.114}$$

where $\xi^M_Y A \in C^0(\mathbb{R}^+, M^\rho_0)$ for some $0 \leq \rho \leq 1$, for $Y \in \Pi'$, and $|Y| \leq N$ for some $N \geq 0$. We introduce the notation

$$\begin{aligned}S^Y(t) = \sup_{0\leq s\leq t} \Big(&(1+s)^{\rho-1}\|(A_Y(s), A_{P_0 Y}(s))\|_{M^\rho} + \|(A_Y(s), A_{P_0 Y}(s))\|_{M^1} \\ &+ (1+s)^{1/2-\rho}\|\delta(s)\Box A_Y(s)\|_{L^2(\mathbb{R}^3,\mathbb{R}^4)} + (1+s)^{1/2-\rho}\|(\partial^\mu A_{Y\mu}(s)\|_{L^2(\mathbb{R}^3,\mathbb{R})}\Big),\end{aligned} \tag{5.115a}$$

for $0 \leq \rho \leq 1$, $Y \in \Pi'$, and

$$S^{\rho,n}(t) = \Big(\sum_{\substack{Y\in\Pi' \\ |Y|\leq n}} (S^Y(t))^2 + \sum_{\substack{Y\in\Pi' \\ |Y|\leq n-1}} \|(B_Y(s), B_{P_0 Y}(s))\|^2_{M^1}\Big)^{1/2}, \tag{5.115b}$$

where $0 \leq \rho \leq 1, n \geq 0$ and $B_\mu(y) = y_\mu \partial^\nu A_\nu(y)$, $A_{Y\mu} = (\xi^M_Y A)_\mu, B_{Y\mu} = (\xi^M_Y B)_\mu$. We also introduce

$$\begin{aligned}[A]^n(t) = &\sum_{\substack{Y\in\Pi' \\ |Y|\leq n}} \sup_{\substack{x\in\mathbb{R}^3 \\ 0\leq s\leq t}} \big((1+s+|x|)^{3/2-\rho}(|A_Y(s,x)| + |B_Y(s,x)|)\big) \\ &+ \sum_{\substack{Y\in\boldsymbol{\sigma}^1 \\ |Y|\leq n}} \sup_{\substack{x\in\mathbb{R}^3 \\ 0\leq s\leq t}} \big((1+s+|x|)(1+\big|s-|x|\big|)^{1/2}(|A_Y(s,x)| + |B_Y(s,x)|)\big),\end{aligned} \tag{5.116a}$$

$n \geq 0, 0 \leq \rho \leq 1$, where B_Y is as in (5.115b), and let

$$[A]'^{n}(t) = \sum_{\substack{Y \in \Pi' \\ |Y| \leq n}} \sup_{x \in \mathbb{R}^3} \big((1+t+|x|)^{3/2-\rho}|A_Y(t,x)|\big) \\ + \sum_{\substack{Y \in \boldsymbol{\sigma}^1 \\ |Y| \leq n}} \sup_{x \in \mathbb{R}^3} \big((1+t+|x|)(1+\big|t-|x|\big|)^{1/2}|A_Y(t,x)|\big), \tag{5.116b}$$

$n \geq 0, 0 \leq \rho \leq 1$. We note that the second sum in (5.116a) and (5.116b) is absent for $n = 0$, since $|Y| \geq 1$ if $Y \in \boldsymbol{\sigma}^1$. It follows from Lemma 4.4 that

$$[G]'^{n}(t) \leq C_n [A]^{n+1}(t), \quad n \geq 0, t \geq 0, 1/2 < \rho \leq 1, \tag{5.116c}$$

where the constant C_n depends only on ρ.

Let $S^{\rho}_{N,n}(t)$ (resp. $[A]_{N,n}(t)$) be defined by b_n in formula (5.87) with $a^{(n)} = S^{\rho,N+n}(t)$ (resp. $[A]^{N+n}(t)$) and let $S^{\rho}_n(t) = S^{\rho}_{0,n}(t)$ (resp. $[A]_n(t) = [A]_{0,n}(t)$). We introduce the notation:

$$\wp^{D}_{n,i}(a) = \Big(\sum_{\substack{Y \in \boldsymbol{\sigma}^i \\ |Y| \leq n}} \|a_Y\|^2_D\Big)^{1/2}, \quad 0 \leq i \leq n, \tag{5.117a}$$

and

$$\wp^{M^\rho}_{n,i}(b) = \Big(\sum_{\substack{Y \in \boldsymbol{\sigma}^i \\ |Y| \leq n}} \|b_Y\|^2_{M^\rho}\Big)^{1/2}, \quad 0 \leq i \leq n, \tag{5.117b}$$

where $Y \mapsto a_Y$ (resp. b_Y) is a function from $\boldsymbol{\sigma}^i$ to D (resp. M^ρ).

In the next lemma we shall estimate the sum over $Y_1 \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$ on the right-hand side in the inequalities of statements i) and ii) of Proposition 5.10. To state the result, let us introduce the following notations:

$$J_Y(t) = 2m^{-1} \sum^{Y}_{\substack{Y_1, Y_2 \\ |Y_2| \leq |Y|-1 \\ Y_1 \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))}} \bigg(\Big((1+t)^{-1}(\|G_{Y_1 0}(t) h_{P_0 Y_2}(t)\|_D \\ + \sum_{1 \leq i \leq 3} \big(\|G_{Y_1 i}(t) h_{M_{0i} Y_2}(t)\|_D + \|G_{Y_1 i}(t) h_{Y_2}(t)\|_D\big)\big)\Big)^{1-a(Y_1,Y_2)} \\ \|G^{\mu}_{Y_1}(t)\partial_\mu h_{Y_2}(t)\|^{a(Y_1,Y_2)}_D + \|(\partial_\mu G^{\mu}_{Y_1}(t)) h_{Y_2}(t)\|_D \\ + \frac{1}{2}\|\gamma^\mu\gamma^\nu\big(\partial_\mu G_{Y_1 \nu}(t) - \partial_\nu G_{Y_1 \mu}(t)\big) h_{Y_2}(t)\|_D \\ + \|\gamma^\mu\gamma^\nu G_\mu(t) G_{Y_1 \nu}(t) h_{Y_2}(t)\|_D + \|\gamma^\mu G_{Y_1 \mu}(t)\theta_{Y_2}(t)\|_D\bigg), \tag{5.118}$$

$Y \in \Pi', t \geq 0$, where θ_Z is defined as in Proposition 5.10, and where $a(Y_1, Y_2) = 1/2$, if $|Y_1| = 1$ and $|Y_2| = |Y| - 1$, and $a(Y_1, Y_2) = 0$ otherwise;

$$k_n^{(1)}(L,t) = \sum_{\substack{n_1+n_2=n \\ 1 \leq n_1 \leq L}} \big([A]^{n_1+1}(t)(1+t)^{1/2} \wp_{n_2}^D(g(t)) + [A]_{n_1+3}(t) \wp_{n_2}^D(h(t))\big), \tag{5.119a}$$

$$l_n^{(1)}(L,t) = \sum_{\substack{n_1+n_2+n_3=n \\ n_2 \leq n_1 \leq n-1 \\ n_3 \leq n-L-1}} S^{\rho,n_1}(t)(1+[A]^{n_2+1}(t)) H_{n_3+1}(t) \tag{5.119b}$$
$$+ \sum_{\substack{Y_1, Y_2 \in \Pi' \\ |Y_1|+|Y_2|=n \\ L+1 \leq |Y_1| \leq n-1}} (1+t)^{2-\rho} \|\gamma^\mu G_{Y_1 \mu} g_{Y_2}(t)\|_D,$$

$$k_n^{(2)}(L,t) = \sum_{\substack{n_1+n_2=n \\ 1 \leq n_1 \leq L}} [A]_{3,n_1}(t)(\wp_{n_2}^D(h(t)) + R_{n_2-1,1}^0(t)) \tag{5.119c}$$

and

$$l_n^{(2)}(L,t) = \sum_{\substack{n_1+n_2+n_3+n_4=n \\ n_1 \leq L, n_2 \leq n-1 \\ n_3+n_4 \leq n-L-2}} (1+[A]_{2,n_1}(t)) S^{\rho,n_2}(t)(1+S_{10,n_3}^\rho(t)) \tag{5.119d}$$
$$\big(R'_{n_4+7}(t) + R_{n_4+9}^2(t) + R_{n_4}^\infty(t) + \wp_{n_4+8}^D(h(t))\big),$$

where $n \geq 0, L \geq 0, t \geq 0$, where $H_n, R_{n,k}^0, R'_n, R_n^2, R_n^\infty$ are defined in Theorem 5.8 and Corollary 5.9.

Lemma 5.11. *Let $1/2 < \rho < 1, h_Z \in C^0(\mathbb{R}^+, D)$, let G be given by (5.114) and $(A_Z, A_{P_0 Z}) \in C^0(\mathbb{R}^+, M^\rho) \cap C^0(\mathbb{R}^+, M^1)$ for $Z \in \Pi'$ and $|Z|$ sufficiently large. Let $L \geq 1, g = (i\gamma^\mu \partial_\mu + m - \gamma^\mu G_\mu)h, \chi_+(s) = 0$ for $s < 0, \chi_+(s) = 1$ for $s \geq 0$ and let $Y \in \boldsymbol{\sigma}^i, Y \notin \boldsymbol{\sigma}^{i+1}, i \geq 0$. If $i = |Y|$, then $J_Y(t) = 0$, if $1 \leq i \leq |Y| - 1$, then*

$$J_Y(t) \leq C'_{|Y|}(1+t)^{-2+\rho} J_{|Y|}^{(i)}(L,t),$$

where

$$J_n^{(l)}(L,t) = [A]_3(t)\Big(\wp_n^D(h(t))^\varepsilon \wp_{n,l+1}^D(h(t))^{1-\varepsilon} + R_{n-1,1}^0(t)^{2(1-\rho)} \wp_{n-1}^D(h(t))^{2\rho-1}\Big)$$
$$+ k_n^{(1)}(L,t) + l_n^{(1)}(L,t) + k_n^{(2)}(L,t) + C''_n l_n^{(2)}(L,t),$$

$n \geq 1$, $0 \leq l \leq n$, $\varepsilon = \max(1/2, 2(1-\rho))$, $\wp_{n,n+1} = 0$, $J_n^{(l)} = 0$ for $l \geq n+1$, and if $i = 0, |Y| \geq 1$, then

$$J_Y(t) \leq (1+t)^{-2+\rho}\Big(C'_{|Y|} J_{|Y|}^{(0)}(L,t)$$
$$+ C\chi_+(|Y|-L-1)\big(S^Y(t)(1+[A]^1(t))H_1(t) + (1+t)^{2-\rho}\|\gamma^\mu G_{Y\mu}(t) g(t)\|_D\big)\Big),$$

for some constant C depending only on ρ, a constant $C'_{|Y|}$ depending only on ρ and $[A]^3(t)$ and a constant $C''_{|Y|}$ depending only on ρ and $[A]^9(t)$.

Proof. Let $Y \in \boldsymbol{\sigma}^i, i \geq 0$. Then the sum in definition (5.118) of $J_Y(t)$ runs over a subset of $\{(Y_1, Y_2) \in \Pi' \times \Pi' \big| Y_1 \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C})), Y_2 \in \boldsymbol{\sigma}^i, i \leq |Y_2| \leq |Y|-1\}$, which shows that $1 \leq |Y_1| \leq |Y| - i$. If $i = |Y| \geq 0$, then it follows that $J_Y = 0$. We therefore only need to consider the case $0 \leq i \leq |Y| - 1$.

Let $Y \in \boldsymbol{\sigma}^i, 0 \leq i \leq |Y| - 1$ and let $L \geq 1$. Since $F_{Z\mu\nu} \equiv \partial_\mu G_{Z\nu} - \partial_\nu G_{Z\mu} = \partial_\mu A_{Z\nu} - \partial_\nu A_{Z\mu}$ for $Z \in \Pi'$, it follows from definition (5.118) of J_Y and from the definitions of norms, that

$$\begin{aligned} J_Y(t) \leq C & \sum^{Y}_{\substack{Y_1,Y_2 \\ 1\leq |Y_1| \leq L \\ Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))}} \Big((1+t)^{-5/2+\rho+a(Y_1,Y_2)} [G]'^{|Y_1|}(t) \\ & \qquad \wp^D_{|Y_2|+1,i}(h(t))^{1-a(Y_1,Y_2)} \wp^D_{|Y_2|+1,i+1}(h(t))^{a(Y_1,Y_2)} \\ & \quad + (1+t)^{-2+\rho}[G]'^{|Y_1|+1}(t) \wp^D_{|Y_2|,i}((1+\lambda_0(t))^{1/2} h(t))^{2(1-\rho)} \wp^D_{|Y_2|,i}(h(t))^{2\rho-1} \\ & \quad + (1+t)^{-3+2\rho}[G]'^0(t)[G]'^{|Y_1|}(t)\wp^D_{|Y_2|,i}(h(t)) \\ & \quad + (1+t)^{-3/2+\rho}[G]'^{|Y_1|}(t)\wp^D_{|Y_2|,i}(g(t)) \Big) \\ & + C \sum^{Y}_{\substack{Y_1,Y_2 \\ 1+L\leq |Y_1| \\ Y_1 \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))}} \Big((1+t)^{-3/2} \|\delta(t)^{-1} G_{Y_1}(t)\|_{L^2} H_{|Y_2|+1}(t) \\ & \quad + (1+t)^{-1/2}\|\delta(t)^{-1}\partial^\mu G_{Y_1\mu}(t)\|_{L^2} H_{|Y_2|}(t) \\ & \quad + (1+t)^{-3/2}\|(A_{Y_1}(t), A_{P_0 Y_1}(t))\|_{M^1} H_{|Y_2|}(t) \\ & \quad + (1+t)^{-2+\rho}[G]'^0(t)\|\delta(t)^{-1}G_{Y_1}(t)\|_{L^2} H_{|Y_2|}(t) \Big) \\ & + 2m^{-1} \sum^{Y}_{\substack{Y_1,Y_2 \\ i \leq |Y_2| \leq |Y|-1}} \|\gamma^\mu G_{Y_1\mu}(t) \xi^D_{Y_2}(\gamma^\nu G_\nu h + g)(t)\|_D, \end{aligned} \tag{5.120}$$

$1/2 < \rho < 1, t \geq 0, 0 \leq i \leq |Y| - 1, Y \in \boldsymbol{\sigma}^i$, for some constant C depending only on ρ.

We shall estimate the different terms on the right-hand side of inequality (5.120). According to definition (5.114) of G, it follows that

$$G_{Y\mu}(y) = A_{Y\mu}(t) - \partial_\mu \vartheta(A_Y, y), \quad 0 \leq \mu \leq 3, \tag{5.121a}$$

$y \in \mathbb{R}^+ \times \mathbb{R}^3$, for $Y \in U(\mathfrak{sl}(2,\mathbb{C}))$. This gives that

$$G_{Y\mu}(y) = A_{Y\mu}(y) - I_{Y\mu}(y) - y^\nu \partial_\mu I_{Y\nu}(y), \tag{5.121b}$$

$Y \in U(\mathfrak{sl}(2,\mathbb{C}))$, where

$$I_\mu(y) = \int_0^1 A_\mu(sy)ds, \quad 0 \leq \mu \leq 3,$$

and $I_{Y\mu} = (\xi_Y^M I)_\mu$ for $Y \in U(\mathfrak{p})$. It follows from equality (5.121b) that for $Z \in \Pi' \cap U(\mathbb{R}^4)$ and $Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$:

$$G_{ZY\mu}(y) = A_{ZY\mu}(y) - y^\nu \partial_\mu I_{ZY\nu}(y) - I_{ZY\mu}(y) - |Z| \sum_{0 \le \nu \le 3} C_\nu(Z) I_{Z_\nu P_\mu Y \nu}(y), \quad (5.121c)$$

for some positive integers $C_\nu(Z)$ and some elements $Z_\nu \in \Pi' \cap U(\mathbb{R}^4), |Z_\nu| = |Z| - 1$. Since

$$\begin{aligned}\|\delta(t)^{-1} A_{ZY}(t)\|_{L^2} &\le \|\delta(t)^{-1}\|_{L^{3/\rho}} \|A_{ZY}(t)\|_{L^{6/(3-2\rho)}} \\ &\le C_\rho (1+t)^{-1+\rho} \||\nabla|^\rho A_{ZY}(t)\|_{L^2}, \quad 0 \le \rho < 1,\end{aligned}$$

it follows from statement ii) of Lemma 4.5 that

$$\begin{aligned}&\|\delta(t)^{-1} G_{ZY}(t)\|_{L^2(\mathbb{R}^3,\mathbb{R}^4)} \quad (5.122) \\ &\le C(1+t)^{-a} \sup_{0 \le s \le t} \Big((1+s)^{b+\rho-1} \||\nabla|^\rho A_{ZY}(s)\|_{L^2} + (1+s)^b \|(A_{ZY}(s), A_{P_0 ZY}(s))\|_{M^1} \\ &\quad + (1+s)^{b-1} |Z| C_{|Z|} \sum_{\substack{X \in \Pi' \cap U(\mathbb{R}^4) \\ |X| = |Z|-1}} \|(A_{XY}(s), A_{P_0 XY}(s))\|_{M^1}\Big),\end{aligned}$$

$0 \le \rho < 1, Z \in \Pi' \cap U(\mathbb{R}^4), Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$, where $a = b$ for $b < |Z| + 1/2$, $a = |Z| + 1/2$ for $b > |Z| + 1/2$, and where the constants $C, C_{|Z|}$ depend only on ρ, a and b.

We introduce the notation

$$\Gamma_n(t) = \Big(\sum_{\substack{Y \in \Pi' \\ |Y| \le n}} \|\delta(t)^{-1} G_Y(t)\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^4)}\Big)^{1/2}, \quad n \ge 0. \quad (5.123)$$

It follows from (5.122) that

$$\begin{aligned}\Gamma_n(t) \le (1+t)^{-a} C \sup_{0 \le s \le t} &\Big((1+s)^{b+\rho-1} \wp_n^{M^\rho}((A(s),0)) \quad (5.124) \\ &+ (1+s)^b \wp_n^{M^1}\big((A(s),\dot{A}(s))\big) + (1+s)^{b-1} n C_n \wp_{n-1}^{M^1}\big((A(s),\dot{A}(s))\big)\Big),\end{aligned}$$

$n \ge 0, 0 \le \rho < 1$, where $a = b$ for $b < 1/2$, $a = 1/2$ for $b > 1/2$ and where the constants C, C_n depend only on ρ, a, b. Inequalities (5.122) and (5.124) with $b = 0$ give, according to (5.115a) and (5.115b),

$$\begin{aligned}&\|\delta(t)^{-1} G_{ZY}(t)\|_{L^2} \quad (5.125a) \\ &\le C S^{ZY}(t) + |Z| C_{|Z|} S^{1,|ZY|}(t), \quad Z \in \Pi' \cap U(\mathbb{R}^4), Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C})),\end{aligned}$$

and

$$\Gamma_n(t) \le C S^{\rho,n}(t) + n C_n S^{1,n-1}(t), \quad n \ge 0, \quad (5.125b)$$

where $0 \leq \rho < 1$ and C, C_n depends only on ρ.

It follows from the definition of G that for $Z \in \Pi' \cap U(\mathbb{R}^4), Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$:

$$\begin{aligned}\partial^\mu G_{ZY\mu}(y) &= \xi_Z \partial^\mu G_{Y\mu}(y) \\ &= \xi_Z\Big(\partial^\mu A_{Y\mu}(y) - \Box \int_0^1 y^\nu A_{Y\nu}(sy)ds\Big) \\ &= \partial^\mu A_{ZY\mu}(y) - I_1(y) - I_2(y) - I_3(y),\end{aligned} \tag{5.126}$$

where

$$I_1(y) = 2\int_0^1 s^{1+|Z|}(\partial^\mu A_{ZY\mu})(sy)ds, \tag{5.127a}$$

$$I_2(y) = \int_0^1 y^\mu s^{2+|Z|}(\Box A_{ZY\mu})(sy)ds, \tag{5.127b}$$

and

$$I_3(y) = |Z| \sum_{0 \leq \nu \leq 3} C_\nu(Z) \int_0^1 s^{1+|Z|}(\Box A_{Z_\nu Y})(sy)ds, \tag{5.127c}$$

for some constants $C_\nu(Z)$ and elements $Z_\nu \in \Pi' \cap U(\mathbb{R}^4), |Z_\nu| = |Z| - 1$. The result, given by (4.86a), (4.86b) and (4.86c) in the proof of Lemma 4.5, implies that

$$\|I_1(t)\|_{L^2} \leq C_{a,b}(1+t)^{-a_1} \sup_{0 \leq s \leq t} \big((1+s)^{b_1} \|\partial^\mu A_{ZY\mu}(s)\|_{L^2}\big), \tag{5.128a}$$

where $a_1 = b_1$ for $b_1 < 1/2 + |Z|$ and $a_1 = 1/2 + |Z|$ for $b > 1/2 + |Z|$,

$$\|\delta(t)^{-\varepsilon} I_2(t)\|_{L^2} \leq C_{a,b,\varepsilon}(1+t)^{-a_2} \sup_{0 \leq s \leq t} \big((1+s)^{b_2} \|\delta(s)^{1-\varepsilon}(\Box A_{ZY})(s)\|_{L^2}\big), \tag{5.128b}$$

where $0 \leq \varepsilon \leq 1, a_2 = b_2$ for $b_2 < 1/2+\varepsilon+|Z|$ and $a_2 = 1/2+\varepsilon+|Z|$ for $b_2 > 1/2+\varepsilon+|Z|$,

$$\|I_3(t)\|_{L^2} \leq |Z| C_{a,b,|Z|}(1+t)^{-a_3} \sum_{\substack{X \in \Pi' \cap U(\mathbb{R}^4) \\ |X| = |Z|-1}} \sup_{0 \leq s \leq t} \big((1+s)^{b_3} \|(\Box A_{XY})(s)\|_{L^2}\big), \tag{5.128c}$$

where $a_3 = b_3$ for $b_3 < 1/2+|Z|$ and $a_3 = 1/2+|Z|$ for $b_3 > 1/2+|Z|$. Let $a_1 = a-\varepsilon, a_2 = a, a_3 = a - \varepsilon$, $b_1 = b - \varepsilon, b_2 = b$ and $b_3 = b - \varepsilon$. It then follows from (5.126), (5.128a), (5.128b) and (5.128c) that, for $Z \in \Pi' \cap U(\mathbb{R}^4), Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$,

$$\begin{aligned}&\|\delta(t)^{-\varepsilon}\partial^\mu G_{ZY\mu}(t)\|_{L^2(\mathbb{R}^3,\mathbb{R})} \\ &\quad \leq C_{a,b,\varepsilon}(1+t)^{-a} \sup_{0 \leq s \leq t}\Big((1+s)^{b-\varepsilon}\|\partial^\mu A_{ZY\mu}(s)\|_{L^2} + (1+s)^b \|\delta(s)^{1-\varepsilon}\Box A_{ZY}(s)\|_{L^2} \\ &\qquad + |Z| C_{|Z|} \sum_{\substack{X \in \Pi' \cap U(\mathbb{R}^4) \\ |X| = |Z|-1}} (1+s)^{b-\varepsilon}\|\Box A_{XY}(s)\|_{L^2}\Big),\end{aligned} \tag{5.129}$$

where $a = b$ for $b < 1/2 + \varepsilon + |Z|$ and $a = 1/2 + \varepsilon + |Z|$ for $b > 1/2 + \varepsilon + |Z|$ and where $0 \le \varepsilon \le 1$. Inequality (5.129) gives that

$$\Big(\sum_{\substack{Y \in \Pi' \\ |Y| \le n}} \| \delta(t)^{-\varepsilon} \partial^\mu G_{Y\mu}(t) \|^2_{L^2(\mathbb{R}^3, \mathbb{R})} \Big)^{1/2} \tag{5.130a}$$
$$\le C_{a,b,\varepsilon}(1+t)^{-a} \sup_{0 \le s \le t} \Big((1+s)^{b-\varepsilon} \Big(\sum_{\substack{Y \in \Pi' \\ |Y| \le n}} \| \partial^\mu A_{Y\mu}(s) \|^2_{L^2} \Big)^{1/2}$$
$$+ (1+s)^b \Big(\sum_{\substack{Y \in \Pi' \\ |Y| \le n}} \| \delta(s)^{1-\varepsilon} \Box A_Y(s) \|^2_{L^2} \Big)^{1/2}$$
$$+ nC_n (1+s)^{b-\varepsilon} \Big(\sum_{\substack{Y \in \Pi' \\ |Y| \le n-1}} \| \Box A_Y(s) \|^2_{L^2} \Big)^{1/2} \Big),$$

where $a = b$ for $b < 1/2 + \varepsilon, a = 1/2 + \varepsilon$ for $b > 1/2 + \varepsilon$ and $0 \le \varepsilon \le 1$. Moreover, with $\varepsilon = 1$ and $b = 3/2 - \rho$, $0 < \rho \le 1$ in inequality (5.129), we obtain that

$$\| \delta(t)^{-1} \partial^\mu G_{Y\mu}(t) \|_{L^2} \le C(1+t)^{-3/2+\rho} S^Y(t), \tag{5.130b}$$

$Y \in \Pi' \cap U(\mathfrak{sl}(2, \mathbb{C})), 0 < \rho \le 1$, where C is a constant depending only on ρ.

In order to estimate the last sum on the right-hand side of inequality (5.120), we note that according to (5.116c)

$$\sum^{Y}_{\substack{Y_1, Y_2 \\ i \le |Y_2| \le |Y|-1}} \sum^{Y_2}_{\substack{Z_1, Z_2 \\ |Y_1| + |Z_1| \le L}} \| G_{Y_1\mu}(t) G_{Z_1\nu}(t) h_{Z_2}(t) \|_{L^2} \tag{5.131a}$$
$$\le C_{|Y|}(1+t)^{-3+2\rho} \sum_{\substack{n_1+n_2+n_3=|Y| \\ 1 \le n_1+n_2 \le L \\ i \le n_2+n_3}} [A]^{n_1+1}(t) [A]^{n_2+1}(t) \wp^D_{n_3}(h(t)),$$

$Y \in \Pi', 1/2 < \rho \le 1$. Moreover, according to (5.120), (5.125a) and (5.125b), we obtain that

$$\sum^{Y}_{\substack{Y_1, Y_2 \\ i \le |Y_2| \le |Y|-1}} \sum^{Y_2}_{\substack{Z_1, Z_2 \\ |Y_1| + |Z_1| \ge L+1}} \| G_{Y_1\mu}(t) G_{Z_1\nu}(t) h_{Z_2}(t) \|_{L^2} \tag{5.131b}$$
$$\le (1+t)^{-2+\rho} \sum^{Y}_{\substack{Y_1, Y_2 \\ i \le |Y_2| \le |Y|-1}} \Big(\sum^{Y_2}_{\substack{Z_1, Z_2 \\ |Y_1|+|Z_1| \ge L+1 \\ |Y_1| \ge |Z_1|}} \| \delta(t)^{-1} G_{Y_1}(t) \|_{L^2} [G]'^{|Z_1|}(t) H_{|Z_2|}(t)$$
$$+ \sum^{Y_2}_{\substack{Z_1, Z_2 \\ |Y_1|+|Z_1| \ge L+1 \\ |Y_1| < |Z_1|}} [G]'^{|Y_1|}(t) \| \delta(t)^{-1} G_{Z_1}(t) \|_{L^2} H_{|Z_2|}(t) \Big)$$

$$\leq (1+t)^{-2+\rho}\Big(C\chi_+(-i)\chi_+(|Y|-L-1)S^Y(t)[A]^1(t)H_0(t)$$
$$+ C_{|Y|} \sum_{\substack{n_1+n_2+n_3=|Y| \\ L+1\leq n_1+n_2 \\ n_2\leq n_1\leq |Y|-1 \\ i\leq n_2+n_3\leq |Y|-1}} S^{\rho,n_1}(t)[A]^{n_2+1}(t)H_{n_3}(t)\Big),$$

$Y \in \boldsymbol{\sigma}^i, Y \notin \boldsymbol{\sigma}^{i+1}, 0 \leq i \leq |Y|-1, 1/2 < \rho < 1$, where the constants $C, C_{|Y|}$ depend only on ρ and where $\chi_+(s) = 0$ for $s < 0$ and $\chi_+(s) = 1$ for $s \geq 0$. It follows from the definition of $[A]_n$ and from (5.131a) and (5.131b), that

$$\sum_{\substack{Y_1,Y_2 \\ |Y_2|\leq|Y|-1 \\ Y_1\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))}}^{Y} \sum_{Z_1,Z_2}^{Y_2} \|G_{Y_1\mu}(t)G_{Z_1\nu}(t)h_{Z_2}(t)\|_D \tag{5.132a}$$
$$\leq C_{|Y|}(1+t)^{-3+2\rho} \sum_{\substack{n_1+n_2=|Y| \\ 1\leq n_1\leq L}} [A]_{n_1+2}(t)\wp^D_{n_2}(h(t))$$
$$+ (1+t)^{-2+\rho}\Big(C\chi_+(-i)\chi_+(|Y|-L-1)S^Y(t)[A]^1(t)H_0(t)$$
$$+ C_{|Y|} \sum_{\substack{n_1+n_2+n_3=|Y| \\ n_1+n_2\geq L+1 \\ n_2\leq n_1\leq |Y|-1}} S^{\rho,n_1}(t)[A]^{n_2+1}(t)H_{n_3}(t)\Big),$$

$Y \in \boldsymbol{\sigma}^i, Y \notin \boldsymbol{\sigma}^{i+1}, 0 \leq i \leq |Y|-1, 1/2 < \rho < 1, L \geq 0$. The first sum on the right-hand side of inequality (5.132a) is bounded by $k^{(1)}_{|Y|}(L,t)$ and the second by $l^{(1)}_{|Y|}(L,t)$. This gives

$$\sum_{\substack{Y_1,Y_2 \\ |Y_2|\leq|Y|-1 \\ Y_1\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))}}^{Y} \sum_{Z_1,Z_2}^{Y_2} \|G_{Y_1\mu}(t)G_{Z_1\nu}(t)h_{Z_2}(t)\|_D \tag{5.132b}$$
$$\leq (1+t)^{-2+\rho}C\chi_+(-i)\chi_+(|Y|-L-1)S^Y(t)[A]^1(t)H_0(t)$$
$$+ C_{|Y|}\big((1+t)^{-3+2\rho}k^{(1)}_{|Y|}(L,t) + (1+t)^{-2+\rho}l^{(1)}_{|Y|}(L,t)\big),$$

$Y \in \boldsymbol{\sigma}^i, Y \notin \boldsymbol{\sigma}^{i+1}, 0 \leq i \leq |Y|-1, 1/2 < \rho < 1, L \geq 0$, where C and $C_{|Y|}$ are constants depending only on ρ.

To use Corollary 5.9, in order to estimate $\wp^D_{|Y_2|}((1+\lambda_1(t))^{1/2}h(t))$ in inequality (5.120), we shall first express the quantities $T^2_n(t)$, defined by (5.89c), in terms of S^Y defined by (5.115a). If $Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$, then $y^\mu G_{Y\mu} = 0$ since $y^\mu G_\mu = 0$. Therefore, if $Z \in \Pi' \cap U(\mathbb{R}^4)$, then

$$0 = \xi_Z y^\mu G_{Y\mu}(y) = y^\mu G_{ZY\mu}(y) + |Z| \sum_{0\leq\mu\leq 3} C_\mu(Z)G_{Z_\mu Y\mu}(y),$$

for some constants $C_\mu(Z)$ and elements $Z_\mu \in \Pi' \cap U(\mathbb{R}^4)$, $|Z_\mu| = |Z|-1$. Let $Q_X(y) = y^\mu G_{X\mu}(y)$, $X \in \Pi'$. Then we obtain that

$$Q_{ZY} = -|Z| \sum_{0\leq\mu\leq 3} C_\mu(Z)G_{Z_\mu Y\mu}, \tag{5.133}$$

for $Z \in \Pi' \cap U(\mathbb{R}^4), Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$. It follows from inequality (5.122) and from equality (5.133) that

$$\|\delta(t)^{-1}Q_{ZY}(t)\|_{L^2} \le C_{a,b,|Z|}|Z|(1+t)^{-a} \tag{5.134}$$
$$\sup_{0\le s\le t}\Big(\sum_{0\le\mu\le 3}\Big((1+s)^{b+\rho-1}\||\nabla|^\rho A_{Z_\mu Y}(s)\|_{L^2} + (1+s)^b\|(A_{Z_\mu Y}(s), A_{P_0 Z_\mu Y}(s))\|_{M^1}\Big)$$
$$+ (1+s)^{b-1}(|Z|-1)\sum_{\substack{X\in\Pi'\cap U(\mathbb{R}^4)\\ |X|=|Z|-2}}\|(A_{XY}(s), A_{P_0XY}(s))\|_{M^1}\Big),$$

where $Z \in \Pi' \cap U(\mathbb{R}^4), Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C})), 0 \le \rho < 1, a = b$ for $b < |Z| - 1/2$ and $a = |Z| - 1/2$ for $b > |Z| - 1/2$. Since

$$\sum_{\substack{|Y|\le n\\ Y\in\Pi'}}\|\delta(t)^{-1}Q_Y(t)\|_{L^2} \le \sum_{n_1+n_2=n}\sum_{\substack{Y\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))\\ Z\in\Pi'\cap U(\mathbb{R}^4)\\ |Y|=n_1, |Z|=n_2}}\|\delta(t)^{-1}Q_{ZY}(t)\|_{L^2},$$

it follows from (5.134), choosing $\rho = 0$ for $|Z| \ge 2$, that

$$\sum_{\substack{Y\in\Pi'\\ |Y|\le n}}\|\delta(t)^{-1}Q_Y(t)\|_{L^2} \tag{5.135}$$
$$\le nC_n(1+t)^{-a}\sup_{0\le s\le t}\Big((1+s)^{b+\rho-1}\wp^{M^\rho}_{n-1}((A(s),0)) + (1+s)^b\wp^{M^1}_{n-1}((A(s),\dot A(s)))\Big),$$

where $a = b$ for $b < 1/2, a = 1/2$ for $b > 1/2, 0 \le \rho < 1$, $n \ge 0$ and where the constant C_n depends only on a, b, ρ. Statement i) of Lemma 4.5 gives that

$$\wp^{M^1}_n\big((G(t),\dot G(t))\big) \tag{5.136}$$
$$\le C(1+t)^{-a}\sup_{0\le s\le t}\Big((1+s)^b\big(\wp^{M^1}_{n+1}((A(s),\dot A(s)))$$
$$+ \wp^{M^1}_n((B(s),\dot B(s))) + C_n\wp^{M^1}_n((A(s),\dot A(s)))\big)\Big),$$

where $a = b$ for $b < 1/2, a = 1/2$ for $b > 1/2, B_\mu = y_\mu\partial^\nu A_\nu$ and where the constants C, C_n depend only on a and b. It follows from inequality (5.130a) with $\varepsilon = 1/2, b = 1-\rho, 0 < \rho \le 1$, that

$$\Big(\sum_{\substack{Y\in\Pi'\\ |Y|\le n}}\|\delta(t)^{-1/2}\partial^\mu G_{Y\mu}(t)\|^2_{L^2}\Big)^{1/2} \tag{5.137}$$
$$\le (1+t)^{-1+\rho}C\big(S^{\rho,n}(t) + nC_nS^{\rho,n-1}(t)\big), \quad 0 < \rho \le 1,$$

where the constants depend only on ρ. It follows from (5.88b), (5.115a), (5.115b), (5.135), (5.136) and (5.137), that

$$T^{2(n)}(t) \le CS^{\rho,n+1}(t) + C_nS^{\rho,n}(t), \quad 0 < \rho < 1, \tag{5.138}$$

where the constants C, C_n depend only on ρ.

Corollary 5.9, (5.116c), (5.125b), (5.138), give that

$$\begin{aligned}
&\wp^D_n\big((1+\lambda_i(t))^{1/2}h(t)\big) \qquad\qquad (5.139a)\\
&\le C'_1\big(\wp^D_{n+1}(h(t))+R^i_{n,1}(t)\big)+C'_{n+1}\sum_{\substack{n_1+n_2=n+1\\1\le n_1\le L_0}}(1+[A]^{n_1+1}(t))\big(\wp^D_{n_2}(h(t))+R^i_{n_2-1,1}(t)\big)\\
&\quad+C''_1\chi_+(n-L_0-1)\big(S^{\rho,n}(t)+nC_{n-1}S^{\rho,n-1}(t)\big)\\
&\qquad\qquad\big(1+S^{\rho,10}(t)+C_9S^{\rho,9}(t)\big)\big(R'_8(t)+R^2_{10}(t)+R^\infty_1(t)+\wp^D_9(h(t))\big)\\
&\quad+C''_{n+1}\sum_{\substack{n_1+n_2+n_3+n_4=n+1\\n_1\le L_0,n_2\le n-1\\n_3+n_4\le n-L_0-1}}(1+[A]^{n_1+1}(t))S^{\rho,n_2}(t)(1+S^\rho_{10,n_3}(t))\\
&\qquad\qquad\qquad\qquad\big(R'_{n_4+7}(t)+R^2_{n_4+9}(t)+R^\infty_{n_4}(t)+\wp^D_{n_4+8}(t)\big),
\end{aligned}$$

where $n\ge 0, L_0\ge 3, 1/2<\rho<1$, $i=0,1$, where the constants C_l depend only on ρ, C'_l depends only on ρ and $[A]^1(t)$, and C''_l depend only on ρ and $[A]^9(t)$. The last inequality shows that

$$\begin{aligned}
&\wp^D_n\big((1+\lambda_0(t))^{1/2}h(t)\big) \qquad\qquad (5.139b)\\
&\quad\le C'_1\big(\wp^D_{n+1}(h(t))+R^0_{n,1}(t)\big)+C'_{n+1}\big(\wp^D_n(h(t))+R^0_{n-1,1}(t)\big)\\
&\qquad+C'_{n+1}k^{(2)}_{n+1}(L_0,t)+C''_{n+1}l^{(2)}_{n+1}(L_0,t),\quad n\ge 0, L_0\ge 3, 1/2<\rho<1,
\end{aligned}$$

where the constants C'_1, C'_{n+1} depend only on ρ and $[A]^1(t)$, the constant C''_{n+1} depends only on ρ and $[A]^9(t)$, and where $k^{(2)}_{n+1}$, respectively $l^{(2)}_{n+1}$, are given by (5.119c) and (5.119d).

Let $Y\in\boldsymbol{\sigma}^i, Y\notin\boldsymbol{\sigma}^{i+1}, 0\le i\le |Y|-1$. It then follows from inequalities (5.116c), (5.120), (5.125a), (5.125b), (5.130b), from the fact that the domain of summation for the sums on the left-hand side of inequality (5.120) is a subset of $\{(Y_1,Y_2)\in\Pi'\times\Pi'\big|Y_1\in U(\mathfrak{sl}(2,\mathbb{C})), Y_2\in\boldsymbol{\sigma}^i, Y_2\notin\boldsymbol{\sigma}^{i+1}\}$ and from the definition of $a(Y_1,Y_2)$ in (5.118),

$$\begin{aligned}
&J_Y(t)\\
&\le C_{|Y|}\sum^Y_{\substack{Y_1,Y_2\\1\le|Y_1|\le L\\Y_1\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))}}\Big((1+t)^{-2+\rho}\big([A]^{|Y_1|+1}(t)+[A]^{|Y_1|+2}(t)+[A]^1(t)[A]^{|Y_1|+1}(t)\big)\\
&\qquad\big(\wp^D_{|Y_2|+1,i}(h(t))^{1-a(Y_1,Y_2)}\wp^D_{|Y_2|+1,i+1}(h(t))^{a(Y_1,Y_2)}\\
&\qquad+\wp^D_{|Y_2|,i}((1+\lambda_0(t))^{1/2}h(t))^{2(1-\rho)}\wp^D_{|Y_2|,i}(h(t))^{2\rho-1}+\wp^D_{|Y_2|,i}(h(t))\big)\\
&\qquad+(1+t)^{-3/2+\rho}[A]^{|Y_1|+1}(t)\wp^D_{|Y_2|,i}(g(t))\Big)\\
&\quad+C\sum^Y_{\substack{Y_1,Y_2\\1+L\le|Y_1|\\Y_1\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))}}(1+t)^{-2+\rho}S^{Y_1}(t)\big(H_{|Y_2|+1}(t)+H_{|Y_2|}(t)+[A]^1(t)H_{|Y_2|}(t)\big)\\
&\quad+2m^{-1}\sum^Y_{\substack{Y_1,Y_2\\|Y_2|\le|Y|-1\\Y_1\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))}}\|\gamma^\mu G_{Y_1\mu}(t)\xi^D_{Y_2}(\gamma^\nu G_\nu h+g)(t)\|_D,\quad 1/2<\rho<1,
\end{aligned}$$

where the constants $C, C_{|Y|}$ depend only on ρ. This inequality gives, since

$$\wp^D_{|Y_2|,i}(h(t)) \leq \wp^D_{|Y_2|,i}((1+\lambda_0(t))^{1/2}h(t))^{2(1-\rho)}\wp^D_{|Y_2|,i}(h(t))^{2\rho-1},$$

that

$$\begin{aligned} J_Y(t) \leq C_{|Y|} & \sum^{Y}_{\substack{Y_1,Y_2 \\ 1\leq |Y_1|\leq L \\ Y_1\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))}} \Big((1+t)^{-2+\rho}[A]_{|Y_1|+2}(t) \\ & \big(\wp^D_{|Y_2|+1,i}(h(t))^{1-a(Y_1,Y_2)}\wp^D_{|Y_2|+1,i+1}(h(t))^{a(Y_1,Y_2)} \\ & + \wp^D_{|Y_2|,i}((1+\lambda_0(t))^{1/2}h(t))^{2(1-\rho)}\wp^D_{|Y_2|,i}(h(t))^{2\rho-1}\big) \\ & + (1+t)^{-3/2+\rho}[A]^{|Y_1|+1}(t)\wp^D_{|Y_2|,i}(g(t))\Big) \\ + C & \sum^{Y}_{\substack{Y_1,Y_2 \\ 1+L\leq |Y_1| \\ Y_1\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))}} (1+t)^{-2+\rho}S^{Y_1}(t)(1+[A]^1(t))H_{|Y_2|+1}(t) \\ + 2m^{-1} & \sum^{Y}_{\substack{Y_1,Y_2 \\ |Y_2|\leq |Y|-1 \\ Y_1\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))}} \|\gamma^\mu G_{Y_1\mu}(t)\xi^D_{Y_2}(\gamma^\nu G_\nu h+g)(t)\|_D, \quad 1/2<\rho<1, \end{aligned} \tag{5.140}$$

where C and $C_{|Y|}$ are constants depending only on ρ.

According to the definition of $a(Y_1,Y_2)$ in (5.118) and according to definitions (5.119a) and (5.119b) of $k^{(1)}_n$ and $l^{(1)}_n$, we obtain from (5.132b) and (5.140) that

$$\begin{aligned} J_Y(t) \leq C_{|Y|}(1+t)^{-2+\rho}\Big(& [A]_3(t)\big(\wp^D_{|Y|,i}(h(t))^{1/2}\wp^D_{|Y|,i+1}(h(t))^{1/2} \\ & + \wp^D_{|Y|-1,i}((1+\lambda_0(t))^{1/2}h(t))^{2(1-\rho)}\wp^D_{|Y|-1,i}(h(t))^{2\rho-1}\big) \\ & + \sum_{\substack{n_1+n_2=|Y| \\ 2\leq n_1\leq L, i\leq n_2}} [A]_{n_1+2}(t)\wp^D_{n_2,i}((1+\lambda_0(t))^{1/2}h(t)) + k^{(1)}_{|Y|}(L,t) + l^{(1)}_{|Y|}(L,t)\Big) \\ & + C(1+t)^{-2+\rho}\chi_+(|Y|-L-1)\chi_+(-i) \\ & \quad \big(S^Y(t)(1+[A]^1(t))H_1(t) + (1+t)^{2-\rho}\|\gamma^\mu G_{Y\mu}(t)g_{\mathbb{I}}(t)\|_D\big), \end{aligned} \tag{5.141}$$

where $Y\in\boldsymbol{\sigma}^i, Y\notin\boldsymbol{\sigma}^{i+1}, 0\leq i\leq |Y|-1, 1/2<\rho<1, L\geq 1$, and where C and $C_{|Y|}$ are constants depending only on ρ.

It follows from inequality (5.139a) that

$$\begin{aligned} & \wp^D_n\big((1+\lambda_i(t))^{1/2}h(t)\big) \\ & \leq C'_{n+1}\sum_{\substack{n_1+n_2=n+1 \\ 0\leq n_1\leq L_0}} (1+[A]^{n_1+1}(t))\big(\wp^D_{n_2}(h(t)) + R^i_{n_2-1,1}(t)\big) \\ & \quad + C''_{n+1}\sum_{\substack{n_1+n_2+n_3+n_4=n+1 \\ n_1\leq L_0, n_2\leq n \\ n_3+n_4\leq n-L_0-1}} (1+[A]^{n_1+1}(t))S^{\rho,n_2}(t) \\ & \qquad (1+S^\rho_{10,n_3}(t))\big(R'_{n_4+7}(t) + R^2_{n_4+9}(t) + R^\infty_{n_4}(t) + \wp^D_{n_4+8}(t)\big), \end{aligned} \tag{5.142}$$

where $n \geq 0, L_0 \geq 3, 1/2 < \rho < 1$, $i = 0, 1$, and where the constant C'_{n+1} depends only on ρ and $[A]^1(t)$ and C''_{n+1} depends only on ρ and $[A]^9(t)$. It follows from (5.142) that

$$\begin{aligned}
&\sum_{\substack{n_1+n_2=|Y| \\ 2\leq n_1\leq L}} [A]_{n_1+2}(t)\wp^D_{n_2,i}\big((1+\lambda_0(t))^{1/2}h(t)\big) \\
&\quad\leq C'_{|Y|} \sum_{\substack{n_1+n_2=|Y| \\ 2\leq n_1\leq L}} \sum_{\substack{n_3+n_4=n_2+1 \\ 0\leq n_3\leq L_0(n_1)}} [A]_{n_1+2}(t)(1+[A]^{n_3+1}(t))\big(\wp^D_{n_4}(h(t))+R^0_{n_4,1}(t)\big) \\
&\qquad + C''_{|Y|} \sum_{\substack{n_1+n_2=|Y| \\ 2\leq n_1\leq L}} \sum_{\substack{n_3+n_4+n_5+n_6=n_2+1 \\ n_3\leq L_0(n_1) \\ n_4\leq n_2+1, n_5+n_6\leq n_2+1-L_0(n_1)-1}} [A]_{n_1+2}(t)(1+[A]^{n_3+1}(t))S^{\rho,n_4}(t) \\
&\qquad\qquad (1+S^\rho_{10,n_5}(t))\big(R'_{n_6+7}(t)+R^2_{n_6+9}(t)+R^\infty_{n_6}(t)+\wp^D_{n_6+8}(t)\big),
\end{aligned}$$

where we have chosen $L_0(n_1) = L-n_1$ for $n_1 \leq L-3$ and $L_0 = 3$ for $n_1 \geq L-2$ and where $1/2 < \rho < 1$, the constant $C'_{|Y|}$ depends on $[A]^1(t)$ and the constant $C''_{|Y|}$ on $[A]^9(t)$. It follows from definition (5.119c) of $k^{(2)}_n$ and the convexity properties (analoguous to those of (5.89d) and (5.89c)) for $[A]_{N,n}$, that

$$\sum_{\substack{n_1+n_2=|Y| \\ 2\leq n_1\leq L}} [A]_{n_1+2}(t)\wp^D_{n_2,i}\big((1+\lambda_0(t))^{1/2}h(t)\big) \leq C'_{|Y|}k^{(2)}_{|Y|}(L,t)+C''_{|Y|}l^{(2)}_{|Y|}(L,t), \tag{5.143}$$

where $1/2 < \rho < 1, Y \in \Pi', L \geq 0$ and where $C'_{|Y|}$ is a constant depending only on $\rho, L, [A]^3(t)$, and $C''_{|Y|}$ is a constant depending only on $\rho, L, [A]^9(t)$.

Using that $(a+b)^e c^{1-e} \leq (a^e+b^e)c^{1-e} \leq a^e c^{1-e}+eb+(1-e)c$ for $a, b, c \geq 0$, $0 \leq e \leq 1$, we obtain from (5.139b) that

$$\begin{aligned}
&\wp^D_{|Y|-1}\big((1+\lambda_0(t))^{1/2}h(t)\big)^{2(1-\rho)}\wp^D_{|Y|-1,i}(h(t))^{2\rho-1} \qquad (5.144)\\
&\quad\leq \big(C'_1(\wp^D_{|Y|}(h(t))+R^0_{|Y|-1,1}(t))\big)^{2(1-\rho)}\wp^D_{|Y|-1,i}(h(t))^{2\rho-1} \\
&\qquad + C'_{|Y|}k^{(2)}_{|Y|}(L,t)+C''_{|Y|}l^{(2)}_{|Y|}(L,t)+C'_{|Y|}\wp^D_{|Y|-1}(h(t))+C'_{|Y|}R^0_{|Y|-2,1}(t),
\end{aligned}$$

where $1/2 < \rho < 1$ and where the constants $C'_1, C'_{|Y|}$ depend only on ρ and $[A]^1(t)$, and the constant $C''_{|Y|}$ depends only on ρ and $[A]^9(t)$.

It follows from inequalities (5.141), (5.143) and (5.144), that

$$\begin{aligned}
J_Y(t) &\leq C(1+t)^{-2+\rho}\chi_+(-i)\chi_+(|Y|-L-1) \qquad (5.145)\\
&\qquad \big(S^Y(t)(1+[A]^1(t))H_1(t)+(1+t)^{2-\rho}\|\gamma^\mu G_{Y\mu}(t)g_{\mathbb{I}}(t)\|_D\big) \\
&\qquad + C^{(1)}_{|Y|}(1+t)^{-2+\rho}\Big([A]_3(t)\wp^D_{|Y|}(h(t))^\varepsilon\wp^D_{|Y|,i}(h(t))^{1-\varepsilon} \\
&\qquad + [A]_3(t)R^0_{|Y|-1,1}(t)^{2(1-\rho)}\wp^D_{|Y|-1,i}(h(t))^{2\rho-1} \\
&\qquad + k^{(1)}_{|Y|}(L,t)+l^{(1)}_{|Y|}(L,t)+C^{(3)}_{|Y|}k^{(2)}_{|Y|}(L,t)+C^{(9)}_{|Y|}l^{(2)}_{|Y|}(L,t)\Big),
\end{aligned}$$

where $Y \in \boldsymbol{\sigma}^i, Y \notin \boldsymbol{\sigma}^{i+1}, 0 \le i \le |Y|-1, 1/2 < \rho < 1, L \ge 1$ and where $C_m^{(n)}$ are constants depending only on $[A]^n(t)$ and ρ. This proves the lemma.

To obtain L^2-estimates of solutions h of equation (5.1), we next estimate the terms on the right-hand side of the inequalities in Proposition 5.10, which have not already been considered in Lemma 5.11.

Lemma 5.12. *Let $1/2 < \rho < 1$, $h_Z \in C^0(\mathbb{R}^+, D)$ and $(A_Z, A_{P_0 Z}) \in C^0(\mathbb{R}^+, M^\rho) \cap C^0(\mathbb{R}^+, M^1)$ for $Z \in \Pi'$ and $|Z|$ sufficiently large. Let G be given by (5.114), $L \ge 1$, $g = (i\gamma^\mu \partial_\mu + m - \gamma^\mu G_\mu)h$, $\chi_+(s) = 0$ for $s < 0$, $\chi_+(s) = 1$ for $s \ge 0$, let $Y \in \boldsymbol{\sigma}^i$, $Y \notin \boldsymbol{\sigma}^{i+1}$, $i \ge 0$, and let $J_n^{(l)}$ be defined as in Lemma 5.11.*

a) *If $i = 0$, then*

$$
\begin{aligned}
\|h_Y(t)\|_D \le{}& \|h_Y(t_0)\|_D + (1+t_0)^{-3/2+\rho} C_{|Y|} \sum_{\substack{n_1+n_2=|Y| \\ 1 \le n_1 \le L}} [A]^{n_1+1}(t_0) \wp^D_{n_2}(h(t_0)) \\
&+ (1+t_0)^{-1/2} C_{|Y|} \sum_{\substack{n_1+n_2=|Y| \\ n_1 \ge L+1}} S^{\rho,n_1}(t_0) H_{n_2}(t_0) + \|f_Y(t)\|_D \\
&+ (1+t)^{-3/2+\rho} C_{|Y|} \sum_{\substack{n_1+n_2=|Y| \\ 1 \le n_1 \le L}} [A]^{n_1+1}(t) \wp^D_{n_2}(h(t)) \\
&+ (1+t)^{-1/2} C_0 \chi_+(|Y|-L-1) S^Y(t) H_0(t) \\
&+ (1+t)^{-1/2} C_{|Y|} \sum_{\substack{n_1+n_2=|Y| \\ L+1 \le n_1 \le |Y|-1}} S^{\rho,n_1}(t) H_{n_2}(t) \\
&+ \int_{\min(t,t_0)}^{\max(t,t_0)} (1+s)^{-2+\rho} \Big(C'_{|Y|} J^{(0)}_{|Y|}(L,s) + \chi_+(|Y|-L-1) \\
&\qquad C_0 \big(S^Y(s)(1+[A]^1(s)) H_1(s) + (1+s)^{2-\rho} \|\gamma^\mu G_{Y\mu}(s) g(s)\|_D \big) \Big) ds.
\end{aligned}
$$

b) *If $1 \le i \le |Y|$, then $Y = P_\nu Z$ for some $Z \in \Pi'$, and*

$$
\begin{aligned}
\|h_Y(t)\|_D \le{}& \|h_Y(t_0)\|_D \\
&+ (1+t_0)^{-3/2+\rho} C_{|Y|} \sum_{\substack{n_1+n_2=|Y|-1 \\ 0 \le n_1 \le L}} [A]^{2+n_1}(t_0) \wp^D_{n_2}(h(t_0)) \\
&+ (1+t_0)^{-1/2} C_{|Y|} \sum_{\substack{n_1+n_2=|Y|-1 \\ n_1 \ge L+1}} S^{\rho,n_1}(t_0) H_{n_2+1}(t_0) \\
&+ (1+t)^{-3/2+\rho} C_{|Y|} \sum_{\substack{n_1+n_2=|Y|-1 \\ 0 \le n_1 \le L}} [A]^{2+n_1}(t) \wp^D_{n_2}(h(t)) \\
&+ (1+t)^{-1/2} C_{|Y|} \sum_{\substack{n_1+n_2=|Y|-1 \\ n_1 \ge L+1}} S^{\rho,n_1}(t) H_{n_2+1}(t)
\end{aligned}
$$

$$+\|f_Y(t)\|_D + \int_{\min(t,t_0)}^{\max(t,t_0)} (1+s)^{-2+\rho} C'_{|Y|} J^{(i)}_{|Y|}(L,s)ds.$$

The constants $C_n, n \geq 0$, depend only on ρ and the constants C'_n depend only on ρ and $[A]^3(\max(t_0,t))$.

Proof. To prove statement a), let $h'_Y(t)$ be as in statement i) of Proposition 5.10. It follows from inequalities (5.116c), (5.125a) and (5.125b), that

$$\begin{aligned}\|h_Y(t) - h'_Y(t)\|_D \leq{}& (1+t)^{-3/2+\rho} C_{|Y|} \sum_{\substack{n_1+n_2=|Y| \\ 1\leq n_1 \leq L}} [A]^{n_1+1}(t) \wp^D_{n_2}(h(t)) \\ &+ (1+t)^{-1/2} C \chi_+(|Y|-L-1) S^Y(t) H_0(t) \\ &+ (1+t)^{-1/2} C_{|Y|} \sum_{\substack{n_1+n_2=|Y| \\ L+1\leq n_1 \leq |Y|-1}} S^{\rho,n_1}(t) H_{n_2}(t),\end{aligned} \tag{5.146}$$

where the constants C and $C_{|Y|}$ depend only on ρ. Since $Y \in \boldsymbol{\sigma}^0$ and $Y \notin \boldsymbol{\sigma}^1$, the sum over $Y_1 \in \boldsymbol{\sigma}^1$ on the right-hand side in the inequality of statement i) of Proposition 5.10, vanish. This inequality then gives, according to definition (5.118) of J_Y, that

$$\|h'_Y(t)\|_D \leq \|h'_Y(t_0)\|_D + \|f_Y(t)\|_D + \int_{\min(t,t_0)}^{\max(t,t_0)} J_Y(s)ds.$$

This inequality, the use of inequality (5.146) to estimate $\|h'_Y(t)\|_D$ and $\|h'_Y(t_0)\|_D$ and the estimate of $J_Y(s)$ in Lemma 5.11, give the estimate of statement a).

To prove statement b), we observe that it follows from the definition of $\boldsymbol{\sigma}^i$ that $Y = P_\nu Z$ for some $Z \in \boldsymbol{\sigma}^{i-1}$ and $Z \notin \boldsymbol{\sigma}^i$. Let $h^{(\nu)}_Z, g^{(\nu)}_{1Z}$ and $g^{(\nu)}_{2Z}$ be as in statement ii) of Proposition 5.10. It follows that

$$\begin{aligned}&\|h_Y(t) - h^{(\nu)}_Z(t) + (2m)^{-1} g^{(\nu)}_Z(t)\|_D \\ &\leq \sum^Z_{Z_1,Z_2} \|G_{Z_1\nu}(t) h_{Z_2}(t)\|_D + (2m)^{-1} \sum^Z_{\substack{Z_1,Z_2 \\ |Z_2|\leq|Z|-1 \\ Z_1\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))}} \|\gamma^\mu G_{Z_1\mu}(t) h_{P_\nu Z_2}(t)\|_D.\end{aligned} \tag{5.147a}$$

This inequality, inequalities (5.116c), (5.125b) and the fact that $S^{1,n}(t) \leq S^{\rho,n}(t)$ for $\rho \leq 1$, give that

$$\begin{aligned}&\|h_Y(t) - h^{(\nu)}_Z(t) + (2m)^{-1} g^{(\nu)}_Z(t)\|_D \\ &\leq C_{|Y|}(1+t)^{-3/2+\rho} \sum_{\substack{n_1+n_2=|Y|-1 \\ 0\leq n_1\leq L}} [A]^{2+n_1}(t) \wp^D_{n_2}(h(t)) \\ &\quad + C_{|Y|}(1+t)^{-1/2} \sum_{\substack{n_1+n_2=|Y|-1 \\ n_1\geq L+1}} S^{\rho,n_1}(t) H_{n_2+1}(t),\end{aligned} \tag{5.147b}$$

where $C_{|Y|}$ depends only on ρ.

Using the gauge invariance of the electromagnetic field, and

$$(i\gamma^\mu \partial_\mu + m - \gamma^\mu G_\mu) h_{Z_2} = g_{Z_2} + \sum_{\substack{Z_3, Z_4 \\ |Z_4| \leq |Z_2| - 1}}^{Z_2} \gamma^\mu G_{Z_3 \mu} h_{Z_4},$$

we obtain from the definition of $g^{(\nu)}_{1Z}$ that

$$\begin{aligned} \|g^{(\nu)}_{1Z}(t)\|_D \leq & \sum_{Z_1, Z_2}^{Z} \|\gamma^\mu(\partial_\mu A_{Z_1 \nu}(t) - \partial_\nu A_{Z_1 \mu}(t)) h_{Z_2}(t)\|_D \\ & + \sum_{\substack{Z_1, Z_2 \\ |Z_2| \leq |Z| - 1 \\ Z_1 \in \boldsymbol{\sigma}^1}}^{Z} \|\gamma^\mu G_{Z_1 \mu}(t) h_{P_\nu Z_2}(t)\|_D \\ & + C \sum_{\substack{Z_1, Z_2, Z_3 \\ |Z_3| \leq |Z| - 1}}^{Z} \sum_{\alpha, \beta} \|G_{Z_1 \alpha}(t) G_{Z_2 \beta}(t) h_{Z_3}(t)\|_D + \sum_{Z_1, Z_2}^{Z} \|G_{Z_1 \nu}(t) g_{Z_2}(t)\|. \end{aligned} \tag{5.148}$$

Let $X \in \Pi'$. Then according to statement i) of Lemma 4.5, with $\rho = 1$,

$$\begin{aligned} \|G_{P_\mu X}(t)\|_{L^2} & \leq \|(G_X(t), G_{P_0 X}(t))\|_{M^1} \\ & \leq C \sup_{0 \leq s \leq t} \Big(\|(A_X(s), A_{P_0 X}(s))\|_{M^1} + \|(B_X(s), B_{P_0 X}(s))\|_{M^1} \\ & \quad + \wp^{M^1}_{|X|+1}((A(s), \dot{A}(s))) + C_{|X|} \wp^{M^1}_{|X|}((A(s), \dot{A}(s))) \Big), \end{aligned}$$

where $B_\mu(y) = y_\mu \partial^\nu A_\nu(y)$ and $\dot{A}_X = A_{P_0 X}$ for $X \in \Pi'$. This gives that

$$\|G_X(t)\|_{L^2} \leq C_0 S^{1,|X|}(t) + C_{|X|} S^{1,|X|-1}(t), \quad \text{for } X \in \boldsymbol{\sigma}^1, \tag{5.149}$$

for two numerical constants C_0 and $C_{|X|}$.

For the first term on the right-hand side of inequality (5.148), we obtain that

$$\begin{aligned} & \sum_{Z_1, Z_2}^{Z} \|\gamma^\mu(\partial_\mu A_{Z_1 \nu}(t) - \partial_\nu A_{Z_1 \mu}(t)) h_{Z_2}(t)\|_D \\ & \leq (1+t)^{-2+\rho} C_{|Z|} \sum_{\substack{n_1 + n_2 = |Z| \\ 0 \leq n_1 \leq L}} [A]^{n_1+1}(t) \wp^D_{n_2}((1 + \lambda_0(t))^{1/2} h(t))^{2(1-\rho)} \wp^D_{n_2}(h(t))^{2\rho - 1} \\ & \quad + (1+t)^{-3/2} C_{|Z|} \sum_{\substack{n_1 + n_2 = |Z| \\ n_1 \geq L+1}} \wp^{M^1}_{n_1}((A(t), \dot{A}(t))) H_{n_2}(t) \\ & \leq (1+t)^{-2+\rho} C_{|Z|} \sum_{\substack{n_1 + n_2 = |Z| \\ 0 \leq n_1 \leq L}} [A]^{n_1+1}(t) \wp^D_{n_2}((1 + \lambda_0(t))^{1/2} h(t))^{2(1-\rho)} \wp^D_{n_2}(h(t))^{2\rho - 1} \\ & \quad + (1+t)^{-3/2} C_{|Z|} l^{(1)}_{|Z|+1}(L+1, t), \quad L \geq 0, Z \in \Pi'. \end{aligned} \tag{5.150}$$

For the second term on the right-hand side of inequality (5.148), we obtain, using (5.149),

$$\sum^{Z}_{\substack{Z_1,Z_2\\|Z_2|\leq|Z|-1\\Z_1\in\boldsymbol{\sigma}^1}} \|\gamma^\mu G_{Z_1\mu}(t)h_{P_\nu Z_2}(t)\|_D \tag{5.151}$$
$$\leq (1+t)^{-2+\rho}C_{|Z|}\chi_+(i-2)\sum_{\substack{n_1+n_2=|Z|\\1\leq n_1\leq L}}[A]^{n_1+2}(t)$$
$$\wp^D_{n_2+1}((1+\lambda_0(t))^{1/2}h(t))^{2(1-\rho)}\wp^D_{n_2+1}(h(t))^{2\rho-1}$$
$$+(1+t)^{-3/2}C_{|Z|}\chi_+(i-2)\sum_{\substack{n_1+n_2=|Z|\\n_1\geq L+1}}S^{1,n_1}(t)H_{n_2+1}(t)$$
$$\leq (1+t)^{-2+\rho}C_{|Z|}\chi_+(i-2)\sum_{\substack{n_1+n_2=|Z|\\1\leq n_1\leq L}}[A]^{n_1+2}(t)$$
$$\wp^D_{n_2+1}((1+\lambda_0(t))^{1/2}h(t))^{2(1-\rho)}\wp^D_{n_2+1}(h(t))^{2\rho-1}$$
$$+(1+t)^{-3/2}C_{|Z|}\chi_+(i-2)l^{(1)}_{|Z|+1}(L+1,t),$$

where $Z\in\boldsymbol{\sigma}^{i-1}$ and $Z\notin\boldsymbol{\sigma}^i, 1\leq i\leq |Z|+1$.

Proceeding like in (5.131a) and (5.131b), we obtain

$$\sum^{Z}_{\substack{Z_1,Z_2,Z_3\\|Z_3|\leq|Z|-1}} \|G_{Z_1\alpha}(t)G_{Z_2\beta}(t)h_{Z_3}(t)\|_D \tag{5.152}$$
$$\leq (1+t)^{-3+2\rho}C_{|Z|}\sum_{\substack{n_1+n_2=|Z|\\1\leq n_1\leq L}}[A]_{n_1+2}(t)\wp^D_{n_2}(h(t))$$
$$+(1+t)^{-2+\rho}C_{|Z|}\sum_{\substack{n_1+n_2+n_3=|Z|\\n_2\leq n_1\\n_3\leq|Z|-L-1}}S^{\rho,n_1}(t)[A]^{n_2+1}(t)H_{n_3}(t)$$
$$\leq (1+t)^{-3+2\rho}C_{|Z|}k^{(1)}_{|Z|+1}(L,t)+(1+t)^{-2+\rho}C_{|Z|}l^{(1)}_{|Z|+1}(L,t),$$

for $Z\in\Pi'$ and $L\geq 0$. It follows from definitions (5.119a) and (5.119b), that

$$\sum^{Z}_{Z_1,Z_2}\|G_{Z_1\nu}(t)g_{Z_2}(t)\|_D \tag{5.153}$$
$$\leq (1+t)^{-3/2+\rho}C_{|Z|}\sum_{\substack{n_1+n_2=|Z|\\0\leq n_1\leq L}}[A]^{n_1+1}(t)\wp^D_{n_2}(g(t))$$
$$+C_{|Z|}\sum_{\substack{|Z_1|+|Z_2|=|Z|\\|Z_1|\geq L+1\\Z_1,Z_2\in\Pi'}}\|\gamma^\mu G_{Z_1\mu}(t)g_{Z_2}(t)\|_D$$
$$\leq (1+t)^{-2+\rho}C_{|Z|}\big(k^{(1)}_{|Z|+1}(L+1,t)+l^{(1)}_{|Z|+1}(L+1,t)\big),\quad L\geq 0, Z\in\Pi'.$$

Inequality (5.148), inequalities (5.150), (5.151) and (5.153), with L instead of $L+1$, and inequality (5.152), give that

$$\begin{aligned}
&\|g^{(\nu)}_{1Z}(t)\|_D \qquad (5.154)\\
&\le (1+t)^{-2+\rho}C_{|Y|}[A]^3(t)\wp^D_{|Y|-1}((1+\lambda_0(t))^{1/2}h(t))^{2(1-\rho)}\wp^D_{|Y|-1}(h(t))^{2\rho-1}\\
&\quad + (1+t)^{-2+\rho}C_{|Y|}\sum_{\substack{n_1+n_2=|Y|-1\\ 1\le n_1\le L-1}}[A]^{n_1+1}(t)\wp^D_{n_2}((1+\lambda_0(t))^{1/2}h(t))\\
&\quad + (1+t)^{-2+\rho}C_{|Y|}\sum_{\substack{n_1+n_2=|Y|-1\\ 2\le n_1\le L-1}}[A]^{n_1+2}(t)\wp^D_{n_2+1}((1+\lambda_0(t))^{1/2}h(t))\\
&\quad + (1+t)^{-2+\rho}C_{|Y|}(k^{(1)}_{|Y|}(L,t)+l^{(1)}_{|Y|}(L,t)),
\end{aligned}$$

where $Z\in\Pi', 1/2<\rho<1$ and where $C_{|Y|}$ is a constant depending only on ρ. Majorizing the second and the third term on the right-hand side of this inequality with the help of (5.143), we obtain that

$$\begin{aligned}
&\|g^{(\nu)}_{1Z}(t)\|_D \qquad (5.155)\\
&\le (1+t)^{-2+\rho}C_{|Y|}[A]^3(t)\wp^D_{|Y|-1}((1+\lambda_0(t))^{1/2}h(t))^{2(1-\rho)}\wp^D_{|Y|-1}(h(t))^{2\rho-1}\\
&\quad + (1+t)^{-2+\rho}C_{|Y|}\big(k^{(1)}_{|Y|}(L,t)+l^{(1)}_{|Y|}(L,t)\big)\\
&\quad + (1+t)^{-2+\rho}\big(C'_{|Y|}k^{(2)}_{|Y|}(L,t)+C''_{|Y|}l^{(2)}_{|Y|}(L,t)\big),
\end{aligned}$$

where $Z\in\Pi', L\ge 1, 1/2<\rho<1$ and where the constant $C_{|Y|}$ depends only on ρ, the constant $C'_{|Y|}$ only on ρ and $[A]^3(t)$ and the constant $C''_{|Y|}$ only on ρ and $[A]^9(t)$. Inequalities (5.144) and (5.155) and the definition of $J^{(l)}_n(L,t)$ in Lemma 5.11, give that

$$\begin{aligned}
&\|g^{(\nu)}_{1Z}(t)\|_D \qquad (5.156)\\
&\le (1+t)^{-2+\rho}C'_{|Y|}[A]^3(t)\big(\wp^D_{|Y|}(h(t))+R^0_{|Y|-1,1}(t)\big)^{2(1-\rho)}\wp_{|Y|-1}(h(t))^{2\rho-1}\\
&\quad + (1+t)^{-2+\rho}C_{|Y|}\big(k^{(1)}_{|Y|}(L,t)+l^{(1)}_{|Y|}(L,t)\big)\\
&\quad + (1+t)^{-2+\rho}\big(C'_{|Y|}k^{(2)}_{|Y|}(L,t)+C''_{|Y|}l^{(2)}_{|Y|}(L,t)\big)\\
&\le (1+t)^{-2+\rho}C'_{|Y|}J^{(i)}_{|Y|}(t),\quad i\ge 1,
\end{aligned}$$

where $Z\in\Pi', Y=P_\nu Z, L\ge 1, 1/2<\rho<1$ and where the constant $C_{|Y|}$ depends only on ρ, $C'_{|Y|}$ only of ρ and $[A]^3(t)$ and the constant $C''_{|Y|}$ only on ρ and $[A]^9(t)$.

Since, according to statement ii) of Proposition 5.10,

$$\begin{aligned}
\|h_Y(t)\|_D &\le \|h_Y(t)-h^{(\nu)}_Z(t)+(2m)^{-1}g^{(\nu)}_{2Z}(t)\|_D+\|h^{(\nu)}_Z(t)-(2m)^{-1}g^{(\nu)}_{2Z}(t)\|_D\\
&\le \|h_Y(t)-h^{(\nu)}_Z(t)+(2m)^{-1}g^{(\nu)}_{2Z}(t)\|_D+\|h^{(\nu)}_Z(t_0)-(2m)^{-1}g^{(\nu)}_{2Z}(t_0)\|_D\\
&\quad + \|f_Y(t)\|_D+\int_{\min(t,t_0)}^{\max(t,t_0)}\big(\|g^{(\nu)}_{1Z}(s)\|_D+J_Y(s)\big)ds,
\end{aligned}$$

where $J_Y(s)$ is given by (5.118), the inequality of statement b) follows from (5.147b), (5.156) and Lemma 5.11. This proves the lemma.

Lemma 5.11 and Lemma 5.12 lead to L^2-estimates of $h_Y(t), Y \in \Pi'$, where h is a solution of (5.1), in terms of L^2 and L^∞ norms of $h_Z(t_0), f_Z(s), g_Z(s)$ and $A_Z(s), Z \in \Pi'$. We recall that $h_Y(t_0)$ is a function of $h_{\mathbb{I}}(t_0)$ for a fixed potential G and an inhomogeneous term g in equation (5.1), which is obtained by eliminating time derivatives in $h_Y(t_0)$ by equation (5.1).

Theorem 5.13. *Let $n \geq 0$, $1/2 < \rho < 1$, $(1-\Delta)^{1/2}(A_X, A_{P_0X}) \in C^0(\mathbb{R}^+, M^1)$ for $X = \mathbb{I}$, $X \in \mathfrak{sl}(2,\mathbb{C})$, let $(1-\Delta)^{1/2}(B, \dot{B}) \in C^0(\mathbb{R}^+, M^1)$, where $B_\mu(y) = y_\mu \partial^\nu A_\nu(y)$, $\dot{B} = \frac{d}{dt}B$, let $A_Y \in C^0(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathbb{R}^4))$ for $Y \in \Pi'$, $|Y| \leq n+3$, let $B_Y \in C^0(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathbb{R}^4))$ for $Y \in \Pi'$, $|Y| \leq n+2$, let G_μ be given by (5.144), f be given by (5.111b), let $g_Y = \xi^D_Y g \in C^0(\mathbb{R}^+, D)$ for $Y \in \Pi'$, $|Y| \leq n$, let*

$$Q_n(t) = \sup_{t_0 \leq s \leq t} \wp^D_n(f(s)) + \sum_{0 \leq l \leq n-1} [A]_{3,n-l}(t)$$
$$\Big(\sup_{t_0 \leq s \leq t} \wp^D_l(f(s)) + \int_{t_0}^t (1+s)^{-3/2+\rho} \wp^D_l((1+\lambda_0(s))^{1/2} g(s)) ds\Big),$$

for $0 \leq t_0 \leq t$, where λ_0 is as in Theorem 5.5 and for $0 \leq t < t_0$ let Q_n be given by the same expression, but with t and t_0 interchanged on the right-hand side. If $h_Y(t_0) \in D$ for $Y \in \Pi'$, $|Y| \leq n$, then h given by (5.3c) is the unique solution of equation (5.1a) in $C^0(\mathbb{R}^+, D)$, with initial data $h(t_0)$. This solution satisfies $h_Y \in C^0(\mathbb{R}^+, D)$ for $Y \in \Pi'$, $|Y| \leq n$, $h_{P_\mu Y} \in C^0(\mathbb{R}^+, (1-\Delta)^{1/2} D)$ for $Y \in \Pi'$, $|Y| \leq n$, and

$$\wp^D_n(h(t)) \leq C_n\Big(\wp^D_n(h(t_0)) + \sum_{0 \leq l \leq n-1} [A]_{3,n-l}(t') \wp^D_l(h(t_0)) + Q_n(t)\Big),$$

for $t, t_0 \geq 0$, where $t' = \max(t, t_0)$ and where the constant C_n depends only on $[A]^3(t')$ and ρ.

If moreover the function $t \mapsto (1+t)^{-3/2+\rho}(1+\lambda_0(t))^{1/2} g_Y(t)$ is an element of $L^1(\mathbb{R}^+, D)$ for $Y \in \Pi'$, $|Y| \leq n$, and if for each $Y \in \Pi'$, $|Y| \leq n$, there exists g_{1Y} and g_{2Y} such that $g_Y = g_{1Y} + g_{2Y}$, and such that:

a) $g_{1Y} \in L^1(\mathbb{R}^+, D)$,

b) $(m - i\gamma^\mu \partial_\mu + \gamma^\mu G_\mu) g_{2Y} \in L^1(\mathbb{R}^+, D)$ *and* $\lim_{t \to \infty} \|g_{2Y}(t)\|_D = 0$,

then there exists a unique solution $h \in C^0(\mathbb{R}^+, D)$ of equation (5.1a) such that $\|h(t)\|_D \to 0$, when $t \to \infty$. This solution satisfies $\wp^D_n(h(t)) \leq C_n Q^\infty_n(t)$, $t \geq 0$, where Q^∞_n is given by the above expression of $Q_n(t)$, with $t_0 = \infty$. Further, $f_Y \in C^0(\mathbb{R}^+, D)$, $f_Y(t) \to 0$ as $t \to \infty$ and $f_Y(t)$ is the limit of $\int_t^T w(t,s) i\gamma^0 g_Y(s) ds$ in D as $T \to \infty$.

Proof. First let $t_0 < \infty$. It follows from statement i) of Lemma 4.5, with $\rho = 1$, that $(G, \dot{G}) \in C^0(\mathbb{R}^+, (1-\Delta)^{-1/2} M^1)$, where $\dot{G}(t) = \frac{d}{dt} G(t)$, since $(1-\Delta)^{1/2}(A_X, A_{P_0X}) \in C^0(\mathbb{R}^+, M^1)$ for $X = \mathbb{I}$, $X \in \mathfrak{sl}(2,\mathbb{C}))$ and $(1-\Delta)^{1/2}(B, \dot{B}) \in C^0(\mathbb{R}^+, M^1)$. Since $h(t_0) \in D$

and $g \in C^0(\mathbb{R}^+, D)$, it follows that conditions (5.1b), (5.1c) and (5.1d) hold for $\tau = 0$, which proves that equation (5.1a) has a unique solution $h \in C^0(\mathbb{R}^+, D)$, that this solution is given by (5.3c) and that $h_{P_\mu} \in C^0(\mathbb{R}^+, (1-\Delta)^{1/2}D)$ for $0 \le \mu \le 3$.

Application of ξ^M_Y to equation (5.1a) gives that $(i\gamma^\mu\partial_\mu + m - \gamma^\mu G_\mu)h_Y = g'_Y$, $Y \in \Pi'$, where

$$g'_Y = \sum_{\substack{Y_1, Y_2 \\ |Y_2| \le |Y|-1}}^{Y} \gamma^\mu G_{Y_1 \mu} h_{Y_2} + g_Y.$$

Since, according to inequality (4.82)

$$\|G_Z(t)\|_{L^\infty} \le C \sup_{0 \le s \le t} \Big(\sum_{|Y| \le |Z|+1} \|A_Y(s)\|_{L^\infty} + \sum_{|Y| \le |Z|} \|B_Y(s)\|_{L^\infty} \Big),$$

for $Z \in \Pi'$, it follows from the hypothesis that $G_Z \in C^0(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathbb{R}^4))$ for $|Z| \le n$. This shows that, if $|Y| \le n$, then $g'_Y \in C^0(\mathbb{R}^+, D)$ if $h_Z \in C^0(\mathbb{R}^+, D)$ for $|Z| \le |Y|-1$. Repeating the argument which proved that $h_{\mathbb{I}} \in C^0(\mathbb{R}^+, D)$ and that $h_{P_\mu} \in C^0(\mathbb{R}^+, (1-\Delta)^{1/2}D)$, it follows that $h_Y \in C^0(\mathbb{R}^+, D)$ and that $h_{P_\mu Y} \in C^0(\mathbb{R}^+, (1-\Delta)^{1/2}D)$ for $|Y| \le n$, $Y \in \Pi'$.

It follows from the definition of $J^{(l)}_n(L,t)$ in Lemma 5.11 and definitions (5.119a), (5.119b), (5.119c) and (5.119d) of $k^{(1)}_n, l^{(1)}_n, k^{(2)}_n$ and $l^{(2)}_n$, that

$$J^{(i)}_n(n,t) \le 2[A]_3(t)(\wp^D_n(h(t)) + R^0_{n-1,1}(t)) + k^{(1)}_n(n,t) + k^{(2)}_n(n,t), \quad n \ge 1, i \ge 0.$$

This inequality and definitions (5.119a) and (5.119c) of $k^{(1)}_n$ and $k^{(2)}_n$ give that

$$\begin{aligned} J^{(i)}_n(n,t) \le C &\sum_{n_1+n_2=n} [A]_{3,n_1}(t)\big(\wp^D_{n_2}(h(t)) + R^0_{n_2-1,1}(t)\big) \\ &+ \sum_{\substack{n_1+n_2=n \\ n_2 \le n-1}} [A]_{3,n_1}(t)(1+t)^{1/2}\wp^D_{n_2}(g(t)), \quad n \ge 1, i \ge 0, \end{aligned} \tag{5.157}$$

where C is a numerical constant. It follows from this inequality, from statement a) of Lemma 4.12 in the case $Y \in \boldsymbol{\sigma}^0, Y \notin \boldsymbol{\sigma}^1$ and from statement b) in the case $Y \in \boldsymbol{\sigma}^1$, with $L = |Y|$, that

$$\begin{aligned} \|h_Y(t)\|_D \le{}& \|h_Y(t_0)\|_D + \|f_Y(t)\|_D \\ &+ C_{|Y|}(1+t_0)^{-3/2+\rho} \sum_{\substack{n_1+n_2=|Y| \\ n_2 \le |Y|-1}} [A]^{n_1+1}(t_0)\wp^D_{n_2}(h(t_0)) \\ &+ C_{|Y|}(1+t)^{-3/2+\rho} \sum_{\substack{n_1+n_2=|Y| \\ n_2 \le |Y|-1}} [A]^{n_1+1}(t)\wp^D_{n_2}(h(t)) \\ &+ |Y|C'_{|Y|} \int_{\min(t,t_0)}^{\max(t,t_0)} (1+s)^{-2+\rho} \Big(\sum_{n_1+n_2=|Y|} [A]_{3,n_1}(s)\big(\wp^D_{n_2}(h(s)) + R^0_{n_2-1,1}(s)\big) \\ &+ \sum_{n_1+n_2=|Y|, n_2 \le |Y|-1} [A]_{3,n_1}(s)(1+s)^{1/2}\wp^D_{n_2}(h(s))\Big) ds, \quad Y \in \Pi', 1/2 < \rho < 1, \end{aligned} \tag{5.158}$$

where $C_{|Y|}$ is a constant depending only on ρ and $C'_{|Y|}$ a constant depending only on ρ and $[A]^3(t')$, $t' = \max(t, t_0)$. Summation over $|Y| \le k \le n$, give that

$$\begin{aligned}\wp^D_k(h(t)) \le{}& C\wp^D_k(h(t_0)) + C\wp^D_k(f(t)) \\ &+ C_k \sum_{\substack{n_1+n_2=k \\ n_2\le k-1}} \Big((1+t_0)^{-3/2+\rho}[A]^{n_1+1}(t_0)\wp^D_{n_2}(h(t_0)) \\ &+ (1+t)^{-3/2+\rho}[A]^{n_1+1}(t)\wp^D_{n_2}(h(t))\Big) \\ &+ kC'_k \int_{\min(t,t_0)}^{\max(t,t_0)} \Big((1+s)^{-2+\rho} \sum_{n_1+n_2=k} [A]_{3,n_1}(s)\big(\wp^D_{n_2}(h(s)) + R^0_{n_2-1,1}(s)\big) \\ &+ (1+s)^{-3/2+\rho} \sum_{\substack{n_1+n_2=k \\ n_2\le k-1}} [A]_{3,n_1}(s)\wp^D_{n_2}(g(s))\Big)ds,\end{aligned} \tag{5.159}$$

$0 \le k \le n, 1/2 < \rho < 1$, where C is a numerical constant, C_k depends only on ρ and C'_k only on ρ and $[A]^3(t')$.

Let, for $0 \le k \le n, t' = \min(t, t_0), t'' = \max(t, t_0)$,

$$\begin{aligned}Q^{(k)}(t) ={}& \sup_{t'\le s\le t''} \Big(C\wp^D_k(h(t_0)) + C\wp^D_k(f(s)) \\ &+ C_k \sum_{\substack{n_1+n_2=k \\ n_2\le k-1}} \Big((1+t_0)^{-3/2+\rho}[A]^{n_1+1}(t_0)\wp^D_{n_2}(h(t_0)) \\ &+ (1+s)^{-3/2+\rho}[A]^{n_1+1}(s)\wp^D_{n_2}(h(s))\Big)\Big) \\ &+ kC'_k \int_{t'}^{t''} \Big(\sum_{\substack{n_1+n_2=k \\ n_2\le k-1}} [A]_{3,n_1}(s)\Big((1+s)^{-2+\rho}\big(\wp^D_{n_2}(h(s)) + R^0_{n_2-1,1}(s)\big) \\ &+ (1+s)^{-3/2+\rho}\wp^D_{n_2}(g(s))\Big) + (1+s)^{-2+\rho}R^0_{k-1,1}(s)\Big)ds,\end{aligned} \tag{5.160}$$

where the constants C, C_k, C'_k are as in (5.159). Inequality (5.159) now reads

$$\wp^D_k(h(t)) \le Q^{(k)}(t) + kC'_k \int_{\min(t,t_0)}^{\max(t,t_0)} (1+s)^{-2+\rho}[A]_{3,0}(s)\wp^D_k(h(s))ds, \tag{5.161}$$

$0 \le k \le n, t \ge 0$, where C'_k is a constant depending only on ρ and $[A]^3(\max(t, t_0))$. Since $Q^{(k)}$ is increasing on the interval $[t_0, t]$, when $t_0 \le t$ and decreasing on the interval $[t, t_0]$, when $t < t_0$, it follows from Grönwall lemma that

$$\wp^D_k(h(t)) \le C'_k Q^{(k)}(t), \quad t \ge 0, k \ge 0, \tag{5.162}$$

where C'_k is a new constant depending only on ρ and $[A]^3(\max(t, t_0))$.

Inequality (5.162) gives for $k = 0$ that

$$\wp_0^D(h(t)) \leq C_0'\Big(\wp_0^D(h(t_0)) + \sup_{t' \leq s \leq t''} \wp_0^D(f(s))\Big),$$

where $t' = \min(t, t_0), t'' = \max(t, t_0)$ and C_0' a constant depending only on ρ and $[A]^3(t'')$, which shows that the statement of the theorem is true for $n = 0$. Suppose that the inequality of the theorem is true with $n-1$ instead of n. Using the convexity property for the functions $[A]_{N,l}(t)$ (analog to (5.89d), (5.89e)), we then obtain that $\wp_n^D(h(t))$ satisfies the inequality of the theorem for n, since

$$\begin{aligned}
&\sum_{n_1+n_2=n} (1+s)^{-2+\rho}[A]_{3,n_1}(s) R^0_{n_2-1,1}(s) \\
&\quad = (1+s)^{-2+\rho} \sum_{\substack{n_1+n_2=n \\ n_2 \geq 1}} [A]_{3,n_1}(s) \wp^D_{n_2-1}((1+\lambda_0(s))g(s)) \\
&\quad \leq C(1+s)^{-3/2+\rho} \sum_{0 \leq l \leq n-1} [A]_{3,n-l-1}(s) \wp_l^D((1+\lambda_0(s))^{1/2} g(s)),
\end{aligned}$$

according to the definition of $R^0_{p,q}$ in Corollary 5.9. This proves the statement of the theorem for $0 \leq t_0 < \infty$.

Let $t_0 = \infty$. If $h_1, h_2 \in C^0(\mathbb{R}^+, D)$ are two solutions of equation (5.1a) such that $\|h_i(t)\|_D \to 0, i = 1, 2$, when $t \to \infty$, then due to the unitarity of $w(t,s)$ in D:

$$0 = \lim_{s \to \infty} \|h_1(s) - h_2(s)\|_D = \|h_1(t) - h_2(t)\|_D, \quad t \geq 0.$$

This proves the uniqueness of the solution h. To prove the existence of h we first prove the existence of f_Y for $Y \in \Pi'$, $|Y| \leq n$, $T \geq 0$. Introduce $f_Y^T, f^T_{(i)Y} \in C^0(\mathbb{R}^+, D)$, $i \in \{0, 1, 2\}$, by $f_Y^T = f^T_{(0)Y}$.

$$\begin{aligned}
&f^T_{(i)Y}(t) = 0 \qquad \text{for } 0 \leq T \leq t, \\
&f^T_{(i)Y}(t) = \int_t^T w(t,s) i\gamma^0 g_{(i)Y}(s) ds \qquad \text{for } 0 \leq t < T,
\end{aligned} \tag{5.163}$$

where $g_{(i)Y} = g_{iY}$ are given by conditions a) and b) of the theorem for $i \in \{1, 2\}$ and where $g_{(0)Y} = g_Y$. f_Y^T has the properties:

$$f_Y^{T_1}(t) - f_Y^{T_2}(t) = w(t, T_2) f_Y^{T_1}, \quad 0 \leq t \leq T_2 \leq T_1,$$

which together with the definition of $f^T_{(i)Y}$ give that

$$\sup_{s \geq 0} \|f^{T_1}_{(i)Y}(s) - f^{T_2}_{(i)Y}(s)\|_D \leq \sup_{T_2 \leq s \leq T_1} \|f^{T_1}_{(i)Y}(s)\|_D, \quad i \in \{0, 1, 2\}, \tag{5.164}$$

for $0 \leq T_2 \leq T_1$. In the case of condition a) we obtain that

$$\sup_{s\geq 0} \|f^{T_1}_{(1)Y}(s) - f^{T_2}_{(1)Y}(s)\|_D \leq \int_{T_1}^{T_2} \|g_{(1)Y}(s)\|_D ds, \quad 0 \leq T_2 \leq T_1. \tag{5.165a}$$

In the case of condition b), statement iib) of Theorem 5.1 gives, since $f^{T_1}_{(2)Y}$ is a solution of equation (5.1a), (with $g_{(2)Y}$ instead of g) in the interval $[0, T_1]$ and which can be extended to a solution $\overline{h} \in C^0(\mathbb{R}^+, D)$, $\overline{h}(t) = \int_t^{T_1} w(t,s) i\gamma^0 g_{(2)Y}(s) ds$, with $\overline{h}(T_1) = 0$:

$$\begin{aligned} &\sup_{s\geq 0} \|f^{T_1}_{(2)Y}(s) - f^{T_2}_{(2)Y}(s)\|_D \\ &\quad \leq \sup_{T_2\leq s\leq T_1} (m^{-1}\|g_{(2)Y}(s)\|_D) + (2m)^{-1} \int_{T_2}^{T_1} \|((m - i\gamma^\mu\partial_\mu + \gamma^\mu G_\mu) g_{(2)Y})(s)\|_D ds, \end{aligned} \tag{5.165b}$$

$0 \leq T_2 \leq T_1$. Inequalities (5.165a) and (5.165b), conditions a) and b), prove that f^T_Y, $T \in \mathbb{N}$, is a Cauchy sequence in $L^\infty(\mathbb{R}^+, D)$ converging to an element $f_Y \in C^0(\mathbb{R}^+, D)$ satisfying $\|f_Y(t)\|_D \to 0$ when $t \to \infty$, for $Y \in \Pi'$, $|Y| \leq n$. Let us define $h(t) = f_{\mathbb{I}}(t)$.

We next construct a sequence of solutions h_k, vanishing for sufficiently large t, of equation (5.1a) converging to h and to which we can apply the estimate of the theorem for finite t_0. Let $\varphi \in C_0^\infty(\mathbb{R}), \varphi(s) = 1$ for $|s| \leq 1$, $\varphi(s) = 0$ for $|s| \geq 2$ and let $\varphi_k(t,x) = \varphi(k^{-1}(1 + t^2 + |x|^2)^{1/2})$, $k \geq 1$. Then $|(\xi_Y\varphi_k)(t,x)| \leq C_{|Y|}(1 + t^2 + |x|^2)^{|Y|/2} k^{-|Y|}$, $Y \in \Pi'$, where $C_{|Y|}$ is independent of t and x. If (t,x) belongs to the support of $\xi_Y\varphi_k$, then $k^{-1}(1 + t^2 + |x|)^{1/2} \leq 2$, which shows that $|(\xi_Y\varphi)(t,x)| \leq C_{|Y|}2^{|Y|}$. Let $g^{(i)}_{kY} = \sum^{Y}_{Y_1,Y_2} (\xi_{Y_1}\varphi_k) g_{(i)Y_2}$ for $Y \in \Pi'$, $|Y| \leq n$, $i \in \{0,1,2\}$. Then $\xi^D_Y(\varphi_k g) = g^{(0)}_{kY} = g^{(1)}_{kY} + g^{(2)}_{kY}$ and

$$\|g^{(i)}_{kY}(t)\|_D \leq C_{|Y|} \sum_{\substack{Z\in\Pi' \\ |Z|\leq|Y|}} \|g_{(i)Z}(t)\|_D, \quad Y \in \Pi', |Y| \leq n, t \geq 0, i \in \{0,1,2\}, \tag{5.166a}$$

for some constant $C_{|Y|}$ independent of t. Similarly we obtain that

$$\begin{aligned} &\|(1 + \lambda_j(t))^{1/2}(\xi^D_Y(\varphi_k g))(t)\|_D \\ &\quad \leq C_{|Y|} \sum_{\substack{Z\in\Pi' \\ |Z|\leq|Y|}} \|(1 + \lambda_j(t))^{1/2}(\xi^D_Z g)(t)\|_D, \quad Y \in \Pi', |Y| \leq n, t \geq 0, j = 0,1, \end{aligned} \tag{5.166b}$$

where $C_{|Y|}$ is independent of t. Moreover

$$|(\xi_{P_\mu Y}\varphi_k)(t,x)| \leq C_{|Y|}(1 + t^2 + |x|^2)^{-1/2},$$

where $C_{|Y|}$ is independent of t, gives that

$$\begin{aligned} &\|\big((m - i\gamma^\mu\partial_\mu + \gamma^\mu G_\mu) g^{(2)}_{kY}\big)(t)\|_D \\ &\quad \leq C_{|Y|} \sum_{\substack{Z\in\Pi' \\ |Z|\leq|Y|}} \Big((1+t)^{-1}\|g_{(2)Z}(t)\|_D + \|\big((m - i\gamma^\mu\partial_\mu + \gamma^\mu G_\mu) g_{(2)Z}\big)(t)\|_D\Big), \end{aligned} \tag{5.166c}$$

$t \geq 0, Y \in \Pi', |Y| \leq n$, where $C_{|Y|}$ is independent of t.

Let $g_k = \varphi_k g, n \geq 1$. Since $\varphi_k(t,x) = 1$ for $(1+t^2+|x|^2)^{1/2} \leq k$ and since $|(\xi_Y \varphi_k)(t,x)|$ is uniformly bounded in k, t, x, it follows from the dominated convergence theorem that $\|(g^{(i)}_{kY} - g_{(i)Y})(t)\|_D \to 0$ for $i \in \{0,1,2\}$, $t \geq 0$, $\|(1+\lambda_0(t))^{1/2}(g^{(0)}_{kY} - g_{(0)Y})(t)\|_D \to 0$, a.e. $t \geq 0$ and $\|\big((m - i\gamma^\mu \partial_\mu + \gamma^\mu G_\mu)(g^{(2)}_{kY} - g_{(2)Y})\big)(t)\|_D \to 0$, a.e. $t \geq 0$, for $Y \in \Pi'$, $|Y| \leq n$. This shows, together with the bound (5.166a) that

$$\lim_{k\to\infty} (\sup_{s\geq 0} \|(g^{(i)}_{kY} - g_{(i)Y})(s)\|_D) = 0, \quad i \in \{0,1,2\}, \tag{5.167a}$$

together with the bound (5.166b) that

$$\lim_{k\to\infty} \int_0^\infty (1+s)^{-3/2+\rho} \|(1+\lambda_0(s))^{1/2}(g^{(0)}_{kY} - g_{(0)Y})(s)\|_D ds = 0, \tag{5.167b}$$

together with the bound (5.166a) and the condition a) of the theorem that

$$\lim_{k\to\infty} \int_0^\infty \|(g^{(1)}_{kY} - g_{(1)Y})(s)\|_D ds = 0, \tag{5.167c}$$

and together with the bounds (5.166a), (5.166c) and the condition b) of the theorem that

$$\lim_{k\to\infty} \int_0^\infty \|\big((m - i\gamma^\mu \partial_\mu + \gamma^\mu G_\mu)(g^{(2)}_{kY} - g_{(2)Y})\big)(s)\|_D ds = 0, \tag{5.167d}$$

and that

$$\lim_{k\to\infty} \lim_{s\to\infty} \|(g^{(2)}_{kY} - g_{(2)Y})(s)\|_D = 0, \tag{5.167e}$$

for $Y \in \Pi'$ and $|Y| \leq n$. Let $f_{k,Y} \in C^0(\mathbb{R}^+, D)$ be the unique solution of equation (5.1a), with $g^{(0)}_{kY}$ instead of g and with initial data $f_{k,Y}(t_0) = 0, k \geq 1$, where $(1+t_0^2)^{1/2} \geq 2k$. Then $g^{(0)}_{kY}(t) = 0$ for $t \geq t_0$, so $f_{k,Y}(t) = 0$ for $t \geq t_0$. It follows from statement iib) of Theorem 5.1 and from limits (5.167a), (5.167c), (5.167d) and (5.167e) that $f_{k,Y}$ is a Cauchy sequence for the uniform convergence topology in $C^0(\mathbb{R}^+, D)$. $f_{k,Y}$ therefore converges to f_Y in this topology, for $Y \in \Pi', |Y| \leq n$. In particular $f_{k,\mathbb{I}}$ converges to $f_{\mathbb{I}} = h$.

Let us define $h_k = f_{k,\mathbb{I}}, k \geq 1$. The support of the function $f_{k,\mathbb{I}}: \mathbb{R}^+ \to D$ is a subset of $[0, 2k[, k \geq 1$. Taking t_0, sufficiently large, it now follows from the inequality of the theorem for finite t_0, that

$$\begin{aligned} &\wp^D_n(h_{k_1}(t) - h_{k_2}(t)) \\ &\quad \leq C_n\Big(\sup_{s\geq t} \wp^D_n(f_{k_1}(s) - f_{k_2}(s)) + \sum_{0\leq l\leq n-1} [A]_{3,n-l}(\infty)\Big(\sup_{s\geq t} \wp^D_l(f_{k_1}(s) - f_{k_2}(s)) \\ &\qquad + \int_t^\infty (1+s)^{-3/2+\rho} \wp^D_l\big((1+\lambda_0(s))^{1/2}(g_{k_1}(s) - g_{k_2}(s))\big) ds\Big)\Big), \end{aligned} \tag{5.168}$$

where C_n depends only on $[A]^3(\infty)$. Since $f_{k,Y}$ is a Cauchy sequence in $C^0(\mathbb{R}^+, D)$ for the uniform convergence topology, it follows from the limit (5.167b) and inequality (5.168) that $\xi^D_Y h_k \in C^0(\mathbb{R}^+, D)$, is a Cauchy sequence, in this topology, of functions with compact support, $|Y| \leq n, Y \in \Pi'$. This proves that $\xi^D_Y h \in C^0(\mathbb{R}^+, D)$ and $\|\xi^D_Y h(t)\|_D \to 0$, when $t \to \infty$, for $Y \in \Pi', |Y| \leq n$. Finally, it follows from inequality (5.168) that $\wp^D_n(h(t)) \leq C_n Q^\infty_n(t)$, where C_n depends only on $[A]^3(\infty)$. This proves the theorem.

In Theorem 5.13 we loose several orders in the seminorms of the electromagnetic potential. We shall derive a result, based on Theorem 5.8, Lemma 5.11, Lemma 5.12 and Theorem 5.13, where no seminorms are lost. We introduce the notation

$$\begin{aligned}\overline{H}_n(t_0, t) = \sum_{n_1+n_2=n} &(1 + S^\rho_{10,n_1}(t))\Big(R'_{n_2+7}(t) + R^2_{n_2+9}(t) \\ &+ R^\infty_{n_2}(t) + \wp^D_{n_2+8}(h(t_0)) + \sum_{0\leq l\leq n_2+7} [A]_{3,n_2+8-l}(t'')\wp^D_l(h(t_0)) + Q_{n_2+8}(t)\Big),\end{aligned} \tag{5.169}$$

where $n \geq 0, t_0, t \geq 0, t'' = \max(t, t_0)$ and where R'_n, R^2_n, R^∞_n are given in Theorem 5.8 and Q_n is given in Theorem 5.13. It follows from Theorem 5.8, definition (5.88a) of $T^{\infty(n)}$, inequality (5.116c), $x_\mu G^\mu(x) = 0$ for G_μ given by (5.114) and inequality (5.138) that

$$H_n(t) \leq a_n \overline{H}_n(t_0, t), \quad n \geq 0, t_0, t \geq 0, \tag{5.170}$$

where a_n depends only on $[A]^{11}(\max(t, t_0))$. Let

$$\overline{\wp}^D_n(t_0, t) = \wp^D_n(h(t_0)) + Q_n(t) + \sum_{0\leq l\leq n-1} [A]_{3,n-l}(t'')\wp^D_l(h(t_0)), \tag{5.171}$$

where $t_0, t \geq 0, n \geq 0, t'' = \max(t_0, t)$ and where Q_n is given in Theorem 5.13. In the situation of Theorem 5.13 it follows that

$$\wp^D_n(h(t)) \leq C_n \overline{\wp}^D_n(t_0, t), \quad n \geq 0, t_0, t \geq 0. \tag{5.172}$$

We introduce, for $n \geq 0, t_0, t \geq 0, t' = \min(t_0, t), t'' = \max(t_0, t), L \geq 0$, the notation

$$\begin{aligned}&k'_n(L, t_0, t) \\ &= (1 + t_0)^{-3/2+\rho} \sum_{\substack{n_1+n_2=n \\ 1\leq n_1\leq L}} [A]^{n_1+1}(t_0)\wp^D_{n_2}(h(t_0)) \\ &\quad + (1 + t_0)^{-1/2} \sum_{\substack{n_1+n_2=n \\ n_1\geq L+1}} S^{\rho,n_1}(t_0)\overline{H}_{n_2}(t_0, t_0) + (1 + t)^{-3/2+\rho} \sum_{\substack{n_1+n_2=n \\ 1\leq n_1\leq L \\ n_2\leq L}} [A]^{n_1+1}(t)\overline{\wp}^D_{n_2}(t_0, t) \\ &\quad + (1 + t)^{-1/2} \sum_{\substack{n_1+n_2=n \\ L+1\leq n_1\leq n-1}} S^{\rho,n_1}(t)\overline{H}_{n_2}(t_0, t) + \int_{t'}^{t''} (1 + s)^{-2+\rho}\overline{k}_n(L, t_0, s)ds,\end{aligned} \tag{5.173a}$$

where

$$\overline{k}_n(L,t_0,t) \tag{5.173b}$$
$$= \sum_{\substack{n_1+n_2=n\\ 1\le n_1\le L\\ n_2\le L}} \Big([A]_{3,n_1}(t)\big(\overline{\wp}^D_{n_2}(t_0,t)+R^0_{n_2-1,1}(t)\big)+(1+t)^{1/2}[A]^{n_1+1}(t)\wp^D_{n_2}(g(t))\Big)$$
$$+\sum_{\substack{n_1+n_2+n_3+n_4=n\\ n_1\le L-1,n_2\le n-1\\ n_3+n_4\le n-L}} (1+[A]_{2,n_1}(t))S^{\rho,n_2}(t)(1+S^\rho_{10,n_3}(t))$$
$$\big(R'_{n_4+7}(t)+R^2_{n_4+9}(t)+R^\infty_{n_4}(t)+\overline{\wp}_{n_4+8}(t_0,t)\big)$$
$$+\sum_{\substack{Y_1,Y_2\in\Pi'\\ |Y_1|+|Y_2|=n\\ L\le |Y_1|=n-1}} (1+t)^{2-\rho}\|\gamma^\mu G_{Y_1\mu}(t)g_{Y_2}(t)\|_D.$$

Here $R^0_{n_2-1,1}$ is given in Corollary 5.9.

Theorem 5.14. *Let $1/2<\rho<1, 18\le L+9\le n+8\le 2L$, $(1-\Delta)^{1/2}(A_X,A_{P_0X})\in C^0(\mathbb{R}^+,M^1)$ for $X=\mathbb{I}$ or $X\in\mathfrak{sl}(2,\mathbb{C})$, let $(1-\Delta)^{1/2}(B,\dot B)\in C^0(\mathbb{R}^+,M^1)$, where $B_\mu(y)=y_\mu\partial^\nu A_\nu$, $\dot B=\frac{d}{dt}B$, let $(A_Y,A_{P_0Y})\in C^0(\mathbb{R}^+,M^\rho)\cap C^0(\mathbb{R}^+,M^1)$ for $Y\in\Pi'$, $|Y|\le n$, let $\Box A_Y\in C^0(\mathbb{R}^+,L^2(\mathbb{R}^3,\mathbb{R}^4))$ for $Y\in\Pi'$, $|Y|\le n$, let $(B_Y,B_{P_0Y})\in C^0(\mathbb{R}^+,M^1)$ for $Y\in\Pi'$, $|Y|\le n-1$, let $\delta^{3/2-\rho}A_Y\in C^0(\mathbb{R}^+,L^\infty(\mathbb{R}^3,\mathbb{R}^4))$ for $Y\in\Pi'$, $|Y|\le L+3$, let the funtion $(t,x)\mapsto(1+|x|+t)(1+\big|t-|x|\big|)^{1/2}A_Y(t,x)$ be an element of $C^0(\mathbb{R}^+,L^\infty(\mathbb{R}^3,\mathbb{R}^4))$ for $Y\in\boldsymbol{\sigma}^1$, $|Y|\le L+3$, let $\delta^{3/2-\rho}B_Y\in C^0(\mathbb{R}^+,L^\infty(\mathbb{R}^3,\mathbb{R}^4))$ for $Y\in\Pi'$, $|Y|\le L+2$, let the function $(t,x)\mapsto(1+t+|x|)(1+\big|t-|x|\big|)^{1/2}B_Y(t,x)$ be an element of $C^0(\mathbb{R}^+,L^\infty(\mathbb{R}^3,\mathbb{R}^4))$ for $Y\in\boldsymbol{\sigma}^1$, $|Y|\le L+2$, let $g_Y\in C^0(\mathbb{R}^+,D)$ for $Y\in\Pi'$, $|Y|\le n$, let $R'_{L-1}(s),R^2_{L+1}(s)$, $R^\infty_{L-8}(s)$ be finite for $t'\le s\le t''$, where $t'=\min(t,t_0)$ and $t''=\max(t,t_0)$, let G_μ be given by (5.114), let f be given by (5.111b) and let*

$$k_n(L,t_0,t)$$
$$=k'_n(L,t_0,t)+(1+t)^{-1/2}S^{\rho,n}(t)\overline{H}_0(t_0,t)+\int_{t'}^{t''}(1+s)^{-2+\rho}\Big([A]^3(s)R^0_{n-1,1}(s)$$
$$+S^{\rho,n}(s)(1+[A]^1(s))\overline{H}_1(t_0,s)+(1+s)^{2-\rho}\Big(\sum_{|Y|=n}\|\gamma^\mu G_{Y\mu}(s)g(s)\|^2_D\Big)^{1/2}\Big)ds.$$

If $h_Y(t_0)\in D$ for $Y\in\Pi',|Y|\le n$ then h given by (5.3c) is the unique solution of equation (5.1a) in $C^0(\mathbb{R}^+,D)$, with initial data $h(t_0)$. This solution satisfies

$$h_Y\in C^0(\mathbb{R}^+,D),\quad h_{P_\mu Y}\in C^0(\mathbb{R}^+,(1-\Delta)^{1/2}D)\quad \text{for } Y\in\Pi',|Y|\le n,$$

and

$$\wp^D_n(h(t))\le C_n\Big(\wp^D_n(h(t_0))+\sup_{t'\le s\le t''}\big(\wp^D_n(f(s))+a_nk_n(L,t_0,s)\big)$$
$$+\sum_{\substack{n_1+n_2=n\\ 1\le n_1\le L\\ n_2\ge L+1}}[A]_{3,n_1}(t'')\Big(\wp^D_{n_2}(h(t_0))+\sup_{t'\le s\le t''}\big(\wp^D_{n_2}(f(s))+a_{n_2}k_{n_2}(L,t_0,s)\big)\Big)\Big),$$

where C_n *depends only on* ρ *and* $[A]^3(t'')$ *and* a_l, $0 \leq l \leq n$, *depends only on* ρ *and* $[A]^{11}(t'')$.

Moreover if $k_n^\infty(L,t)$ *is given by* $k_n(L,t_0,t)$ *with* $\wp_n^D(h(t_0)) = 0$ *and* $t_0 = \infty$, *if* $k_n^\infty(L) = \sup_{t\geq 0} k_n^\infty(L,t) < \infty$ *and if for each* $Y \in \Pi'$, $|Y| \leq n$, *there exists* g_{1Y} *and* g_{2Y} *such that* $g_Y = g_{1Y} + g_{2Y}$, *and such that:*

a) $g_{1Y} \in L^1(\mathbb{R}^+, D)$,

b) $(m - i\gamma^\mu\partial_\mu + \gamma^\mu G_\mu)g_{2Y} \in L^1(\mathbb{R}^+, D)$ *and* $\lim_{t\to\infty} \|g_{2Y}(t)\|_D = 0$,

then there exists a unique solution $h \in C^0(\mathbb{R}^+, D)$ *of equation (5.1a) such that* $\|h(t)\|_D \to 0$, *when* $t \to \infty$. *This solution satisfies*

$$\wp_n^D(h(t)) \leq C_n\Big(\sup_{t\leq s}\big(\wp_n^D(f(s)) + a_n k_n^\infty(L,s)\big) + \sum_{\substack{n_1+n_2=n\\ 1\leq n_1\leq L\\ n_2\geq L+1}} [A]_{3,n_1}(\infty)\sup_{t\leq s}\big(\wp_{n_2}^D(f(s)) + a_{n_2}k_{n_2}^\infty(L,s)\big)\Big),$$

where the constants C_n *and* $a_l, 0 \leq l \leq n$ *are as above. Further,* $f_Y \in C^0(\mathbb{R}^+, D)$, $f_Y(t) \to 0$ *as* $t \to \infty$ *and* $f_Y(t)$ *is the limit of* $\int_t^T w(t,s)i\gamma^0 g_Y(s)ds$ *in* D *when* $T \to \infty$.

Proof. First let $0 \leq t_0 < \infty$. According to the hypothesis, it follows from Theorem 5.13 that equation (5.1a) has a unique solution $h \in C^0(\mathbb{R}^+, D)$, that

$$h_Y \in C^0(\mathbb{R}^+, D), \quad (1-\Delta)^{-1/2}h_{P_\mu Y} \in C^0(\mathbb{R}^+, D) \quad \text{for } Y \in \Pi', |Y| \leq L, 0 \leq \mu \leq 3,$$

and that

$$\wp_j^D(h(t)) \leq C_j(\wp_j^D(h(t_0)) + \sum_{0\leq l\leq j-1} [A]_{3,j-l}(t'')\wp_l^D(h(t_0)) + Q_n(t)), \quad j \leq L,$$

where $t'' = \max(t,t_0)$ and where C_j depends only on $[A]^3(t'')$.

Like in the beginning of the proof of Theorem 5.13, let

$$g'_Y = \sum_{\substack{Y_1,Y_2\\ |Y_2|\leq|Y|-1}}^{Y} \gamma^\mu G_{Y_1\mu}h_{Y_2} + g_Y.$$

As in the proof of Theorem 5.13 it follows that $G_Y \in C^0(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathbb{R}^4))$ for $Y \in \Pi', |Y| \leq L$. It follows from the hypothesis of the theorem, from definition (5.115d) of $S^{\rho,n}$ and from inequality (5.125b) that $\delta^{-1}G_Y \in C^0(\mathbb{R}^+, L^2(\mathbb{R}^3, \mathbb{R}^4))$ for $Y \in \Pi', |Y| \leq n$. Moreover since $\|\delta(t)h_Z(t)\|_{L^\infty} \leq H_{|Z|}(t) \leq a_{|Z|}\overline{H}_{|Z|}(t_0,t)$, where $a_{|Z|}$ depends only on $[A]^{11}(\max(t,t_0))$, according to (5.170), it follows from the hypothesis that $\delta h_Z \in C^0(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathbb{C}^4))$ for $Z \in \Pi', |Z| \leq n-L \leq L-8$. Since $g_Y \in C^0(\mathbb{R}^+, D)$ for $|Y| \leq n$, $Y \in \Pi'$, it follows that $g'_Y \in C^0(\mathbb{R}^+, D)$ for $Y \in \Pi'$, $|Y| \leq n$, which together with $h_Y(t_0) \in D$ for $Y \in \Pi', |Y| \leq n$, give that $h_Y \in C^0(\mathbb{R}^+, D)$ and $h_{P_\mu Y} \in C^0(\mathbb{R}^+, (1-\Delta)^{1/2}D)$ for $Y \in \Pi'$, $|Y| \leq n$.

Statement a) of Lemma 5.12 and statement b), but with $L-1$ instead of L, give, for $0 \leq i \leq n$,

$$\begin{aligned}
&\wp^D_{n,i}(h(t)) \qquad (5.174)\\
&\leq \wp^D_{n,i}(h(t_0)) + \wp^D_{n,i}(f(t)) + (1+t_0)^{-3/2+\rho} C_n \sum_{\substack{n_1+n_2=n\\ 1\leq n_1\leq L}} [A]^{n_1+1}(t_0)\wp^D_{n_2}(h(t_0))\\
&\quad + (1+t_0)^{-1/2} C_n \sum_{\substack{n_1+n_2=n\\ n_1\geq L+1}} S^{\rho,n_1}(t_0) H_{n_2}(t_0) + (1+t)^{-3/2+\rho} C_n \sum_{\substack{n_1+n_2=n\\ 1\leq n_1\leq L}} [A]^{n_1+1}(t)\wp^D_{n_2}(h(t))\\
&\quad + (1+t)^{-1/2} C_n \sum_{\substack{n_1+n_2=n\\ L+1\leq n_1\leq n-1}} S^{\rho,n_1}(t) H_{n_2}(t) + (1+t)^{-1/2}\chi_+(-i) C_0 S^{\rho,n}(t) H_0(t)\\
&\quad + C'_n \int_{t'}^{t''} (1+s)^{-2+\rho}\Big(\sum_{l\geq \max(i,1)} J^{(l)}_n(L-1,s) + \chi_+(-i) J^{(0)}_n(L,s)\Big) ds\\
&\quad + \chi_+(-i) C_0 \int_{t'}^{t''} (1+s)^{-2+\rho}\Big(S^{\rho,n}(s)(1+[A]^1(s)) H_1(s)\\
&\quad + (1+s)^{2-\rho}\Big(\sum_{|Y|=n} \|\gamma^\mu G_{Y\mu}(s) g(s)\|^2_D\Big)^{1/2}\Big) ds,
\end{aligned}$$

where C_n depends only on ρ and C'_n depends on ρ and $[A]^3(t'')$. It follows from definitions (5.119a)–(5.119d) of $k^{(1)}_n, l^{(1)}_n, k^{(2)}_n$ and $l^{(2)}_n$ and from definition (5.173b) of $\overline{k}_n$, that

$$\begin{aligned}
&k^{(1)}_n(L,t) + l^{(1)}_n(L,t) + k^{(2)}_n(L,t) + l^{(2)}_n(L,t) \qquad (5.175)\\
&\quad + k^{(1)}_n(L-1,t) + l^{(1)}_n(L-1,t) + k^{(2)}_n(L-1,t) + l^{(2)}_n(L-1,t)\\
&\qquad \leq 2\Big(k^{(1)}_n(L,t) + l^{(1)}_n(L-1,t) + k^{(2)}_n(L,t) + l^{(2)}_n(L-1,t)\Big)\\
&\qquad \leq C \sum_{\substack{n_1+n_2=n\\ 1\leq n_1\leq L\\ n_2\geq L+1}} [A]_{3,n_1}(t)\wp^D_{n_2}(h(t)) + a_n \overline{k}_n(L,t_0,t),
\end{aligned}$$

where C is a numerical constant and a_n depends only on $[A]^{11}(t'')$. The definition of $J^{(l)}_n$ in Lemma 5.11, inequalities (5.170) and (5.172), definition (5.173a) of $k'_n(L,t_0,t)$ and inequalities (5.174) and (5.175) give

$$\begin{aligned}
&\wp^D_n(h(t)) \qquad (5.176)\\
&\leq \wp^D_n(h(t_0)) + \wp^D_n(f(t)) + a_n k'_n(L,t_0,t) + (1+t)^{-1/2} a_0 S^{\rho,n_1}(t)\overline{H}_0(t_0,t)\\
&\quad + (1+t)^{-3/2+\rho} C_n \sum_{\substack{n_1+n_2=n\\ 1\leq n_1\leq L\\ n_2\geq L+1}} [A]^{n_1+1}(t)\wp^D_{n_2}(h(t))\\
&\quad + C_0 \int_{t'}^{t''} (1+s)^{-2+\rho}\Big(S^{\rho,n}(s)(1+[A]^1(s)) a_1 \overline{H}_1(t_0,s)
\end{aligned}$$

$$+ (1+s)^{2-\rho}\Big(\sum_{|Y|=n} \|\gamma^\mu G_{Y\mu}(s) g(s)\|_D^2\Big)^{1/2}\Big) ds$$

$$+ C_n \int_{t'}^{t''} (1+s)^{-2+\rho}\Big(\sum_{\substack{n_1+n_2=n \\ 1\le n_1\le L \\ n_2\ge L+1}} [A]_{3,n_1}(s) \wp^D_{n_2}(h(s))$$

$$+ C'_n [A]^3(s)\big(\wp^D_n(h(s))^\varepsilon \wp^D_{n,1}(h(s))^{1-\varepsilon} + R^0_{n-1,1}(s)^{2(1-\rho)} \wp^D_{n_1}(h(s))^{2\rho-1}\big)\Big) ds,$$

and for $1 \le i \le n$, that

$$\begin{aligned} &\wp^D_{n,i}(h(t)) \\ &\quad\le \wp^D_{n,i}(h(t_0)) + \wp^D_{n,i}(f(t)) + a_n k'_n(L,t_0,t) \\ &\qquad + (1+t)^{-3/2+\rho} C_n \sum_{\substack{n_1+n_2=n \\ 1\le n_1\le L \\ n_2\ge L+1}} [A]^{n_1+1}(t) \wp^D_{n_2}(h(t)) \\ &\qquad + C'_n \int_{t'}^{t''} (1+s)^{-2+\rho}\Big(\sum_{\substack{n_1+n_2=n \\ 1\le n_1\le L \\ n_2\ge L+1}} [A]_{3,n_1}(s) \wp^D_{n_2}(h(s)) \\ &\qquad + [A]^3(s)\big(\wp^D_n(h(s))^\varepsilon \wp^D_{n,i+1}(h(s))^{1-\varepsilon} + R^0_{n-1,1}(s)^{2(1-\rho)} \wp^D_{n_1}(h(s))^{2\rho-1}\big)\Big) ds, \end{aligned} \tag{5.177}$$

where $18 \le L+9 \le n+8 \le 2L, \varepsilon = \max(1/2, 2(1-\rho)), 1/2 < \rho < 1$, $t' = \min(t,t_0), t'' = \max(t,t_0)$ and where C_n depends only on ρ, C' only on ρ and $[A]^3(t'')$ and a_n only on ρ and $[A]^{11}(t'')$. Let

$$\begin{aligned} k_n(L,t_0,t) = k'_n(L,t_0,t) &+ (1+t)^{-1/2} S^{\rho,n}(t) \overline{H}_0(t_0,t) \\ &+ \int_{t'}^{t''} (1+s)^{-2+\rho}\Big(S^{\rho,n}(s)(1+[A]^1(s))\overline{H}_1(t_0,s) \\ &+ (1+s)^{2-\rho}\Big(\sum_{|Y|=n} \|\gamma^\mu G_{Y\mu}(s) g(s)\|_D^2\Big)^{1/2} + [A]^3(s) R^0_{n_1,1}(s)\Big) ds, \end{aligned} \tag{5.178}$$

for $n \ge 0, L \ge 0, t, t_0 \ge 0, t' = \min(t,t_0), t'' = \max(t,t_0)$. It follows from inequalities (5.176) and (5.177) that

$$\begin{aligned} \wp^D_{n,i}(h(t)) \le \wp^D_{n,i}(h(t_0)) &+ \wp^D_{n,i}(f(t)) + a_n k_n(L,t_0,t) \\ &+ C_n (1+t)^{-3/2+\rho} \sum_{\substack{n_1+n_2=n \\ 1\le n_1\le L \\ n_2\ge L+1}} [A]^{n_1+1}(t) \wp^D_{n_2}(h(t)) \\ &+ C'_n \int_{t'}^{t''} (1+s)^{-2+\rho}\Big([A]^3(s) \wp^D_n(h(s))^\varepsilon \wp^D_{n,i+1}(h(s))^{1-\varepsilon} \\ &+ \sum_{\substack{n_1+n_2=n \\ 1\le n_1\le L \\ n_2\ge L+1}} [A]_{3,n_1}(s) \wp^D_{n_2}(h(s))\Big) ds, \end{aligned} \tag{5.179}$$

where $0 \le i \le n, 18 \le L+9 \le n+8 \le 2L, \varepsilon = \max(1/2, 2(1-\rho))$, $1/2 < \rho < 1$ and where C_n depends only on ρ, C' only on ρ and $[A]^3(t'')$ and a_n only on ρ and $[A]^{11}(t'')$.

To solve the system of inequalities (5.179) in the variables $\wp^D_{n,i}(h(t))$, we introduce the variables

$$\xi_{l,j} = \sup_{t' \le s \le t''} \wp^D_{l,j}(h(s)), \quad \text{for } 0 \le j \le l \le n, \tag{5.180a}$$
$$\xi_{l,j} = 0 \quad \text{for } j \ge l+1,$$

where $t' = \min(t, t_0), t'' = \max(t, t_0)$, and we introduce the positive real numbers

$$\eta_l = \wp^D_l(h(t_0)) + \sup_{t' \le s \le t''} \big(\wp^D_l(f(s)) + a_l k_l(L, t_0, s)\big), \quad 0 \le l \le n. \tag{5.180b}$$

It then follows from (5.179) that

$$\xi_{n,i} \le \eta_n + b_n \sum_{\substack{n_1+n_2=n \\ 1 \le n_1 \le L \\ n_2 \ge L+1}} [A]_{3,n_1}(t'')\xi_{n_2,0} + b_n [A]^3(t'')\xi^{\varepsilon}_{n,0}\xi^{1-\varepsilon}_{n,i+1}, \tag{5.181}$$

where $0 \le i \le n, L+9 \le n+8 \le 2L$ and where b_n is a constant depending only on ρ and on $[A]^3(t'')$. Let

$$\alpha_n = \eta_n + b_n \sum_{\substack{n_1+n_2=n \\ 1 \le n_1 \le L \\ n_2 \ge L+1}} [A]_{3,n_1}(t'')\xi_{n_2,0}, \tag{5.182a}$$

and

$$\beta_n = b_n [A]^3(t''). \tag{5.182b}$$

Then inequalities (5.181) give that

$$\xi_{n,i} \le \alpha_n + \beta_n \xi^{\varepsilon}_{n,0}\xi^{1-\varepsilon}_{n,i+1}, \tag{5.182c}$$

where $0 \le i \le n, L+9 \le n+8 \le 2L$. We note that $\alpha_{L+1} = \eta_{L+1}$ because of (5.182a). The solutions $x \ge 0$ of the inequality

$$x \le a + bx^{\varepsilon}y^{1-\varepsilon}, \tag{5.183a}$$

where $y \ge 0, a \ge 0, b \ge 0, 0 \le \varepsilon < 1$ satisfy

$$x \le 2a + 2b(1+2b)^{k-1}y, \quad k \in \mathbb{N}, k \ge 1/(1-\varepsilon) \ge 1. \tag{5.183b}$$

As a matter of fact for $0 < \varepsilon < 1$:

i) if $\varepsilon b \le 1/2$ then $x \le a + bx^{\varepsilon}y^{1-\varepsilon} \le a + \varepsilon bx + (1-\varepsilon)by$ gives that $x \le 2a + 2(1-\varepsilon)by$,

ii) if $\varepsilon b > 1/2$ and $(2b)^{1/(1-\varepsilon)}y \le x$ then $bx^{\varepsilon}y^{1-\varepsilon} \le x/2$ and inequality (5.183a) give that $x \le 2a$,

iii) if $\varepsilon b > 1/2$ and $(2b)^{1/(1-\varepsilon)}y > x$ then $x < (2b)^k y$ since $2b > 1/\varepsilon > 1$.

Applying the result (5.183b) to the solutions $\xi_{n,0}$ of inequality (5.182c), with $i = 0$, it follows that $\xi_{n,0} \leq 2\alpha_n + 2\beta_n(1+2\beta_n)^{k-1}\xi_{n,1}$. Majorizing $\xi_{n,1}$ by the right-hand side of inequality (5.182c), with $i = 1$, we obtain that

$$\xi_{n,0} \leq 2\alpha_n + 2\beta_n(1+2\beta_n)^{k-1}\alpha_n + 2\beta_n(1+2\beta_n)^{k-1}\beta_n \xi_{n,0}^{\varepsilon}\xi_{n,2}^{1-\varepsilon},$$

to which we apply inequality (5.183b). Continuing this iteration, we obtain after a finite number of steps, since $\xi_{n,n+1} = 0$, that $\xi_{n,0} \leq \gamma_n \alpha_n$, where γ_n is a polynomial in β_n. Since $\alpha_{L+1} = \eta_{L+1}$, we obtain, using $\xi_{l,0} \leq \gamma_l \alpha_l$ for $L+1 \leq l \leq n$ and using expression (5.182a) of α_n and (5.182b) of β_n, that

$$\alpha_l \leq \eta_l + b_l \sum_{\substack{n_1+n_2=l \\ 1\leq n_1 \leq L \\ n_2 \geq L+1}} [A]_{3,n_1}(t'')\alpha_{n_2}, \alpha_{L+1} = \eta_{L+1},$$

where $L + 10 \leq l + 8 \leq 2L$ and where b_n depends only on ρ and $[A]^3(t'')$. Iteration of this inequality for $L + 2 \leq l \leq n$, the convexity property $[A]_{3,n_1}(t'')[A]_{3,n_2}(t'') \leq C_{n_1+n_2}[A]_{3,n_1+n_2}(t'')$, where $C_{n_1+n_2}$ depends only on $[A]_{3,0}(t'')$ and the fact that $n-L-1 \leq L-9 \leq L$ give that

$$\alpha_n \leq \eta_n + b_n \sum_{\substack{n_1+n_2=n \\ 1\leq n_1 \leq L \\ n_2 \geq L+1}} [A]_{3,n_1}(t'')\eta_{n_2}, \quad L+9 \leq n+8 \leq 2L,$$

for some constants b_n depending only on ρ and $[A]^3(t'')$. The last inequality and the fact that $\xi_{n,0} \leq \gamma_n \alpha_n$ prove the inequality of the theorem, when $t_0 < \infty$. The proof of the case $t_0 = \infty$, is done by the same limit procedure as in the proof of the case $t_0 = \infty$ of Theorem 5.13. We omit the details since they are so similar to those of that proof. This proves the theorem.

Remark 5.15. For the case of $t_0 = \infty$ in Theorem 5.14 expressions (5.173a) of k'_n and (5.173b) of $\overline{k}_n$ are simplified. Let

$$\overline{H}_n^{\infty}(t) = \sum_{n_1+n_2=n} (1 + S^{\rho}_{10,n_1}(t))\big(R'_{n_2+7}(t) + R^2_{n_2+9}(t) + R^{\infty}_{n_2}(t) + Q^{\infty}_{n_2+8}(t)\big), \tag{5.184}$$

where $Q^{\infty}_l(t)$ is given in Theorem 5.13 and let

$$\begin{aligned} k'^{\infty}_n(L,t) = (1+t)^{-3/2+\rho} &\sum_{\substack{n_1+n_2=n \\ 1\leq n_1 \leq L \\ n_2 \leq L}} [A]^{n_1+1}(t) Q^{\infty}_{n_2}(t) \qquad (5.185) \\ &+ (1+t)^{-1/2} \sum_{\substack{n_1+n_2=n \\ L+1\leq n_1 \leq n-1}} S^{\rho,n_1}(t)\overline{H}^{\infty}_{n_2}(t) + \int_t^{\infty} (1+s)^{-2+\rho}\overline{k}^{\infty}_n(L,s)ds, \end{aligned}$$

where $\overline{k}_n^\infty(L,t)$ is given by expression (5.173b) of $\overline{k}_n(L,t_0,t)$, with $\overline{\wp}_l^D(t_0,t)$ replaced by $Q_l^\infty(t)$ and with $t_0=\infty$. Inequalities (5.176) and (5.177) are then true with $\overline{H}_l(t_0,t)$ replaced by $\overline{H}_l^\infty(t), k_n'(L,t_0,t)$ replaced by $k_n'^\infty(L,t)$, with $\wp_n^D(h(t_0))=0$ and with $t_0=\infty$. These inequalities will be useful for the nonlinear case, where A is a function of the Dirac field.

In order to use Theorem 5.13 and 5.14 we need an estimate to f_Y, $Y\in\Pi'$, given by (5.111b). To state the result we introduce first certain notations. Let $1/2<\rho<1, \eta\in [0,\rho[, \eta\neq 1/2$, let $\varepsilon=\eta$ if $\eta<1/2$ and $\varepsilon=1/2$ if $\eta>1/2$, let $0\leq\rho'<1$, let

$$\begin{aligned}\tau_n^M(t_0,t) &= (1+t_0)^{-1/2-\varepsilon}\sup_{0\leq s\leq t_0}\big((1+s)^\eta\wp_n^{M^1}(a(s),\dot a(s))\big) \\ &\quad+\int_{t_0}^t(1+s)^{-2+\rho-\varepsilon}\sup_{0\leq s'\leq s}\Big((1+s')^{\eta+\rho'-1}\wp_n^{M^{\rho'}}(a(s'),\dot a(s')) \\ &\quad+(1+s')^\eta\wp_n^{M^1}(a(s'),\dot a(s'))+(1+s')^{\varepsilon+1/2-\rho}\wp_n^{M^1}(0,\partial_\mu a^\mu(s')) \\ &\quad+(1+s')^{\eta+3/2-\rho}\wp_n^{M^1}(0,\Box a(s'))\Big)ds,\end{aligned}\tag{5.186}$$

for $n\geq 0, 0\leq t_0\leq t<\infty$, let $\tau_n^M(t_0,t)=\tau_n^M(t,t_0)$ for $0\leq t<t_0<\infty$ and let $\tau_n^M(t,\infty)=\lim_{t_0\to\infty}\tau_n^M(t,t_0)$ when this limit exists. Here $a_Y=\xi_Y^M a, Y\in\Pi'$. Let

$$\begin{aligned}\tau_n^D(t_0,t) &= (1+t_0)^{\rho-3/2}\wp_n^D(r(t_0)) \\ &\quad+\int_{\min(t,t_0)}^{\max(t,t_0)}\Big((1+s)^{-5/2+\rho}\wp_{n+1}^D(r(s))+(1+s)^{-3/2}\wp_n^D((1+\lambda_0(s))^{1/2}r(s)) \\ &\quad+(1+s)^{-3+2\rho}\wp_n^D(r(s))+(1+s)^{-3/2+\rho}\wp_n^D(((i\gamma^\mu\partial_\mu+m)r)(s))\Big)ds,\end{aligned}\tag{5.187}$$

for $n\geq 0, t\geq 0, t_0\geq 0, 1/2<\rho<1$ and let $\tau_n^D(\infty,t)$ be given by (5.187), but without the first term on the right-hand side. When $t_0\in\mathbb{R}^+\cup\{\infty\}$ is fixed we shall write $\tau_n^M(t)$ and $\tau_n^D(t)$ instead of $\tau_n^M(t_0,t)$ and $\tau_n^D(t_0,t)$.

Proposition 5.16. *Let $t_0\in\mathbb{R}^+\cup\{\infty\}, \rho\in]1/2,1[, \eta\in[0,1/2[\cup]1/2,\rho[, \rho'\in[0,1[$ and let $\varepsilon=\eta$ if $\eta<1/2$ and $\varepsilon=1/2$ if $\eta>1/2$. Let $g=\gamma^\mu(a_\mu-\partial_\mu\vartheta(a))r, g_Y=\xi_Y^D g$ for $Y\in\Pi'$, let $f_Y, Y\in\Pi'$ be given by (5.111b) and let τ_n^M and τ_n^D be defined by (5.186) and (5.187). Let $(1-\Delta)^{1/2}(A_X,A_{P_0X})\in C^0(\mathbb{R}^+,M^1)$ for $X=\mathbb{I}$ or $X\in sl(2,\mathbb{C})$, let $(1-\Delta)^{1/2}(B,\dot B)\in C^0(\mathbb{R}^+,M^1)$ and let G_μ be given by (5.114). Let $B_\mu(y)=y_\mu\partial_\nu A^\nu(y)$, and let*

$$\begin{aligned}\theta_n^D(t) &= \sum_{\substack{Y\in\Pi'\\|Y|\leq n+1}}\sup_{t'\leq s\leq t''}\big((1+[A]^1(s))\|\delta(s)^{3/2}r_Y(s)\|_{L^\infty}\big) \\ &\quad+\sum_{\substack{Y\in\Pi'\\|Y|\leq n}}\sup_{t'\leq s\leq t''}\big((1+s)^{2-\rho}\|\delta(s)((i\gamma^\mu\partial_\mu+m)r_Y)(s)\|_{L^\infty}\big),\quad n\geq 0, t\geq 0,\end{aligned}$$

where $t'=\min(t,t_0), t''=\max(t,t_0)$ and let $\theta_n^M(t)=[a]^{n+2}(t'')(1+[A]^1(t''))$.

i) *If* $(a_Y, a_{P_0 Y}) \in C^0(\mathbb{R}^+, M^1) \cap C^0(\mathbb{R}^+, M^{\rho'})$ *for* $Y \in \Pi'$, $|Y| \leq n$, $\delta^{3/2} r_Y \in C^0(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathbb{C}^4))$ *for* $|Y| \leq n$, *if* $\delta(i\gamma^\mu \partial_\mu + m) r_Y \in C^0(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathbb{R}^4))$ *for* $|Y| \leq n$, *where* $\delta(t,x) = (\delta(t))(x)$, *and if* $0 \leq j \leq n$, *then*

$$\wp^D_{n,j}(f(t)) \leq C^{(j)} \tau^M_n(t) \theta^D_0(t) + C_n \sum_{0 \leq l \leq n-1} \tau^M_l(t) \theta^D_{n-l}(t),$$

where $C^{(j)} = 0$ *if* $1 \leq j \leq n$.

ii) *If* $r_Y \in C^0(\mathbb{R}^+, D)$ *for* $Y \in \Pi'$, $|Y| \leq n+1$, $(1+\lambda_1)^{1/2} r_Y \in C^0(\mathbb{R}^+, D)$ *for* $|Y| \leq n$ *and, if* $a_Y \in C^0(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathbb{C}^4))$ *for* $Y \in \Pi'$, $|Y| \leq n+2$ *and if* $[a]^{n+2}(t'') < \infty$, *then*

$$\wp^D_n(f(t)) \leq C_n \sum_{0 \leq l \leq n} \theta^M_{n-l}(t) \tau^D_l(t), \quad n \geq 0.$$

iii) *If* $0 \leq L \leq n, (a_Y, a_{P_0 Y}) \in C^0(\mathbb{R}^+, M^1) \cap C^0(\mathbb{R}^+, M^{\rho'})$ *for* $Y \in \Pi'$, $|Y| \leq n$, $r_Y \in C^0(\mathbb{R}^+, D)$ *for* $|Y| \leq n+1$, $(1+\lambda_1)^{1/2} r_Y \in C^0(\mathbb{R}^+, D)$ *for* $|Y| \leq n$, *if* $a_Y \in C^0(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathbb{C}^4))$ *for* $Y \in \Pi'$, $|Y| \leq n+2$ *and if* $[a]^{L+2}(t'') < \infty$, *if* $\delta^{3/2} r_Y \in C^0(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathbb{C}^4))$ *for* $|Y| \leq n-L$, $\delta(i\gamma^\mu \partial_\mu + m) r_Y \in C^0(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathbb{R}^4))$ *for* $|Y| \leq n$, *then*

$$\begin{aligned} &\wp^D_{n,j}(f(t)) \\ &\quad \leq C^{(j)} \tau^M_n(t) \theta^D_0(t) + C_n \Big(\sum_{\substack{n_1+n_2=n \\ 0 \leq n_1 \leq L}} \theta^M_{n_1}(t) \tau^D_{n_2}(t) + \sum_{\substack{n_1+n_2=n \\ L+1 \leq n_1 \leq n-1}} \tau^M_{n_1}(t) \theta^D_{n_2}(t) \Big), \quad n \geq 0, \end{aligned}$$

where $C^{(j)}, C_n$ *are constants depending only on* ρ, ρ', η, *and* $C^{(j)} = 0$ *for* $1 \leq j \leq n$.

Proof. Let $b_\mu = a_\mu - \partial_\mu \vartheta(a), 0 \leq \mu \leq 3$ and $b_{Y\mu} = (\xi^M_Y b)_\mu, Y \in \Pi'$. If

$$f_{Y_1,Y_2}(t) = \int_{t_0}^t w(t,s)(-i\gamma^0)\gamma^\mu b_{Y_1 \mu}(s) r_{Y_2}(s) ds, \tag{5.188a}$$

then by definition (5.111b) of f_Y

$$f_Y(t) = \sum_{Y_1,Y_2}^{Y} f_{Y_1,Y_2}(t). \tag{5.188b}$$

According to the hypothesis, it follows that $\gamma^\mu b_{Y_1\mu} r_{Y_2} \in C^0(\mathbb{R}^+, D)$, when Y_1, Y_2 are in the domain of summation in (5.188b) and $|Y| \leq n, Y \in \Pi'$.

Let $t_0 < \infty$. Since the two cases $0 \leq t_0 \leq t$ and $0 \leq t < t_0$ are so similar, we only consider the situation where $0 \leq t_0 \leq t$. Like in the beginning of the proof of Theorem 5.13, it follows that $f_{Y_1,Y_2} \in C^0(\mathbb{R}^+, D)$ is the unique solution of $(i\gamma^\mu \partial_\mu + m - \gamma^\mu G_\mu) f_{Y_1,Y_2} = \gamma^\mu b_{Y_1\mu} r_{Y_2}$, with $f_{Y_1,Y_2}(t_0) = 0$.

For $Y_1 \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$ denote by

$$I_{Y_1,Y_2}(t) \tag{5.189a}$$
$$= \big|\|f_{Y_1,Y_2}(t) - (2m)^{-1}\gamma^\mu b_{Y_1\mu}(t) r_{Y_2}(t)\|_D - \|(2m)^{-1}\gamma^\mu b_{Y_1\mu}(t_0) r_{Y_2}(t_0)\|_D\big|,$$

where (Y_1, Y_2) is in the domain of summation in (5.188b) and $Y \in \Pi', |Y| \leq n$. It follows from Corollary 5.2, with $a_l = 0$ and using the gauge invariance of the electromagnetic field that

$$\begin{aligned} &I_{Y_1,Y_2}(t) \qquad (5.189\text{b})\\ &\leq C\int_{t'}^{t''}\Big((1+s)^{-1}\Big(\|b_{Y_1 0}(s) r_{P_0 Y_2}(s)\|_D + \sum_{1\leq i\leq 3}\big(\|b_{Y_1 i}(s) r_{M_{0i}Y_2}(s)\|_D + \|b_{Y_1 i}(s) r_{Y_2}(s)\|_D\big)\Big)\\ &\quad + \|(\partial_\mu b^\mu_{Y_1}(s)) r_{Y_2}(s)\|_D + \|\gamma^\mu\gamma^\nu(\partial_\mu a_{Y_1\nu}(s) - \partial_\nu a_{Y_1\mu}(s)) r_{Y_2}(s)\|_D\\ &\quad + \|\gamma^\mu\gamma^\nu G_\mu(s) b_{Y_1\nu}(s) r_{Y_2}(s)\|_D + \|\gamma^\mu b_{Y_1\mu}(s) R_{Y_2}(s)\|_D\Big)ds, \end{aligned}$$

where $R_{Y_2} = (i\gamma^\mu\partial_\mu + m) r_{Y_2}$ and C is a numerical constant. Since $[G]'^0(t) \leq C_0[A]^1(t)$, according to (5.116c), it follows from (5.189b) that

$$\begin{aligned} &I_{Y_1,Y_2}(t)\\ &\leq C\int_{t'}^{t''}(1+s)^{-2+\rho-\varepsilon}\Big((1+s)^{1/2-\rho+\varepsilon}\|(a_{Y_1}(s), a_{P_0 Y_1}(s))\|_{M^1}\|\delta(s)^{3/2} r_{Y_2}(s)\|_{L^\infty}\\ &\quad + (1+s)^\varepsilon\|\delta(s)^{-1} b_{Y_1}(s)\|_{L^2}\Big((1+s)^{1/2-\rho}\Big(\|\delta(s)^{3/2} r_{P_0 Y_2}(s)\|_{L^\infty}\\ &\quad + \sum_{1\leq i\leq 3}\big(\|\delta(s)^{3/2} r_{M_{0i}Y_2}(s)\|_{L^\infty} + \|\delta(s)^{3/2} r_{Y_2}(s)\|_{L^\infty}\big)\Big)\\ &\quad + [A]^1(s)\|\delta(s)^{3/2} r_{Y_2}(s)\|_{L^\infty} + (1+s)^{2-\rho}\|\delta(s) R_{Y_2}(s)\|_{L^\infty}\Big)\\ &\quad + (1+s)^{3/2-\rho+\varepsilon}\|\delta(s)^{-1}\partial_\mu b^\mu_{Y_1}(s)\|_{L^2}\|\delta(s)^{3/2} r_{Y_2}(s)\|_{L^\infty}\Big)ds, \end{aligned}$$

where C is a constant depending only on ρ. Since $1/2 < \rho < 1$, this inequality gives, according to the definition of θ^D_n,

$$\begin{aligned} &I_{Y_1,Y_2}(t) \qquad (5.190)\\ &\leq C\int_{t'}^{t''}(1+s)^{-2+\rho-\varepsilon}\Big((1+s)^{1/2-\rho+\varepsilon}\|(a_{Y_1}(s), a_{P_0 Y_1}(s))\|_{M^1} + (1+s)^\varepsilon\|\delta(s)^{-1} b_{Y_1}(s)\|_{L^2}\\ &\quad + (1+s)^{3/2-\rho+\varepsilon}\|\delta(s)^{-1}\partial_\mu b^\mu_{Y_1}(s)\|_{L^2}\Big)ds\,\theta^D_{|Y_2|}(t), \end{aligned}$$

where C is a constant depending only on ρ. Inequality (5.122) gives, since $Y_1 \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$, that

$$(1+t)^\varepsilon\|\delta(t)^{-1} b_{Y_1}(t)\|_{L^2} \tag{5.191a}$$
$$\leq C\sup_{0\leq s\leq t}\Big((1+s)^{\eta+\rho'-1}\||\nabla|^{\rho'} a_{Y_1}(s)\|_{L^2} + (1+s)^\eta\|(a_{Y_1}(s), a_{P_0 Y_1}(s))\|_{M^1}\Big),$$

since $\varepsilon=\eta$ for $\eta<1/2, \varepsilon=1/2$ for $\eta>1/2$ and $0\leq\rho'<1$. Moreover inequality (5.129) gives that

$$(1+t)^{\varepsilon+3/2-\rho}\|\delta(t)^{-1}\partial_\mu b^\mu_{Y_1}(t)\|_{L^2} \\ \leq C\sup_{0\leq s\leq t}\Big((1+s)^{\varepsilon+1/2-\rho}\|\partial_\mu a^\mu_{Y_1}(s)\|_{L^2}+(1+s)^{\varepsilon+3/2-\rho}\|\Box a_{Y_1}(s)\|_{L^2}\Big), \tag{5.191b}$$

since $\varepsilon+3/2-\rho<3/2$. In inequality (5.191a), C depends only on ε,η,ρ' and in (5.191b) only on ε,ρ. It follows from inequalities (5.190), (5.191a) and (5.191b), since $1/2-\rho+\varepsilon<\varepsilon\leq\eta$, that

$$I_{Y_1,Y_2}(t)\leq C\tau^{1M}_{Y_1}(t)\theta^D_{|Y_2|}(t), \tag{5.192a}$$

where C depends only on $\varepsilon,\eta,\rho',\rho$, where (Y_1,Y_2) is in the domain of summation in (5.188b), $Y\in\Pi', |Y|\leq n$, $Y_1\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))$, and where

$$\tau^{1M}_{Y_1}(t)=\int_{t'}^{t''}(1+s)^{-2+\rho-\varepsilon}\sup_{0\leq s'\leq s}\Big((1+s')^{\eta+\rho'-1}\|(a_{Y_1}(s'),a_{P_0Y_1}(s'))\|_{M^{\rho'}} \\ +(1+s')^{\eta}\|(a_{Y_1}(s'),a_{P_0Y_1}(s'))\|_{M^1}+(1+s')^{\varepsilon+1/2-\rho}\|\partial_\mu a^\mu_{Y_1}(s')\|_{L^2} \\ +(1+s')^{\eta+3/2-\rho}\|\Box a_{Y_1}(s')\|_{L^2}\Big)ds. \tag{5.192b}$$

It follows from (5.116c) and (5.189b) that

$$I_{Y_1,Y_2}(t)\leq C_{|Y_1|}\int_{t'}^{t''}\Big((1+s)^{-5/2+\rho}[a]^{|Y_1|+1}(s)\wp^D_{|Y_2|+1}(r(s)) \\ +(1+s)^{-3/2}[a]^{|Y_1|+2}(s)\wp^D_{|Y_2|}((1+\lambda_1(s))^{1/2}r(s)) \\ +(1+s)^{-3/2}[a]^{|Y_1|+1}(s)\wp^D_{|Y_2|}((1+\lambda_1(s))^{1/2}r(s)) \\ +(1+s)^{-3+2\rho}[A]^1(s)[a]^{|Y_1|+1}(s)\wp^D_{|Y_2|}(r(s)) \\ +(1+s)^{-3/2+\rho}[a]^{|Y_1|+1}\wp^D_{|Y_2|}(R(s))\Big)ds,$$

which shows that

$$I_{Y_1,Y_2}(t) \\ \leq C_{|Y_1|}\theta^M_{|Y_1|}(t)\int_{t'}^{t''}\Big((1+s)^{-5/2+\rho}\wp^D_{|Y_2|+1}(r(s))+(1+s)^{-3/2}\wp^D_{|Y_2|}((1+\lambda_1(s))^{1/2}r(s)) \\ +(1+s)^{-3+2\rho}\wp^D_{|Y_2|}(r(s))+(1+s)^{-3/2+\rho}\wp^D_{|Y_2|}(R(s))\Big)ds, \tag{5.193}$$

where $C_{|Y_1|}$ depends only on ρ and where (Y_1,Y_2) is in the domain of summation in (5.88b), $Y\in\Pi'$, $|Y|\leq n$.

Let $q_\mu(y)=\int_0^1 a_\mu(sy)ds, 0\leq\mu\leq 3$, and let $q_{Y\mu}=(\xi^M_Y q)_\mu$ for $Y\in U(\mathfrak{p})$. Similarly as (5.121c) was obtained, it follows that

$$b_{ZY\mu}(y)=a_{ZY\mu}(y)-y^\nu\partial_\mu q_{ZY\nu}(y)-q_{ZY\mu}(y)-|Z|\sum_{0\leq\nu\leq 3}C_\nu(Z)q_{Z_\nu P_\mu Y^\nu}(y), \tag{5.194}$$

for some positive integers $C_\nu(Z)$ and some elements $Z_\nu \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C})), |Z_\nu| = |Z| - 1$, when $Z \in \Pi' \cap U(\mathbb{R}^4)$ and $Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$. Using the estimate

$$\begin{aligned}\|\delta(t)^{-3/2}\varphi\|_{L^2} &\le \|\delta(t)^{-3/2}\|_{L^3}\|\varphi\|_{L^6} \\ &\le C'(1+t)^{-1/2}\||\nabla|\varphi\|_{L^2} \\ &\le C(1+t)^{-1/2}\sum_{0\le i\le 3}\|\partial_i\varphi\|_{L^2}\end{aligned}$$

for the first and third term on the right-hand side of inequality (5.194), we obtain that

$$\begin{aligned}&\|\delta(t)^{-3/2}b_{ZY}(t)\|_{L^2} \\ &\quad\le C(1+t)^{-1/2}\sum_{0\le\mu\le 3}\big(\|a_{P_\mu ZY}(t)\|_{L^2} + \|q_{P_\mu ZY}(t)\|_{L^2}\big) \\ &\qquad + |Z|(1+t)^{-3/2}\sum_{\substack{0\le\nu\le 3\\ 0\le\mu\le 3}} C_\nu(Z)\|q_{Z_\nu P_\mu Y}(t)\|_{L^2}.\end{aligned}$$

This inequality and the result given by (4.86a)–(4.86b), show that

$$\begin{aligned}&\|\delta(t)^{-3/2}b_{ZY}(t)\|_{L^2} \\ &\quad\le C(1+t)^{-1/2-\chi}\sup_{0\le s\le t}\big((1+s)^{\chi'}\|(a_{ZY}(s), a_{P_0ZY}(s))\|_{M^1}\big) \\ &\qquad + |Z|C(1+t)^{-1/2-\chi}\sum_{0\le\nu\le 3} C_\nu(Z)\sup_{0\le s\le t}\big((1+s)^{\chi'-1}\|(a_{Z_\nu Y}(s), a_{P_0Z_\nu Y}(s))\|_{M^1}\big),\end{aligned} \tag{5.195}$$

where $\chi = \chi'$ for $\chi' < |Z| + 1/2, \chi = |Z| + 1/2$ for $\chi' > |Z| + 1/2$, where C depends only on χ and χ' and where $Z \in \Pi' \cap U(\mathbb{R}^4), Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$.

Due to the unitarity of the linear time evolution in M^1, we have

$$\|(a_Y(t), a_{P_0Y}(t))\|_{M^1} \le \|(a_Y(t_0), a_{P_0Y}(t_0))\|_{M^1} + \int_{t'}^{t''}\|(0, J_Y(s))\|_{M^1}ds, \tag{5.196}$$

where $J_Y = \Box a_Y$ and $Y \in \Pi'$. It follows from inequalities (5.195) and (5.196) that

$$\begin{aligned}&\|\delta(t)^{-3/2}b_Y(t)\|_{L^2} \le C(1+t)^{-1/2-\chi} \\ &\quad\sup_{0\le s\le t_0}\Big((1+s)^{\chi'}\|(a_Y(s), a_{P_0Y}(s))\|_{M^1} + |Y|(1+s)^{\chi'-1}\wp^{M^1}_{|Y|-1}((a(s),\dot a(s)))\Big) \\ &\qquad + C\int_{t'}^{t''}\Big((1+s)^{-1/2-\chi+\chi'}\|\Box a_Y(s)\|_{L^2} + |Y|(1+s)^{-3/2-\chi+\chi'}\wp^{M^1}_{|Y|-1}((0,\Box a(s)))\Big)ds,\end{aligned} \tag{5.197}$$

where $Y \in \Pi'$, $\chi = \chi'$ for $\chi' < 1/2$, $\chi = 1/2$ for $\chi' > 1/2$ and $-1/2 - \chi + \chi' \le 0$. Since these conditions are satisfied with $\chi = \varepsilon$ and $\chi' = \eta$ it follows from (5.186), (5.192b) and (5.197) that

$$\|\delta(t)^{-3/2}b_Y(t)\|_{L^2} \le C(\tau^{0M}_Y(t) + \tau^{1M}_Y(t) + |Y|\tau^M_{|Y|-1}(t)), \quad Y \in \Pi', \tag{5.198}$$

where

$$\tau_Y^{0M}(t) = (1+t_0)^{-1/2-\varepsilon} \sup_{0\le s\le t_0} \big((1+s)^\eta \|(a_Y(s), a_{P_0Y}(s))\|_{M^1}\big). \tag{5.199}$$

(We note that inequality (5.197) is also true for $t_0 > t$).

Since $\frac{d}{dt}e^{-\mathcal{D}t} r_Y(t) = e^{-i\mathcal{D}t}(-i\gamma^0)(i\gamma^\mu\partial_\mu + m) r_Y(t)$, we obtain using (5.187) that

$$\begin{aligned}&(1+t)^{-3/2+\rho}\wp_n^D(r(t)) \\ &\quad\le (1+t_0)^{-3/2+\rho}\wp_n^D(r(t_0)) + \int_{t'}^{t''}\Big((1+s)^{-5/2+\rho}(\rho-3/2)\wp_n^D(r(s)) \\ &\qquad + (1+s)^{-3/2+\rho}\wp_n^D((i\gamma^\mu\partial_\mu + m)r(s))\Big)ds \\ &\quad\le \tau_n^D(t).\end{aligned} \tag{5.200}$$

It follows from inequalities (5.189a), (5.192a) and (5.198), that

$$\|f_{Y_1,Y_2}(t)\|_D \le C\big(\tau_{Y_1}^{0M}(t) + \tau_{Y_1}^{1M}(t) + |Y_1|\tau^M_{|Y_1|-1}(t)\big)\theta^D_{|Y_2|}(t) \tag{5.201a}$$

and from (5.187), (5.189a), (5.193) and (5.200), that

$$\|f_{Y_1,Y_2}(t)\|_D \le C_{|Y_1|,|Y_2|}\theta^M_{|Y_1|}(t)\tau^D_{|Y_2|}(t), \tag{5.201b}$$

where $Y_1 \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C})), (Y_1, Y_2)$ are in the domain of summation in (5.188b) and $Y \in \Pi', |Y| \le n$.

For $Y_1 \in \boldsymbol{\sigma}^1$, there is ν such that $Y_1 = P_\nu Z$, $Z \in \Pi', |Z| = |Y_1| - 1$. It follows from (5.7b′) and from the gauge invariance of the electromagnetic field, that

$$\begin{aligned}&(i\gamma^\mu\partial_\mu + m - \gamma^\mu G_\mu)(f_{Y_1,Y_2} + ib_{Z\nu}r_{Y_2}) \\ &\qquad = \gamma^\mu(\partial_\nu a_{Z\mu} - \partial_\mu a_{Z\nu})r_{Y_2} + b_{Z\nu}(i\gamma^\mu\partial_\mu + m - \gamma^\mu G_\mu)r_{Y_2}.\end{aligned} \tag{5.202}$$

With the notation

$$I_{Y_1,Y_2}(t) = \|f_{Y_1,Y_2}(t) + ib_{Z\nu}(t)r_{Y_2}(t)\|_D - \|b_{Z\nu}(t_0)r_{Y_2}(t_0)\|_D, \tag{5.203}$$

it follows from (5.202) that

$$\begin{aligned}I_{Y_1,Y_2}(t) \le \int_{t'}^{t''}\Big(&\|\gamma^\mu(\partial_\nu a_{Z\mu}(s) - \partial_\mu a_{Z\nu}(s))r_{Y_2}(s)\|_D \\ &+ \|b_{Z\nu}(s)R_{Y_2}(s)\|_D + \|b_{Z\nu}(s)\gamma^\mu G_\mu(s)r_{Y_2}(s)\|_D\Big)ds,\end{aligned} \tag{5.204}$$

for $Y_1 = P_\nu Z, Z \in \Pi'$. This inequality gives that

$$\begin{aligned}I_{Y_1,Y_2}(t) \le C\int_{t'}^{t''}&((1+s)^{-3/2}\|(a_Z(s), a_{P_0Z}(s))\|_{M^1} \\ &+ (1+s)^{-2+\rho}\|\delta(s)^{-1}b_{Z^\nu}(s)\|_{L^2})ds\,\theta^D_{Y_2}(t).\end{aligned}$$

It follows then from inequality (5.122), that

$$I_{Y_1,Y_2}(t) \le C\theta^D_{|Y_2|}(t) \int_{t'}^{t''} \Big((1+s)^{-3/2}\|(a_Z(s), a_{P_0Z}(s))\|_{M^1}$$
$$+ (1+s)^{-2+\rho-\varepsilon} \sup_{0\le s'\le s} \Big((1+s')^{\eta+\rho'-1}\||\nabla|^{\rho'} a_Z(s')\|_{L^2}$$
$$+ (1+s')^{\eta}\|(a_Z(s'), a_{P_0Z}(s'))\|_{M^1}$$
$$+ (1+s')^{\eta-1}C_{|Z|}|Z|\wp^{M^1}_{|Z|-1}((a(s'), \dot{a}(s')))\Big)\Big)ds,$$

and then from definition (5.186) of τ^M, that

$$I_{Y_1,Y_2}(t) \le C_{|Y_1|}\tau^M_{|Y_1|-1}(t)\theta^D_{|Y_2|}(t), \tag{5.205a}$$

where $Y_1 \in \boldsymbol{\sigma}^1$. It also follows from inequalities (5.116c) and (5.204) that

$$I_{Y_1,Y_2}(t) \le C_{|Y_1|} \int_{t'}^{t''} \Big((1+s)^{-3/2}[a]^{|Y_1|}(s)\wp^D_{|Y_2|}((1+\lambda_1(s))^{1/2}r(s))$$
$$+ (1+s)^{-3/2+\rho}[a]^{|Y_1|}(s)\wp^D_{|Y_2|}(R(s))$$
$$+ (1+s)^{-3+2\rho}[A]^1(s)[a]^{|Y_1|}(s)\wp^D_{|Y_2|}(r(s))\Big)ds,$$

which gives that

$$I_{Y_1,Y_2}(t) \le C_{|Y_1|}\theta^M_{|Y_1|-1}(t)\tau^D_{|Y_2|}(t), \tag{5.205b}$$

where $Y_1 \in \boldsymbol{\sigma}^1$. It follows from definition (5.203) of I_{Y_1,Y_2} and from inequalities (5.198) and (5.205a), that

$$f_{Y_1,Y_2}(t) \le C_{|Y_1|,|Y_2|}\tau^M_{|Y_1|-1}(t)\theta^D_{|Y_2|} \tag{5.206a}$$

and that

$$f_{Y_1,Y_2}(t) \le C_{|Y_1|,|Y_2|}\theta^M_{|Y_1|-1}(t)\tau^D_{|Y_2|}(t), \tag{5.206b}$$

where $Y_1 \in \boldsymbol{\sigma}^1$.

Decomposition (5.188b), inequality (5.201a) and inequality (5.206a) prove statement i) of the proposition, since

$$\|f_Y(t)\|_D \le \|f_{Y,\mathbb{I}}(t)\| + C_{|Y|} \sum^{Y}_{\substack{Y_1,Y_2\\|Y_1|\le|Y|-1}} \|f_{Y_1,Y_2}(t)\|_D.$$

Statement ii) follows from inequalities (5.201b) and (5.206b). Statement iii) follows by using (5.201a) and (5.206a) for $|Y_1| \ge L+1$ and inequalities (5.201b) and (5.206b) for $0 \le |Y_1| \le L$. This proves the proposition.

Finally in this chapter we shall prove two corollaries, which are particular cases of Theorem 5.13 and Proposition 5.16 and which are obtained by using the convexity property of the seminorms $\|\cdot\|_{E_n}$ given by Corollary 2.6.

Corollary 5.17. *Let $t_0 \in [0,\infty], n \geq 0, 1/2 < \rho < 1, (1-\Delta)^{1/2}(A_X, A_{P_0 X}) \in C^0(\mathbb{R}^+, M^1)$ for $X = \mathbb{I}$ or $X \in \mathfrak{sl}(2,\mathbb{C})$, let $(1-\Delta)^{1/2}(B, \dot{B}) \in C^0(\mathbb{R}^+, M^1)$, where $B_\mu(y) = y_\mu \partial^\nu A_\nu(y)$, $\dot{B} = \frac{d}{dt}B$, let $A_Y \in C^0(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathbb{R}^4))$, $Y \in \Pi'$, let*

$$[A]^l(t) \leq C_l \|u\|_{E_{N_0+l}}, \quad l \geq 0, t \geq 0,$$

for some element $u \in E^\rho_\infty$, integers N_0 and L_0 independent of u and l, and some constants C_l depending only on $\|u\|_{E_{L_0}}$, $L_0 \geq N_0$, let G_μ be given by (5.114), f be given by (5.111b), let

$$\begin{aligned} Q_n(t) = \sup_{t' \leq t \leq t''} \wp^D_n(f(s)) + \sum_{0 \leq l \leq n-1} \|u\|_{E_{N_0+3+n-l}} \\ \Big(\sup_{t' \leq s \leq t''} \wp^D_l(f(s)) + \int_{t'}^{t''} (1+s)^{-3/2+\rho} \wp^D_l((1+\lambda_0(s))^{1/2} g(s)) ds \Big), \end{aligned}$$

where $t' = \min(t, t_0)$, $t'' = \max(t, t_0)$. If $t_0 \in \mathbb{R}^+$ and $h_Y(t_0) \in D$ for $Y \in \Pi'$, $|Y| \leq n$, then h given by (5.3c) is the unique solution of equation (5.1a) in $C^0(\mathbb{R}^+, D)$ with initial data $h(t_0)$ at t_0. This solution satisfies $h_Y, (1-\Delta)^{-1/2} h_{P_\mu Y} \in C^0(\mathbb{R}^+, D), 0 \leq \mu \leq 3$, and

$$\wp^D_n(h(t)) \leq C_n \Big(\wp^D_n(h(t_0)) + \sum_{0 \leq l \leq n-1} \|u\|_{E_{N_0+3+n-l}} \wp^D_l(h(t_0)) + Q_n(t) \Big),$$

for $t \in \mathbb{R}^+$, where the constant C_n depends only on $\|u\|_{E_L}$, $L = \max(L_0, N_0+3)$ and ρ. If $t_0 = \infty$, if the function $t \mapsto (1+t)^{-3/2+\rho}(1+\lambda_0(t))^{1/2} g_Y(t)$ is an element of $L^1(\mathbb{R}^+, D)$ for $Y \in \Pi', |Y| \leq n$, and if for each $Y \in \Pi'$, $|Y| \leq n$, there exists g_{1Y} and g_{2Y} such that $g_Y = g_{1Y} + g_{2Y}$, and such that:

a) $g_{1Y} \in L^1(\mathbb{R}^+, D)$,

b) $(m - i\gamma^\mu \partial_\mu + \gamma^\mu G_\mu) g_{2Y} \in L^1(\mathbb{R}^+, D)$ *and* $\lim_{t \to \infty} \|g_{2Y}(t)\|_D = 0$,

then there exists a unique solution $h \in C^0(\mathbb{R}^+, D)$ of equation (5.1a) such that $\|h(t)\|_D \to 0$, when $t \to \infty$. This solution satisfies $\wp^D_n(h(t)) \leq C_n Q^\infty_n(t)$, where $Q^\infty_n(t)$ is given by the above expression of $Q_n(t)$ with $t_0 = \infty$. Further $f_Y \in C^0(\mathbb{R}^+, D)$, $f_Y(t) \to 0$ in D when $t \to \infty$ and $f_Y(t)$ is the limit of $\int_t^T w(t,s) i\gamma^0 g_Y(s) ds$ in D, when $T \to \infty$, for $Y \in \Pi'$, $|Y| \leq n$.

Proof. Since $[A]^{q+l}(t) \leq C_{l+q} \|u\|_{E_{N_0+q+l}}, l \geq 0, q \geq 0$, according to the hypothesis, it follows from the definition of $[A]_{q,l}$ before formula (5.117a) and from (5.87), that

$$[A]_{q,0}(t) \leq C_q \|u\|_{E_{N_0+q}}$$

and that

$$[A]_{q,l}(t) \leq C^{(l)}_q \sum_{1 \leq p \leq l} \sum_{\substack{n_1 + \cdots + n_p = l \\ n_i \geq 1}} \|u\|_{E_{N_0+q+n_1}} \cdots \|u\|_{E_{N_0+q+n_p}}, \quad l \geq 1,$$

where $C_q^{(l)}$ is a constant depending only on ρ and $\|u\|_{E_{L_0}}$. Repeated use of Corollary 2.6 shows that

$$\|u\|_{E_{N_0+q+n_1}} \cdots \|u\|_{E_{N_0+q+n_p}} \leq K_{N_0+q+l}(\|u\|_{E_{N_0+q}})^{p-1}\|u\|_{E_{N_0+q+l}},$$

since $n_1 + \cdots + n_p = l$, where K_{N_0+q+l} is a constant independent of u. This shows that

$$[A]_{q,l}(t) \leq C_{q,l}\|u\|_{E_{N_0+q+l}}, \tag{5.207}$$

where $C_{q,l}$ is a constant depending only on ρ and $\|u\|_{E_M}$, $M = \max(L_0, N+q)$. Using inequality (5.207) with $q = 3$, the corollary follows from Theorem 5.13 by redefining $Q_N(t)$. This proves the corollary.

Corollary 5.18. *Let $n \geq 0, t_0 \in [0, \infty]$, let A, $h(t_0)$, f and G be as in Corollary 5.17, let $\rho \in]1/2, 1[, \eta \in [0, 1/2[\cup]1/2, \rho[, \rho' \in [0, 1[$ and let $\varepsilon = \eta$ if $\eta < 1/2$ and $\varepsilon = 1/2$ if $\eta > 1/2$. Let $g = \gamma^\mu(a_\mu - \partial_\mu\vartheta(a))r$ and $g_Y = \xi_Y^D g$ for $Y \in \Pi'$.*
i) *If $(a_Y, a_{P_0 Y}) \in C^0(\mathbb{R}^+, M^1) \cap C^0(\mathbb{R}^+, M^{\rho'})$ for $Y \in \Pi', |Y| \leq n, \partial_\mu a^\mu = 0$,*

$$\|\delta(t)^{3/2}(1+\lambda_1(t))^{\rho-1/2}r_Y(t)\|_{L^\infty} + \|(1+\lambda_1(t))^{\rho-1/2}r_Y(t)\|_D \leq C_{|Y|}\|u\|_{E_{N_0+|Y|}}$$

and

$$\|\delta(t)^{3-\rho}(i\gamma^\mu\partial_\mu + m)r_Y(t)\|_{L^\infty} \leq C_{|Y|}\|u\|_{E_{N_0}}\|u\|_{E_{N_0+|Y|}}, \quad t \geq 0, Y \in \Pi'$$

where $C_{|Y|}, N_0$ and u are as in Corollary 5.17, then

$$\begin{aligned}&\wp_n^D(h(t)) \\ &\quad \leq C_n'\Big(\wp_n^D(h(t_0)) + \sum_{0\leq l\leq n-1}\|u\|_{E_{N_0+3+n-l}}\wp_l^D(h(t_0)) + \sum_{0\leq l\leq n}\|u\|_{E_{N_0+3+n-l}}\tau_l^M(t)\Big),\end{aligned}$$

for $t_0 < \infty$, and

$$\wp_n^D(h(t)) \leq C_n'\sum_{0\leq l\leq n}\|u\|_{E_{N_0+5+n-l}}\tau_l^M(t)$$

for $t_0 = \infty$, where C_n' is a constant depending only on $\|u\|_{E_{N_0+3}}$.
ii) *If $r_Y \in C^0(\mathbb{R}^+, D)$ for $Y \in \Pi', |Y| \leq n+1$, $(1+\lambda_1)^{1/2}r_Y \in C^0(\mathbb{R}^+, D)$ for $|Y| \leq n$ and if $[a]^l(t) \leq C_l\|u\|_{E_{N_0+l}}, l \geq 0$, where C_l, N_0 and u are as in Corollary 5.17, then*

$$\begin{aligned}&\wp_n^D(h(t)) \\ &\quad \leq C_n'\Big(\wp_n^D(h(t_0)) + \sum_{0\leq l\leq n-1}\|u\|_{E_{N_0+3+n-l}}\wp_l^D(h(t_0)) + \sum_{0\leq l\leq n}\|u\|_{E_{N_0+3+n-l}}\tau_l^D(t)\Big),\end{aligned}$$

for $t_0 < \infty$, and

$$\wp_n^D(h(t)) \leq C_n'\sum_{0\leq l\leq n}\|u\|_{E_{N_0+3+n-l}}\tau_l^D(t),$$

for $t_0 = \infty$, where C_n' is a constant depending only on $\|u\|_{E_{N_0+3}}$.

Proof. It follows from the definitions of θ_j^D in Proposition 5.16 that

$$\theta_j^D(t) \leq K_j \Big((1 + C_1 \|u\|_{E_{N_0+1}}) C_{j+1} \|u\|_{E_{N_0+j+1}} + C_j \|u\|_{E_{N_0}} \|u\|_{E_{N_0+j}} \Big),$$

when the hypothesis of statement i) are satisfied. Here K_j is a numerical constant independent of u and $C_l, l \geq 0$, are constants depending only on $\|u\|_{E_{N_0}}$. Thus for new constants depending only on $\|u\|_{E_{N_0+1}}$, we have that $\theta_j^D(t) \leq C_j \|u\|_{E_{N_0+j+1}}$, which together with statement i) of Proposition 5.16 gives that

$$\wp_j^D(f(t)) \leq C_n \sum_{0 \leq l \leq j} \|u\|_{E_{N_0+j+1-l}} \tau_l^M(t). \tag{5.208}$$

With $b_\mu = a_\mu - \partial_\mu \vartheta(a)$, Hölder inequality give that

$$\|(1+\lambda_1(t))^{1/2} g_Y(t)\|_D \leq \sum_{Y_1, Y_2}^{Y} \|b_{Y_1}(t)\|_{L^p} \|(1+\lambda_1(t))^{1/2} r_{Y_2}(t)\|_{L^q},$$

$p = 6/(3-2\rho), q = 3/\rho$. We have that

$$\begin{aligned}
&\|(1+\lambda_1(t))^{1/2} r_{Y_2}(t)\|_{L^{3/\rho}} \\
&\quad \leq \|(1+\lambda_1(t))^{\rho-1/2} \delta(t)^{1-\rho} r_{Y_2}(t)\|_{L^{3/\rho}} \\
&\quad \leq \big(\|\delta(t)^{(1-\rho)3/\rho} |(1+\lambda_1(t))^{\rho-1/2} r_{Y_2}(t)|^{3/\rho-2}\|_{L^\infty} \|(1+\lambda_1(t))^{\rho-1/2} r_{Y_2}(t)\|_{L^2}^2 \big)^{\rho/3} \\
&\quad \leq \big(\|\delta(t)^{-3/2\rho} |\delta(t)^{3/2} (1+\lambda_1(t))^{\rho-1/2} r_{Y_2}(t)|^{3/\rho-2}\|_{L^\infty} \|(1+\lambda_1(t))^{\rho-1/2} r_{Y_2}(t)\|_D^2 \big)^{\rho/3} \\
&\quad \leq (1+t)^{-1/2} \|\delta(t)^{3/2} (1+\lambda_1(t))^{\rho-1/2} r_{Y_2}(t)\|_{L^\infty}^{1-2\rho/3} \|(1+\lambda_1(t))^{\rho-1/2} r_{Y_2}(t)\|_D^{2\rho/3} \\
&\quad \leq C_{|Y_2|} \|u\|_{E_{N_0+|Y_2|}} (1+t)^{-1/2},
\end{aligned}$$

where $C_{|Y_2|}$ depends only on $\|u\|_{E_{N_0}}$ and where the last step follows from the hypothesis of statement i). Since $\|b_{Y_1}(t)\|_{L^p} \leq C_\rho \||\nabla|^\rho b_{Y_1}(t)\|_{L^2}$, it now follows from statement i) of Lemma 4.5 (with $\rho = 1$) that

$$\begin{aligned}
&\|(1+\lambda_1(t))^{1/2} g_Y(t)\|_D \\
&\quad \leq C_n \sum_{n_1+n_2=|Y|} (1+t)^{-1/2-\varepsilon} \sup_{0 \leq s \leq t} \Big((1+s)^\eta \wp_{n_1+1}^{M^1}\big((a(s), \dot{a}(s))\big) \Big) \|u\|_{E_{N_0+n_2}},
\end{aligned}$$

where we have used that $\partial_\mu a^\mu = 0$. This shows that

$$\int_{t'}^{t''} (1+s)^{-3/2+\rho} \wp_j^D((1+\lambda_1(s))^{1/2} g(s)) ds \leq C_j \sum_{n_1+n_2=j} \tau_{n_1+1}^M(t) \|u\|_{E_{N_0+n_2}}, \tag{5.209}$$

where C_j depends only on $\|u\|_{E_{N_0}}$. It follows from inequalities (5.208) and (5.209) that Q_n defined in Corollary 5.17 satisfies

$$\begin{aligned} Q_n(t) \leq C_n \Big(& \sum_{0 \leq l \leq n} \|u\|_{E_{N_0+n+1-l}} \tau_l^M(t) \\ & + \sum_{0 \leq l \leq n-1} \|u\|_{E_{N_0+3+n-l}} \Big(\sum_{0 \leq j \leq l} \|u\|_{E_{N_0+l+1-j}} \tau_j^M(t) + \sum_{0 \leq j \leq l} \|u\|_{N_0+l-j} \tau_{j+1}^M(t) \Big) \Big) \\ \leq C'_n & \sum_{0 \leq l \leq n} \|u\|_{E_{N_0+3+n-l}} \tau_l^M(t), \end{aligned}$$

where we have used Corollary 2.6. Here C_n depends only on $\|u\|_{E_{N_0}}$ and C'_n only on $\|u\|_{E_{N_0+3}}$. Statement i) of the corollary now follows from Corollary 5.17. Since the proof of statement ii) is so similar, we omit it.

6. Construction of the modified wave operator and its inverse.

The properties, given by Theorem 4.9, Theorem 4.10 and Theorem 4.11 of A_J, ϕ'_J, where we choose $J \geq (3/2-\rho)/(1-\rho), 1/2 < \rho < 1$, of the approximate solution A_J, ϕ_J defined by (4.135a) and (4.135b), permit to prove the existence of modified wave operators for the M-D equations. To do this we shall first introduce *new functions* A^*_n, ϕ^*_n, $n \geq 0$, slightly different from the above approximate solutions, but having the advantage of satisfying the *Lorentz gauge condition* $\partial_\mu A^{*\mu}_n = 0$. Then, we shall prove that $(A^*_n, \exp(i\vartheta(A^*_n))\phi^*_n), n \geq 0$, *converges to a solution of the M-D equations*.

Let us fix once for all $J \geq (3/2-\rho)/(1-\rho)+2$. Given $u = (f, \dot{f}, \alpha) \in E^{\circ\rho}_\infty$, we shall prove that there is a unique solution $\phi^*_0 \in C^0(\mathbb{R}^+, D)$ such that $\|\phi^*_0(t) - e^{t\mathcal{D}}\alpha\|_D \to 0$, when $t \to \infty$, of the equation

$$\phi^*_0(t) = e^{t\mathcal{D}}\alpha + i\int_t^\infty e^{(t-s)\mathcal{D}}((A_{J,\mu} + B_{J,\mu})\gamma^0\gamma^\mu\phi^*_0)(s)ds, \quad t \geq 0, \tag{6.1a}$$

where $B_{J,\mu} = -\partial_\mu\vartheta(A_J), 0 \leq \mu \leq 3$. $A^*_{0,\mu}$ is defined by

$$\begin{aligned} A^*_{0,\mu}(t) &= \cos(|\nabla|t)f_\mu + |\nabla|^{-1}\sin(t|\nabla|)\dot{f}_\mu \\ &\quad - \int_t^\infty |\nabla|^{-1}\sin(|\nabla|(t-s))((\phi^*_0)^+\gamma^0\gamma_\mu\phi^*_0)(s)ds, \quad t \geq 0. \end{aligned} \tag{6.1b}$$

For $n \geq 0$, we then prove that the equations

$$\phi^*_{n+1}(t) = e^{t\mathcal{D}}\alpha + i\int_t^\infty e^{(t-s)\mathcal{D}}((A^*_{n,\mu} + B^*_{n,\mu})\gamma^0\gamma^\mu\phi^*_{n+1})(s)ds, \quad t \geq 0, \tag{6.2a}$$

where $B^*_{n,\mu} = -\partial_\mu\vartheta(A^*_n)$ and

$$\begin{aligned} A^*_{n+1,\mu}(t) &= \cos(|\nabla|t)f_\mu + |\nabla|^{-1}\sin(t|\nabla|)\dot{f}_\mu \\ &\quad - \int_t^\infty |\nabla|^{-1}\sin(|\nabla|(t-s))((\phi^*_{n+1})^+\gamma^0\gamma_\mu\phi_{n+1})(s)ds, \quad t \geq 0, \end{aligned} \tag{6.2b}$$

have a unique solution $(\phi^*_{n+1}, A^*_{n+1}, \frac{d}{dt}A^*_{n+1}) \in C^0(\mathbb{R}^+, E^\rho_0)$ and that $(\Delta^{*M}_n, \Delta^{*D}_n)$, where

$$\Delta^{*M}_{n,\mu} = A^*_{n+1,\mu} - A^*_{n,\mu}, \quad \Delta^{*D}_n = \phi^*_{n+1} - \phi^*_n, \quad n \geq 0, \tag{6.3}$$

converges to zero in an appropriate space when $n \to \infty$.

To begin with we complete Theorem 4.9 and Theorem 5.10 by *decrease properties*, established in chapter 5, of solutions of the inhomogeneous Dirac equation. We adapt the notation used in Theorem 4.9, Theorem 4.10, Corollary 5.9 and use $(\delta(t))(x) = 1 + t + |x|$. We recall that λ_0 and λ_1 are defined in Theorem 5.5.

Lemma 6.1. *If $n \geq 0$, and $1/2 < \rho < 1$ then there exists $N_0 \geq 0$ such that*

$$\begin{aligned}
&\sup_{t\geq 0}\Big(\|(1+\lambda_1(t))^{k/2}(D^l\phi'_{n,Y}(u;v_1,\ldots,v_l))(t)\|_D \\
&\quad+(1+t)^{\chi_{n+1}}\|(1+\lambda_1(t))^{k/2}(D^l(\phi'_{n+1,Y}-\phi'_{n,Y}))(u;v_1,\ldots,v_l))(t)\|_D \\
&\quad+\|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2}(D^l\phi'_{n,Y}(u;v_1,\ldots,v_l))(t)\|_{L^\infty} \\
&\quad+\|(\delta(t))^{3/2+\chi_{n+1}}(1+\lambda_1(t))^{k/2}(D^l(\phi'_{n+1,Y}-\phi'_{n,Y}))(u;v_1,\ldots,v_l))(t)\|_{L^\infty}\Big) \\
&\quad\leq F_{L,l,k}(\|u\|_{E^\rho_{N_0}})\mathcal{R}^l_{N_0,L+k}(v_1,\ldots,v_l)+F'_{L,l,k}(\|u\|_{E^\rho_{N_0}})\|u\|_{E^\rho_{N_0+L+k}}\|v_1\|_{E^\rho_{N_0}}\cdots\|v_l\|_{E^\rho_{N_0}},
\end{aligned}$$

for all $L \geq 0, l \geq 0, k \geq 0, Y \in \Pi', |Y| \leq L, u, v_1, \ldots, v_l \in E^{\circ\rho}_\infty$, where $F_{L,l,k}$ and $F'_{L,l,k}$ are increasing polynomials on $[0,\infty[$.

Proof. If $k = 0$ then the statement follows from Theorem 4.9 and Theorem 4.10. Let $k \geq 1$. It follows from (4.137c) that $(i\gamma^\mu\partial_\mu + m)\phi'_0 = 0$ and that

$$(i\gamma^\mu\partial_\mu+m)\phi'_n=\gamma^\mu(A_{n-1,\mu}+B_{n-1,\mu})\phi'_{n-1},\quad n\geq 1. \tag{6.4}$$

Corollary 5.9 (with $G = 0, g = g_n = \gamma^\mu(A_{n-1,\mu} + B_{n-1,\mu})\phi'_{n-1}$ for $n \geq 1$ and $g = g_0 = 0$ for $n = 0$ and $L \geq \max(3, n+k-1)$) gives

$$\begin{aligned}
&\wp^D_L\big((1+\lambda_1(t))^{k/2}\phi'_n(t)\big) \\
&\quad\leq C_{L+k}\Big(\wp^D_{L+k}(\phi'_n(t))+\sum_{0\leq j\leq k-1}\wp^D_{L+j}\big((1+\lambda_1(t))^{(k+1-j)/2}g_n(t)\big)\Big),
\end{aligned}\tag{6.5}$$

$n \geq 0$, $L \geq 0$, $k \geq 1$, where C_{L+k} is a numerical constant. This gives that

$$\wp^D_L\big((1+\lambda_1(t))^{k/2}\phi'_0(t)\big)\leq C_{L+k}\wp^D_{L+k}(\phi'_0(t))=C_{L+k}\|\alpha\|_{D_{L+k}}, \tag{6.6}$$

$L \geq 0, k \geq 1$, where we have used that $\phi'_0(t) = e^{t\mathcal{D}}\alpha$. Moreover using, for $n \geq 1$, that

$$\begin{aligned}
&\wp^D_{L+j}\big((1+\lambda_1(t))^{(k+1-j)/2}g_n\big) \\
&\quad\leq C'_{L+j}\sum_{i_1+i_2=L+j}\|u\|_{E^\rho_{N_0+i_1}}\wp^D_{i_2}\big((1+\lambda_1(t))^{(k-j)/2}\phi'_{n-1}(t)\big),
\end{aligned}\tag{6.7}$$

C'_{L+j} being a polynomial in $\|u\|_{E^\rho_{N_0}}$ which follows from Theorem 4.9 and Corollary 4.12, inequality (6.5) gives that

$$\begin{aligned}
&\wp^D_L\big((1+\lambda_1(t))^{k/2}\phi'_n(t)\big) \\
&\quad\leq C^{(n)}_{L+k}\Big(\wp^D_{L+k}(\phi'_n(t))+\sum_{\substack{i_1+i_2=L+j\\0\leq j\leq k-1}}\|u\|_{E^\rho_{N_0+i_1}}\wp^D_{i_2}\big((1+\lambda_1(t))^{(k-j)/2}\phi'_{n-1}(t)\big)\Big),
\end{aligned}\tag{6.8}$$

$n \geq 1$, $L \geq 0$, $k \geq 1$, where $C^{(n)}_{L+k}$ is a polynomial in $\|u\|_{E^\rho_{N_0}}$. We make the induction hypothesis,

$$\wp^D_L\big((1+\lambda_1(t))^{k/2}\phi'_N(t)\big) \leq C^{(N)}_{L+k}\|u\|_{E^\rho_{N_0+L+k}}, \quad L \geq 0, k \geq 1, \tag{6.9}$$

for $0 \leq N \leq n-1$, where C^N_{L+k} is some polynomial in $\|u\|_{E^\rho_{N_0}}$. According to inequality (6.6) the hypothesis is true for $N = 0$ since $\|\alpha\|_{D_{L+k}} \leq \|u\|_{E^\rho_{N_0+L+k}}$. It follows from inequality (6.8) and Theorem 4.9 that (for a new polynomial $C^{(n)}_{L+k}$)

$$\begin{aligned}&\wp^D_L\big((1+\lambda_1(t))^{k/2}\phi'_n(t)\big)\\ &\qquad \leq C^{(n)}_{L+k}\Big(\|u\|_{E^\rho_{N_0+L+k}} + \sum_{\substack{i_1+i_2=L+j\\ 0\leq j\leq k-1}} \|u\|_{E^\rho_{N_0+i_1}}\|u\|_{E^\rho_{N_0+i_2+k-j}}\Big), \quad n \geq 1,\end{aligned}$$

which together with Corollary 2.6 proves that inequality (6.9) is true for every $N \geq 0$.

Similarly differentiation of both members of equation (6.4) l times and induction give that

$$\begin{aligned}&\wp^D_L\big((1+\lambda_1(t))^{k/2}(D^l\phi'_n(u;v_1,\ldots,v_l))(t)\big) \\ &\quad \leq F_{L,l,k}(\|u\|_{E^\rho_{N_0}})\mathcal{R}^l_{N_0,L+k}(v_1,\ldots,v_l) + F'_{L,l,k}(\|u\|_{E^\rho_{N_0}})\|u\|_{E^\rho_{N_0+L+k}}\|v_1\|_{E^\rho_{N_0}}\cdots\|v_l\|_{E^\rho_{N_0}},\end{aligned} \tag{6.10}$$

$L \geq 0, l \geq 0, k \geq 1$, where $F_{L,l,k}$ and $F'_{L,l,k}$ are polynomials depending on $n \geq 0$.

To prove the estimate of $\|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2}\phi'_{n,Y}(t)\|_{L^\infty}, Y \in \Pi'$, we use Theorem 5.8 with $G = 0$, equation (6.4) and note that the hypothesis of Theorem 5.8 are satisfied due to Theorem 4.9. With the notation

$$Q_{n,k,L}(t) = \sum_{\substack{Y\in\Pi'\\ |Y|\leq L}} \|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2}\phi'_{n,Y}(t)\|_{L^\infty}, \tag{6.11}$$

$n \geq 0, k \geq 0, L \geq 0$, we then obtain that

$$\begin{aligned}Q_{n,k,L}(t) \leq C_{k+L}\Big(&\wp^D_{k+L}(\phi'_n(t)) + \sum_{i+j\leq k+L+7}\wp^D_i\big(\delta(t)(1+\lambda_1(t))^{j/2}g'_n(t)\big)\\ &+ \sum_{i+j\leq k+L+9}\wp^D_i((1+\lambda_1(t))^{j/2}g_n(t))\\ &+ \sum_{\substack{Y\in\Pi'\\ |Y|+j\leq k+L}}\|(\delta(t))^{3/2}(1+\lambda_1(t))^{j/2}g_{n,Y}(t)\|_{L^\infty}\Big), \quad n \geq 0, k \geq 0, L \geq 0,\end{aligned} \tag{6.12}$$

where $g'_n = (2m)^{-1}(m - i\gamma^\mu\partial_\mu)g_n$, $g_{n,Y} = \xi^D_Y g_n$, $g'_{n,Y} = \xi^D_Y g'_n$ and where C_{k+L} is a numerical constant. Similarly as we obtained (6.7) it follows that

$$\begin{aligned}&\sum_{i+j\leq k+L+9}\wp^D_i\big((1+\lambda_1(t))^{j/2}g_n(t)\big)\\ &\qquad \leq C'_{L+k}\sum_{i+j\leq k+L+9}\;\sum_{i_1+i_2=i}\|u\|_{E^\rho_{N_0+i_1}}\wp^D_{i_2}\big((1+\lambda_1(t))^{(j-1)/2}\phi'_{n-1}(t)\big),\end{aligned}$$

where C'_{L+k} is a polynomial in $\|u\|_{E^{\rho}_{N_0}}$. This inequality, Corollary 2.6 and inequality (6.9), which we have proved is true for all $N \geq 0$, give that

$$\sum_{i+j\leq L+k+9} \wp^D_i\big((1+\lambda_1(t))^{j/2}g_n(t)\big) \leq C^{(n)}_{L+k}\|u\|_{E^{\rho}_{N_0}}\|u\|_{E^{\rho}_{N_0+k+L+9}}, \quad n\geq 0, k\geq 0, L\geq 0, \tag{6.13}$$

where $C^{(n)}_{L+k}$ is a polynomial in $\|u\|_{E^{\rho}_{N_0}}$. Similarly it follows that

$$\begin{aligned}\sum_{\substack{Y\in\Pi' \\ |Y|+j\leq k+L}} &\|(\delta(t))^{3/2}(1+\lambda_1(t))^{j/2}g_{n,Y}(t)\|_{L^\infty} \\ &\leq C'_{k+L}\sum_{i+j\leq k+L}\ \sum_{i_1+|Y_2|=i}\|u\|_{E^{\rho}_{N_0+i_1}}\|\delta(t)(1+\lambda_1(t))^{j/2}\phi'_{n-1,Y_2}(t)\|_{L^\infty},\end{aligned} \tag{6.14}$$

$n\geq 0, k\geq 0, L\geq 0$, where C'_{k+L} is a polynomial in $E^{\rho}_{N_0}$. To estimate the term with g'_n in (6.12), we note that according to equality (5.7a) (with $G=0$)

$$\begin{aligned}(m-i\gamma^\mu\partial_\mu)g_n = {}&\gamma^\nu(A_{n-1,\nu}+B_{n-1,\nu})(m+i\gamma^\mu\partial_\mu)\phi'_{n-1} \\ &-2i(A^\mu_{n-1}+B^\mu_{n-1})\partial_\mu\phi'_{n-1} - i\phi'_{n-1}\partial_\mu(A^\mu_{n-1}+B^\mu_{n-1}) \\ &-\frac{i}{4}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)\phi'_{n-1}(\partial_\mu A_{n-1,\nu}-\partial_\nu A_{n-1,\mu}), \quad n\geq 1,\end{aligned} \tag{6.15}$$

where we have used the gauge invariance of the last term. The first term on the right-hand side vanish when $n=1$, since $(m+i\gamma^\mu\partial_\mu)\phi'_0=0$. According to Theorem 4.9 and Corollary 4.12 it follows from equality (6.15) that

$$\begin{aligned}&\|\delta(t)(1+\lambda_1(t))^{j/2}g'_{n,Y}(t)\|_D \\ &\quad\leq C^{(n)}_{j,|Y|}\sum^{Y}_{Y_1,Y_2}\Big(\|u\|_{E^{\rho}_{N_0+|Y_1|}}\|(\delta(t))^{1/2}(1+\lambda_1(t))^{j/2}(i\gamma^\mu\partial_\mu+m)\phi'_{n-1,Y_2}(t)\|_D \\ &\qquad+\|u\|_{E^{\rho}_{N_0+|Y_1|}}\|(1+\lambda_1(t))^{j/2}\phi'_{n-1,Y_2}(t)\|_D\Big) \\ &\qquad+2\sum^{Y}_{Y_1,Y_2}\|\delta(t)(1+\lambda_1(t))^{j/2}(A^\mu_{n-1,Y_1}(t)+B^\mu_{n-1,Y_1}(t))\partial_\mu\phi'_{n-1,Y_2}(t)\|_D,\end{aligned} \tag{6.16}$$

where $Y\in\Pi', n\geq 1$, and where $C^{(n)}_{j,|Y|}$ is a polynomial in $\|u|_{E^{\rho}_{N_0}}$. If $Y_1\in\boldsymbol{\sigma}^1$ in the last term in the right-hand side of inequality (6.16), then it follows from Theorem 4.9 and Corollary 4.12 that

$$\begin{aligned}&\|\delta(t)(1+\lambda_1(t))^{j/2}(A^\mu_{n-1,Y_1}(t)+B^\mu_{n-1,Y_1}(t))\partial_\mu\phi'_{n-1,Y_2}(t)\|_D \\ &\quad\leq C^{(n)}_{|Y_1|}\|u\|_{E^{\rho}_{N_0+|Y_1|}}\sum_{0\leq\mu\leq 3}\|(1+\lambda_1(t))^{j/2}\phi'_{n-1,P_\mu Y_2}(t)\|_D,\end{aligned} \tag{6.17}$$

where $Y_1 \in \boldsymbol{\sigma}^1$ and where $C^{(n)}_{|Y_1|}$ is a polynomial in $\|u\|_{E^\rho_{N_0}}$. If $Y_1 \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$, then $y_\mu(A_{n-1,Y_1} + B_{n-1,Y_1})^\mu(y) = 0$. It therefore follows from inequality (5.7d), the definition of $\xi^D_{M_{\mu\nu}}$, Theorem 4.9 and Corollary 4.12 that

$$\begin{aligned}
&\|\delta(t)(1+\lambda_1(t))^{j/2}(A^\mu_{n-1,Y_1}(t) + B^\mu_{n-1,Y_1}(t))\partial_\mu \phi'_{n-1,Y_2}(t)\|_D \qquad (6.18)\\
&\quad \le C^{(n)}_{|Y_1|}\|u\|_{E^\rho_{N_0+|Y_1|}}\Big(\sum_{0\le\mu\le3}\|(1+\lambda_1(t))^{j/2}\phi'_{n-1,P_\mu Y_2}(t)\|_D \\
&\qquad + \sum_{0\le\mu<\nu\le3}\|(1+\lambda_1(t))^{j/2}\phi'_{n-1,M_{\mu\nu}Y_2}(t)\|_D + \|(1+\lambda_1(t))^{j/2}\phi'_{n-1,Y_2}(t)\|_D\Big),
\end{aligned}$$

where $Y_1 \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$ and where $C^{(n)}_{|Y_1|}$ is a polynomial in $\|u\|_{E^\rho_{N_0}}$. Since $(i\gamma^\mu\partial_\mu + m)\phi'_0 = 0$, it follows from equation (6.4), Theorem 4.9 and Corollary 4.12 that

$$\begin{aligned}
&\|(\delta(t))^{1/2}(1+\lambda_1(t))^{j/2}(i\gamma^\mu\partial_\mu + m)\phi'_{n-1,Z}(t)\|_D \qquad (6.19)\\
&\quad \le C^{(n)}_{|Z|}\sum_{i_1+i_2=|Z|}|u|_{E^\rho_{N_0+i_1}}\wp^D_{i_2}\big((1+\lambda_1(t))^{j/2}\phi'_{n-2}(t)\big), \quad n\ge1,
\end{aligned}$$

where $Z \in \Pi'$, $C^{(n)}_{|Z|}$ is a polynomial in $\|u\|_{E^\rho_{N_0}}$ and where the right-hand side is given the value zero when $n=1$. Inequalities (6.16), (6.17), (6.18) and (6.19) give that

$$\begin{aligned}
&\|\delta(t)(1+\lambda_1(t))^{j/2}g'_{n,Y}(t)\|_D \qquad (6.20)\\
&\quad \le C^{(n)}_{j,|Y|}\Big(\sum_{i_1+i_2+i_3=|Y|}\|u\|_{E^\rho_{N_0+i_1}}\|u\|_{E^\rho_{N_0+i_2}}\wp^D_{i_3}\big((1+\lambda_1(t))^{j/2}\phi'_{n-2}(t)\big) \\
&\qquad + \sum_{i_1+i_2=|Y|}\|u\|_{E^\rho_{N_0+i_1}}\wp^D_{i_2}\big((1+\lambda_1(t))^{j/2}\phi'_{n-1}(t)\big)\Big), \quad n\ge1, Y\in\Pi',
\end{aligned}$$

where the sum over $i_1+i_2+i_3=|Y|$ is absent when $n=1$ and where $C^{(n)}_{j,|Y|}$ is a polynomial in $\|u\|_{E^\rho_{N_0}}$. The already proved inequality (6.9), inequality (6.20) and Corollary 2.6 give that

$$\|\delta(t)(1+\lambda_1(t))^{j/2}g'_{n,Y}(t)\|_D \le C^{(n)}_{j+|Y|}\|u\|_{E^\rho_{N_0}}\|u\|_{E^\rho_{N_0+j+|Y|+1}}, \qquad (6.21)$$

$Y\in\Pi'$, $n\ge1$, $j\ge0$, where $C^{(n)}_{j+|Y|}$ is a polynomial in $\|u\|_{E^\rho_{N_0}}$. It follows from inequalities (6.9), (6.12), (6.13), (6.14) and (6.21) that

$$\begin{aligned}
&Q_{n,k,L}(t) \qquad (6.22a)\\
&\quad \le C^{(n)}_{k+L}\Big(\|u\|_{E^\rho_{N_0+L+k+9}} + \sum_{\substack{Z\in\Pi' \\ i+j+|Z|\le k+L}}\|u\|_{E^\rho_{N_0+i}}\|\delta(t)(1+\lambda_1(t))^{j/2}\phi'_{n-1,Z}(t)\|_{L^\infty}\Big),
\end{aligned}$$

$n\ge1$, and

$$Q_{0,k,L}(t) \le C^{(0)}_{k+L}\|u\|_{E^\rho_{N_0+L+k+9}}. \qquad (6.22b)$$

It follows by induction from (6.22a) and (6.22b) that

$$Q_{n,k,L}(t) \leq C^{(n)}_{k+L} \|u\|_{E^\rho_{N_0(n)+L+k}}, \tag{6.23}$$

where $n \geq 0, k \geq 0, L \geq 0$, where $C^{(n)}_{k+L}$ is a polynomial in $\|u\|_{E^\rho_{N_0}}$ and where $N_0(n) \in \mathbb{N}$ depends on n. Inequality (6.23) shows that the estimate of

$$\|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2}\phi'_{n,Y}(t)\|_{L^\infty}$$

given by the lemma is true.

The proof of the announced L^∞-estimates of the derivatives $D^l\phi'_{n,Y}$ is so similar to the proof of inequality (6.23) that we omit it. Also the L^2- and L^∞-estimates of $D^l(\phi'_{n+1,Y} - \phi'_{n,Y})$ are obtained so similarly by using Theorem 4.10, that we omit the proofs. This ends the proof of Lemma 6.1.

Next we shall prove the existence of ϕ^*_j and A^*_j for $j=0$ and $j=1$, which permits to prove the existence for $j \geq 2$ by a contraction theorem. Denote $A^*_{j,Y} = \xi^M_Y A^*_j$ and $\phi^*_{j,Y} = \xi^D_Y \phi^*_j$, (c.f. (4.81d)).

Proposition 6.2. *Let $u = (f, \dot{f}, \alpha) \in E^{\circ\rho}_\infty, 1/2 < \rho < 1$, and $0 \leq \rho' \leq 1$. Then equation (6.1a) has a unique solution $\phi^*_0 \in C^0(\mathbb{R}^+, D)$ and equation (6.2a) has a unique solution $\phi^*_1 \in C^0(\mathbb{R}^+, D)$. Moreover there exists N_0 such that*

$$\begin{aligned}
&\sup_{t\geq 0}\Big(\|\big((D^l(A^*_{j,Y}, A^*_{j,P_0Y}))(u;v_1,\ldots,v_l)\big)(t)\|_{M^\rho_0} \\
&\quad + \sum_{n+k\leq L} \wp^D_n\big((1+\lambda_1(t))^{k/2}\big((D^l\phi^*_j)(u;v_1,\ldots,v_l)\big)(t)\big) \\
&\quad + (1+t)^{1+\rho'-\rho}\|\big((D^l(\Delta^{*M}_{0,Y}, \Delta^{*M}_{0,P_0Y}))(u;v_1,\ldots,v_l)\big)(t)\|_{M^{\rho'}_0} \\
&\quad + (1+t)^{3/2-\rho}\|\big((D^l\Delta^{*D}_{0,Y})(u;v_1,\ldots,v_l)\big)(t)\|_D \\
&\quad + (1+t)^{2-\rho}\|(1+t+|\cdot|)\big(\Box\big((D^l\Delta^{*M}_{0,Y})(u;v_1,\ldots,v_l)\big)\big)(t)\|_{L^2} \\
&\quad + (1+t)^{1-\rho}\|(1+t+|\cdot|)\big(\Box\big((D^l\Delta^{*M}_{0,Y})(u;v_1,\ldots,v_l)\big)\big)(t)\|_{L^{6/5}}\Big) \\
&+ \sup_{t\geq 0, x\in\mathbb{R}^3}\Big((1+t+|x|)^{3/2-\rho}|\big((D^lA^*_{j,Y})(u;v_1,\ldots,v_l)\big)(t,x)| \\
&\quad + (1+t+|x|)(1+\big|t-|x|\big|)^{1/2}(1+\big|t-|x|\big|_\delta)^{1-\rho}|\big((D^lA^*_{j,P_0Y})(u;v_1,\ldots,v_l)\big)(t,x)| \\
&\quad + \sum_{\substack{Z\in\Pi' \\ |Z|+k\leq L}} \big|(1+t+|x|)^{3/2}(1+(\lambda_1(t))(x))^{k/2}\big((D^l\phi^*_{j,Z})(u;v_1,\ldots,v_l)\big)(t,x)\big|\Big) \\
&\quad \leq C_{L,l}\Big(\mathcal{R}^l_{N_0,L}(v_1,\ldots,v_l) + \|u\|_{E^\rho_{N_0+L}}\|v_1\|_{E^\rho_{N_0}}\cdots\|v_l\|_{E^\rho_{N_0}}\Big),
\end{aligned}$$

for every $j = 0,1, \delta > 0, Y \in \Pi', |Y| \leq L, l \geq 0,\ u, v_1, \ldots, v_l \in E^{\circ\rho}_\infty$, where $C_{L,l}$ is a constant depending only on $\|u\|_{E^\rho_{N_0}}$ and δ. Here $\big|t-|x|\big|_\delta = \big|t-|x|\big|$ for $|x| \leq \delta t$ and

$\big|t-|x|\big|_\delta = 0$ *for* $|x| > \delta t$. *Moreover* $\partial_\mu A_j^{*\mu} = 0$, $j = 0, 1$, *and, as functions of* $u \in E_\infty^{0\rho}$, (A^*, ϕ^*) *has a zero of order at least* 1 *and* $(\Delta^{*M}, \Delta^{*D})$ *has a zero of order at least* 2 *at* $u = 0$.

Proof. If $\phi_0^* \in C^0(\mathbb{R}^+, D)$, satisfying $\|\phi_0^*(t) - e^{t\mathcal{D}}\alpha\|_D \to 0$ when $t \to \infty$, exists, then it is unique. In fact, if w is the difference between two solutions of equation (6.1a) then $(i\gamma^\mu\partial_\mu + m - \gamma^\mu(A_{J,\mu} + B_{J,\mu}))w = 0$ and $\|w(t)\|_D \to 0$ when $t \to \infty$. It then follows from Theorem 4.9 and Corollary 5.17 that $w = 0$.

It follows from equation (4.137c) that

$$(\phi_0^* - \phi'_{J+1})(t) = i\int_t^\infty e^{(t-s)\mathcal{D}}\big((A_{J,\mu} + B_{J,\mu})\gamma^0\gamma^\mu(\phi_0^* - \phi'_J)\big)(s)ds,$$

and then from the relation (4.140b) that

$$\begin{aligned}(\phi_0^* - \phi'_J - \Delta_J^D)(t) = i\int_t^\infty & e^{(t-s)\mathcal{D}}\big((A_{J,\mu} + B_{J,\mu})\gamma^0\gamma^\mu\Delta_J^D\big)(s)ds \\ &+ i\int_t^\infty e^{(t-s)\mathcal{D}}\big((A_{J,\mu} + B_{J,\mu})\gamma^0\gamma^\mu(\phi_0^* - \phi'_J - \Delta_J^D)\big)(s)ds.\end{aligned}$$

This shows that

$$\big(i\gamma^\mu\partial_\mu + m - \gamma^\mu(A_{J,\mu} + B_{J,\mu})\big)(\phi_0^* - \phi'_J - \Delta_J^D) = \gamma^\mu(A_{J,\mu} + B_{J,\mu})\Delta_J^D. \tag{6.24}$$

It follows from Theorem 4.9 that $[A_J]^l(t) \le C_l\|u\|_{E_{N_0+l}}$, where N_0 is an integer independent of u and l and where C_l is a constant depending only on $\|u\|_{E_{N_0}}$. Statement ii) of Corollary 5.18, equation (6.24) and the relation $\phi_0^* - \phi'_J - \Delta_J^D = \phi_0^* - \phi'_{J+1}$, then give

$$\begin{aligned}\wp_n^D\big(\phi_0^*(t) - \phi'_{J+1}(t)\big) \le C_n \sum_{0\le l\le n} \|u\|_{E_{N_0+5+n-l}} \int_t^\infty & \Big((1+s)^{-5/2+\rho}\wp_{l+1}^D(\Delta_J^D(s)) \\ &+ (1+s)^{-3/2}\wp_l^D\big((1+\lambda_1(s))^{1/2}\Delta_J^D(s)\big) \\ &+ (1+s)^{-3+2\rho}\wp_l^D(\Delta_J^D(s)) \\ &+ (1+s)^{-3/2+\rho}\wp_l^D\big(((i\gamma^\mu\partial_\mu + m)\Delta_J^D)(s)\big)\Big)ds.\end{aligned} \tag{6.25}$$

Since $\Delta_J^D = \phi'_{J+1} - \phi'_J$, it follows from (4.137c) that

$$(i\gamma^\mu\partial_\mu + m)\Delta_J^D = \gamma^\mu(A_{J,\mu} + B_{J,\mu})\phi'_J - \gamma^\mu(A_{J-1,\mu} + B_{J-1,\mu})\phi'_{J-1}.$$

This equality, Theorem 4.9, Theorem 4.10, statement ii) of Lemma 4.5 and Corollary 2.6 give that

$$\wp_n^D\big(((i\gamma^\mu\partial_\mu + m)\Delta_J^D)(t)\big) \le C_n(1+t)^{-1}\|u\|_{E_{N_0}}\|u\|_{E_{N_0+n}},$$

where C_n is a polynomial in $\|u\|_{E_{N_0}}$. This inequality, inequality (6.25), Theorem 4.9, Lemma 6.1 and Corollary 2.6 then give (after redefining N_0)

$$\wp_n^D\big(\phi_0^*(t)-\phi'_{J+1}(t)\big)\leq(1+t)^{-3/2+\rho}C_n\|u\|_{E_{N_0}}\|u\|_{E_{N_0+n}},\quad n\geq 0,t\geq 0. \tag{6.26a}$$

Differentiation of equation (6.24) gives similarly by induction

$$\begin{aligned}&\wp_n^D\big(\big((D^l(\phi_0^*-\phi'_{J+1}))(u;v_1,\dots,v_l)\big)(t)\big)\\&\quad\leq C_{n,l}(1+t)^{-3/2+\rho}\Big(\mathcal{R}^l_{N_0,n}(v_1,\dots,v_l)+\|u\|_{E_{N_0+n}}\|v_1\|_{E_{N_0}}\cdots\|v_l\|_{E_{N_0}}\Big),\end{aligned} \tag{6.26b}$$

$n\geq 0,l\geq 0,t\geq 0$, where $C_{n,l}$ depends only on $\|u\|_{E_{N_0}}$. Inequality (6.26b), the estimates of $\wp_n^D\big(\big((D^l\phi'_J)(u;v_1,\dots,v_l)\big)(t)\big)$ given by Theorem 4.9 prove the estimate of $\|\big((D^l\phi^*_{0,Y})(u;v_1,\dots,v_l)\big)(t)\|_D$ given by the inequality of the proposition.

The announced L^2- and L^∞-estimates of $(1+\lambda_1(t))^{k/2}\big((D^l\phi^*_{0,Y})(u;v_1,\dots,v_l)\big)(t)$ follow in a such a similar way as in the proof of Lemma 6.1 that we omit the details. The M^ρ-estimate of $(A^*_{0,Y},A^*_{0,P_0Y})$ follows from the definition of A_0^* and from the already proved estimates of ϕ^*. It follows from definition (4.138a) that

$$\begin{aligned}&A^*_{0,\mu}(t)-A_{J+1,\mu}(t)-\Delta^M_{J+1,\mu}(t)\\&\quad=-\int_t^\infty|\nabla|^{-1}\sin(|\nabla|(t-s))(\overline{\phi}^*_0\gamma_\mu\phi_0^*-\overline{\phi}'_{J+1}\gamma_\mu\phi'_{J+1})(s)ds.\end{aligned} \tag{6.27}$$

This gives

$$\begin{aligned}&\wp_n^{M^{\rho'}}\big((A_0^*-A_{J+1}-\Delta^M_{J+1})(t),\frac{d}{dt}(A_0^*-A_{J+1}-\Delta^M_{J+1})(t)\big)\\&\quad\leq C_n(1+t)^{-1-\rho'+\rho}\|u\|_{E_{N_0+n}},\quad t\geq 0,0\leq\rho\leq\rho',\end{aligned} \tag{6.28}$$

where C_n depends only on $\|u\|_{E_{N_0}}$ and where we have used (6.26a), (6.27), the already proved L^2- and L^∞-estimates of ϕ_0^*, those given by Theorem 4.9 of ϕ'_{J+1} and the convexity properties of the seminorms given by Corollary 2.6. Estimate (6.28), Theorem 4.9 and Theorem 4.11 give the L^2-estimates of $\Delta^{*M}_{0,Y}$ in the proposition. Using statement ii), statement iii) and statement iv) of Proposition 5.6, we obtain that

$$\begin{aligned}&(1+t+|x|)(1+\big|t-|x|\big|)^{1/2}|(A^*_{0,Y}-A_{J+1,Y}-\Delta^M_{J+1,Y})(t,x)|\\&\quad\leq C\Big(\wp^M_{|Y|+2}\big((A_0^*-A_{J+1}-\Delta^M_{J+1})(t),\frac{d}{dt}(A_0^*-A_{J+1}-\Delta^M_{J+1})(t)\big)\\&\qquad+\sum_{|Z|\leq|Y|+2}(1+t)\|(\xi^M_ZG)(t)\|_{L^{6/5}(\mathbb{R}^3,\mathbb{R}^4)}\Big),\end{aligned}$$

where $G_\mu=\overline{\phi}^*_0\gamma_\mu\phi_0^*-\overline{\phi}'_{J+1}\gamma_\mu\overline{\phi}'_{J+1}$ is obtained using equation (6.27). The estimates for ϕ_0^* and ϕ'_{J+1} give

$$(1+t)\|(\xi^M_ZG)(t)\|_{L^{6/5}}\leq C_{|Z|}\|u\|_{E_{N_0+|Z|+2}}(1+t)^{\rho-1}.$$

It then follows from inequality (6.28) that

$$(1+t)^{1-\rho}(1+t+|x|)(1+\big|t-|x|\big|)^{1/2}|(A^*_{0,Y}-A_{J+1,Y}-\Delta^M_{J+1,Y})(t,x)| \qquad (6.29)$$
$$\le C_{|Y|}\|u\|_{E_{N_0+|Y|}}, \quad t\ge 0, x\in\mathbb{R}^3, Y\in\Pi',$$

where $C_{|Y|}$ depends only on $\|u\|_{E_{N_0}}$, and where N_0+2 has been redefined by N_0. Since $(1+t)^{1-\rho}(1+t+|x|)(1+\big|t-|x|\big|)^{1/2}\ge(1+t+|x|)^{3/2-\rho}$ for $(t,x)\in\mathbb{R}^+\times\mathbb{R}^3$ the estimate for L^∞-norms of $A^*_{0,Y}$ and $A^*_{0,P_\mu Y}$ in the proposition follows from inequality (6.29) and from Theorem 4.9 and Theorem 4.11. Using Theorem 4.10, Theorem 4.11 with interpolation and inequality (6.8) we obtain that

$$\wp^{M^{\rho'}}_n\big((A^*_0-A_J)(t),\frac{d}{dt}(A^*_0-A_J)(t)\big)\le C_n(1+t)^{-1-\rho'+\rho}\|u\|_{E_{N_0+n}},$$

$n\ge 0, t\ge 0, 0\le\rho'\le 1$. The proof of the existence and the properties of ϕ^*_1, A^*_1 and Δ^*_0 are so similar to that of ϕ^*_0 and A^*_0 that we omit it. That the gauge conditions $\partial_\mu A^{*\mu}_j=0$ is satisfied follows directly from the fact that

$$\square i\partial_\mu A^{*\mu}_j=(\overline{i\gamma^\mu\partial_\mu\phi^*_j})\phi^*_j+\overline{\phi}^*_j(i\gamma^\mu\partial_\mu\phi^*_j)=0,$$

by the equation satisfied by ϕ^*_j and by the fact that $(f,\dot f)\in M^{\circ\rho}$. This proves the proposition.

We now make the following variable transformation in the M-D equations:

$$\psi'=e^{i\vartheta(A)}\psi,\quad \Phi=\psi'-\phi^*,\quad K=A-A^*,\quad \phi^*=\phi^*_1,\quad A^*=A^*_1, \qquad (6.30)$$

and denote $\Delta^{*M}_\mu=\Delta^{*M}_{0,\mu}$. The M-D equations are then transformed into

$$\big(i\gamma^\mu\partial_\mu+m-\gamma^\mu(K_\mu+A^*_\mu-\partial_\mu\vartheta(K+A^*))\big)\Phi \qquad (6.31a)$$
$$=\gamma^\mu\big(K_\mu+\Delta^{*M}_\mu-\partial_\mu\vartheta(K+\Delta^{*M})\big)\phi^*,$$
$$\square K_\mu=\overline{\Phi}\gamma_\mu\Phi+\overline{\Phi}\gamma_\mu\phi^*+\overline{\phi}^*\gamma_\mu\Phi \qquad (6.31b)$$

and

$$\partial_\mu K^\mu=0. \qquad (6.31c)$$

We introduce the Banach space $\mathcal{F}_N, N\ge 0$, which is the completion of the space of all functions (K,Φ) such that $K_Y\in C^0(\mathbb{R}^+,L^2)$, $K_{P_0Y}\in C^0(\mathbb{R}^+,|\nabla|L^2)$, $K_{P_\mu Y}\in C^0(\mathbb{R}^+,L^2)$, $\delta\square K_Y\in C^0(\mathbb{R}^+,L^2)$,$\delta\square K_Y\in C^0(\mathbb{R}^+,L^{6/5})$, $\Phi_Y\in C^0(\mathbb{R}^+,D)$ for $Y\in\Pi', |Y|\le N$, and such that $\|(K,\Phi)\|_{\mathcal{F}_N}<\infty$, with respect to the norm $\|\cdot\|_{\mathcal{F}_N}$ defined by

$$(e_N(K,\Phi))(t) \qquad (6.32a)$$
$$=(1+t)^{1-\rho}\wp^{M^0}_N(K(t),\dot K(t))+(1+t)^{2-\rho}\wp^{M^1}_N(K(t),\dot K(t))+(1+t)^{3/2-\rho}\wp^D_N(\Phi(t))$$
$$+\Big(\sum_{\substack{Y\in\Pi'\\|Y|\le N}}\big((1+t)^{2-\rho}\|\delta(t)\square K_Y(t)\|_{L^2}+(1+t)^{1-\rho}\|\delta(t)\square K_Y(t)\|_{L^{6/5}}\big)^2\Big)^{1/2}$$

and

$$\|(K,\Phi)\|_{\mathcal{F}_N} = \sup_{t\geq 0}(e_N(K,\Phi))(t), \tag{6.32b}$$

where $(\delta(t))(x) = 1+t+|x|, 1/2<\rho<1, K_Y = \xi_Y^M K, \dot{K}_Y = K_{P_0 Y}, \Phi_Y = \xi_Y^D \Phi$ for $Y \in \Pi'$ (c.f. (4.81d)). We denote by $\mathcal{F}_N^M$ the subspace of elements $(K,0)$ and by $\mathcal{F}_N^D$ the subspace of elements $(0,\Phi)$.

If N is sufficiently large and $K \in \mathcal{F}_N^M$, then equation (6.31a) has a unique solution $\Phi \in C^0(\mathbb{R}^+, D)$ satisfying $\lim_{t\to\infty}\|\Phi(t)\|_D = 0$. It will be proved that the equation

$$\Box K'_\mu = \overline{\Phi}\gamma_\mu\Phi + \overline{\Phi}\gamma_\mu\phi^* + \overline{\phi}^*\gamma_\mu\Phi, \quad K' \in \mathcal{F}_N^M, \tag{6.33}$$

has a unique solution for N sufficiently large and that $(u,K)\mapsto K' = \mathcal{N}(u,K)$ defines a map $\mathcal{N}\colon E_\infty^{\circ\rho}\times\mathcal{F}_N^M \to \mathcal{F}_N^M$. K' depends on K via Φ in equation (6.33) and depends on u via A^*, ϕ^* and Φ. For u and K sufficiently small this map turns out to be a contraction map (in the variable K).

The definition of the space $\mathcal{F}_N^M$, gives space-time decrease properties of the absolute value of its elements.

Lemma 6.3. *There exists $C_\rho > 0$ such that*

$$(1+t+|x|)(1+t)^{1-\rho}(1+\big|t-|x|\big|)^{1/2}|K_Y(t,x)| \leq C_\rho\|K\|_{\mathcal{F}^M_{|Y|+2}},$$

for $Y\in\Pi', |Y|\leq N, t\in\mathbb{R}^+, x\in\mathbb{R}^3, K\in\mathcal{F}^M_{N+2}$.

Proof. Statement ii) of Proposition 5.6, with $F_{Y(t)} = \Box K_Y(t)$, gives that

$$\begin{aligned}(1+t)^{3/2}|K_Y(t,x)| \leq C_\delta\Big(\sum_{\substack{|Y|\leq 2\\ Z\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C}))}} &\big(\|(K_{ZY}(t), K_{P_0 ZY}(t))\|_{M^0} \\ &+ (1+t)\|\Box K_{ZY}(t)\|_{L^{6/5}}\big) + \|(1-\Delta)K_Y(t)\|_{L^2}\Big),\end{aligned} \tag{6.34}$$

for $0\leq|x|\leq\delta t, t\geq 0$, where $0<\delta<1$ and where C_δ is a numerical constant. Definitions (6.32a) and (6.32b) of the norm in $\mathcal{F}_N^M$, then give that

$$(1+t)^{3/2}|K_Y(t,x)| \leq C_\delta(1+t)^{-1+\rho}\|K\|_{\mathcal{F}^M_{|Y|+2}}, \tag{6.35a}$$

for $0\leq|x|\leq\delta t, t\geq 0$, where C_δ is a new constant.

Similarly it follows from statement iii) and iv) of Proposition 5.6 that

$$(1+t)(1+\big|t-|x|\big|)^{1/2}|K_Y(t,x)| \leq C_{\delta_1,\delta_2}(1+t)^{-1+\rho}\|K\|_{\mathcal{F}^M_{|Y|+2}}, \tag{6.35b}$$

for $\delta_1 t\leq|x|\leq\delta_2 t, t\geq 0$, where $0<\delta_1<1<\delta_2$, and that

$$(1+|x|)^{3/2}|K_Y(t,x)| \leq C_\delta(1+t)^{-1+\rho}\|K\|_{\mathcal{F}^M_{|Y|+2}}, \tag{6.35c}$$

for $0 \le t \le \delta|x|, |x| \ge 0$, where $0 < \delta < 1$. The inequality of the lemma now follows by choosing suitably δ, δ_1 and δ_2. This proves the lemma.

In order to study the map $\mathcal{N}$ we consider the equation

$$\big(i\gamma^\mu \partial_\mu + m - \gamma^\mu (K_\mu + A^*_\mu - \partial_\mu \vartheta(K + A^*))\big)\Psi = g, \tag{6.36}$$

for $\Psi \in \mathcal{F}^D_N$, where $K \in \mathcal{F}^M_N$. Most often $g = \gamma^\mu\ (L_\mu - \partial_\mu \vartheta(L)\phi^*, L \in \mathcal{F}^M_N$. We shall use in next proposition the notation

$$\chi^{(n)} = \|u\|_{E^\rho_{N_0+n}} + \|K\|_{\mathcal{F}^M_n}, \quad n \ge 0, \tag{6.37}$$

where N_0 is given by Proposition 6.2 and where $u \in E^\rho_\infty$ and $K \in \mathcal{F}^M_n$ and $\chi_{N,n}$, $N \ge 0$, $n \ge 0$ is defined by $\chi_{N,n} = b_n$ given by formula (5.87) with $a^{(n)} = \chi^{(N+n)}$. It follows from this definition that $\chi_{N,n} < \infty$ if $u \in E^\rho_\infty$ and $K \in \mathcal{F}^M_{N+n}$. We note that according to Proposition 6.2 and Lemma 6.3

$$[A^*]^n(t) \le C_n \|u\|_{E^\rho_{N_0+n}}, \quad [K]^n(t) \le C\|K\|_{\mathcal{F}^M_{n+2}}, \quad t \ge 0, \tag{6.38a}$$

where C_n depends only on ρ and (polynomially) on $\|u\|_{E^\rho_{N_0}}$ and C depends only on ρ, $1/2 < \rho < 1$. Using Corollary 2.6 we then obtain that

$$[A^*]_{N,n}(t) \le C_{n+N}\|u\|_{E^\rho_{N_0+N+n}}, \quad N, n \ge 0, t \ge 0, \tag{6.38b}$$

where C_{n+N} depends only on ρ and (polynomially) on $\|u\|_{E^\rho_{N_0+N}}$. Moreover, it follows from inequalities (6.38a) that

$$[A^* + K]^n(t) \le C_n \chi^{(n+2)}, \quad [A^* + K]_{N,n}(t) \le C_{N+n}\chi_{N+2,n}, \tag{6.38c}$$

for $t \ge 0, n \ge 0, N \ge 0$, where C_n and C_{N+n} depends only on ρ and (polynomially) on $\|u\|_{E^\rho_{N_0}}$.

In order to state next proposition we introduce the following notations:

$$\overline{Q}_n(t) = \sup_{s \ge t} \big((1+s)^{3/2-\rho} \wp^D_n(f(s))\big) + Q'_n(t), \tag{6.39a}$$

$$Q'_n(t) = \sum_{\substack{n_1+n_2=n \\ n_2 \le n-1}} \chi_{5,n_1} \sup_{s \ge t} \big((1+s)^{3/2-\rho} \wp^D_{n_2}(f(s)) + (1+s)\wp^D_{n_2}((1+\lambda_0(s))^{1/2} g(s))\big), \quad n \ge 0, t \ge 0, \tag{6.39b}$$

where f is defined by (5.111b) for equation (6.36),

$$R^{(1)}_n(t) = \sup_{s \ge t} \big((1+s)^{3/2-\rho}(R'_{n+7}(s) + R^2_{n+9}(s) + R^\infty_n(s))\big) + \overline{Q}_{n+8}(t), \tag{6.39c}$$

$n \geq 0$, $t \geq 0$, where R'_n, R^2_n, R^∞_n are defined in Theorem 5.8,

$$h'_n(L,t) = \sum_{\substack{n_1+n_2+n_3+n_4=n \\ n_1 \leq n-1, n_2 \leq L-1 \\ n_3+n_4 \leq n-L}} \overline{\chi}^{(n_1)}(1+\chi_{4,n_2})(1+\chi_{10,n_3})R^{(1)}_{n_4}(t) \tag{6.39d}$$
$$+ \sum_{\substack{n_1+n_2=n \\ 1 \leq n_1 \leq L \\ n_2 \leq L}} \chi_{5,n_1}\big(\overline{Q}_{n_2}(t) + \sup_{s \geq t}\big((1+s)\wp^D_{n_2}((1+\lambda_0(s))^{1/2}g(s))\big)\big),$$

$n \geq 0, L \geq 0, t \geq 0$, where $\overline{\chi}^{(n)} = \|u\|_{E^\rho_{N_0+n+1}} + \|K\|_{\mathcal{F}^M_n}, n \geq 0$,

$$h''_n(t) = \|K\|_{\mathcal{F}^M_n}(1+\chi^{(3)}) \sum_{n_1+n_2=1}(1+\chi_{10,n_1})R^{(1)}_{n_2}(t), \quad n \geq 0, \tag{6.39e}$$

$$k''_n(t) = (1+t)^{-1/2}S^{\rho,n}(t)\overline{H}_0(t) + \int_t^\infty \Big((1+s)^{-2+\rho}S^{\rho,n}(s)(1+[A]^1(s))\overline{H}_1(s) \tag{6.39f}$$
$$+ \big(\sum_{\substack{Y \in \Pi' \\ |Y|=n}} \|\gamma^\mu G_{Y\mu}(s)g(s)\|^2_D\big)^{1/2}\Big)ds, \quad n \geq 0,$$

where $\overline{H}_n(t) = \overline{H}_n(\infty,t)$, which makes sense by formula (5.169) since $\Psi \in \mathcal{F}^D_N$.

Proposition 6.4. *Let $1/2 < \rho < 1, n \geq 19, u \in E^{\circ\rho}_\infty, K \in \mathcal{F}^M_n$ and let n_0 be the integer part of $n/2+5$. Let $g_Y \in C^0(\mathbb{R}^+, D)$ for $Y \in \Pi', |Y| \leq n$, and let*

$$\sup_{t \geq 0}\big((1+t)^{3/2-\rho}(R'_{n_0-1}(t) + R^2_{n_0+1}(t) + R^\infty_{n_0}(t))\big) < \infty.$$

If for each $Y \in \Pi', |Y| \leq n$, there are two functions $e_1, e_2 \in C^0(\mathbb{R}^+, D)$ such that $g_Y = e_1 + e_2$, where:

a) $e_1 \in L^1(\mathbb{R}^+, D)$

b) $(m - i\gamma^\mu\partial_\mu + \gamma^\mu G_\mu)e_2 \in L^1(\mathbb{R}^+, D)$ *and* $\lim_{t\to\infty}\|e_2(t)\|_D = 0$,

and if $\sup_{t\geq 0}\big((1+t)\wp^D_{n-1}\big((1+\lambda_0(t))^{1/2}g(t)\big)\big) < \infty$ *and* $\sup_{t\geq 0}\big((1+t)^{3/2-\rho}\wp^D_n(f(t))\big) < \infty$, *then there exists a unique solution $\Psi \in C^0(\mathbb{R}^+, D)$ of equation (6.36) such that $\|\Psi(t)\|_D \to 0$, when $t \to \infty$. Moreover $\Psi \in \mathcal{F}^D_n, k'_l(n_0,t) \leq (1+t)^{-3/2+\rho}C_l h'_l(n_0,t)$, for $0 \leq l \leq n$, $k'_l(n_0,t) + k''_l(t) \leq (1+t)^{-3/2+\rho}(C_l h'_l(n_0,t) + C h''_l(t))$ for $n_0+1 \leq l \leq n$, where C_l and C depend only on ρ and $\|u\|_{E^\rho_{N_0}}$, the functions $t \mapsto h'_l(n_0,t), 0 \leq l \leq n$, and h''_l, $0 \leq l \leq n$, are uniformly bounded on $\mathbb{R}^+$ and the following estimates are satisfied:*

i) $\wp_l(\Psi(t)) \leq (1+t)^{-3/2+\rho}C_l\overline{Q}_l(t)$,

for $0 \leq l \leq n-5$, where C_l depends only on ρ and $\chi^{(5)}$,

ii) $$\wp^D_l(\Psi(t)) \leq (1+t)^{-3/2+\rho}C_l\Big(\sup_{s \geq t}\big((1+s)^{3/2-\rho}\wp^D_l(f(s))\big) + a_l h^\infty_l(n_0,t)$$
$$+ \sum_{\substack{n_1+n_2=l \\ 1 \leq n_1 \leq n_0 \\ n_2 \geq n_0+1}}(1+\chi_{5,n_1})\big(\sup_{s\geq t}\big((1+s)^{3/2-\rho}\wp^D_{n_2}(f(s))\big) + a_{n_2}h^\infty_{n_2}(n_0,t)\big)\Big)$$

and $k^\infty(n_0,t) \le (1+t)^{-3/2+\rho} C h^\infty(n_0,t)$, *for* $n_0+1 \le l \le n$, *where* $h_j^\infty(n_0,t) = h_j'(n_0,t) + h_j''(t) + \chi^{(5)} \sup_{s\ge t}\big((1+s)\wp_{j-1}^D((1+\lambda_0(s))^{1/2} g(s))\big)$ *and where* C_j *depends only on* ρ *and* $\chi^{(5)}$, *and* a_j *depends only on* ρ *and* $\chi^{(13)}$,

iii) $$\sum_{\substack{Y\in\Pi' \\ |Y|+k\le l}} \|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2}\Psi_Y(t)\|_{L^\infty} \le (1+t)^{-3/2+\rho} a_l \sum_{n_1+n_2=l} (1+\chi_{10,n_1}) R_{n_2}^{(1)}(t),$$

for $0 \le l \le n-10$, *where* a_l *depends only on* ρ *and* $\chi^{(13)}$ *and where it is supposed that*

$$\sup_{t\ge 0}\big((1+t)^{3/2-\rho}(R'_{l+7}(t) + R^2_{l+9}(t) + R^\infty_l(t))\big) < \infty,$$

iv) $$\begin{aligned}\wp_{l,i}^D(\Psi(t)) \le \wp_{l,i}^D(f(t)) &+ (1+t)^{-3/2+\rho}(a_l h_l'(n_0,t) + C h_{l,i}''(t)) \\ &+ (1+t)^{-3/2+\rho} C_l \sum_{\substack{n_1+n_2=l \\ 1\le n_1\le n_0 \\ n_2\ge n_0+1}} \chi_{5,n_1} \sup_{s\ge t}\big((1+s)^{1/2}\wp_{n_2}^D(\Psi(s))\big) \\ &+ (1+t)^{-3/2+\rho} C_l \chi^{(5)} \sup_{s\ge t}\big((1+s)^{1/2} \\ &\quad \big(\wp_l^D(\Psi(s))^\varepsilon \wp_{l,i+1}^D(\Psi(s))^{1-\varepsilon} + R^0_{l-1,1}(s)^{2(1-\rho)}\wp_{l-1}^D(\Psi(s))^{2\rho-1}\big)\big),\end{aligned}$$

for $n_0+1 \le l \le n, 0 \le i \le l$, *where* $h_{l,0}'' = h_l''$, $h_{l,i}'' = 0$ *for* $1 \le i \le l$, *where* C *depends only on* ρ *and* $\chi^{(13)}$, C_l *depends only on* ρ *and* $\chi^{(5)}$ *and* a_l *depends only on* ρ *and* $\chi^{(13)}$, *and where* $\varepsilon = \max(1/2, 2(1-l))$.

Proof. To prove the inequality of statement i) of the proposition, we first note that the hypothesis on $A = A^* + K$ of Theorem 5.13, with l instead of n, are satisfied for $0 \le l \le n-5$, according to the definition of the norm $\|\cdot\|_{\mathcal{F}_n^M}$, according to Proposition 6.2 and according to inequalities (6.38c). Since also the hypothesis on g are satisfied for $0 \le l \le n-1$ it follows from Theorem 5.13 that there exists a unique solution $\Psi \in C^0(\mathbb{R}^+, D)$ of equation (6.36), such that $\|\Psi(t)\|_D \to 0$, when $t \to \infty$, and that this solution satisfies $\wp_l^D(\Psi(t)) \le C_l Q_l^\infty(t)$, $0 \le l \le n-5$, where $Q_l^\infty(t)$ is defined in Theorem 5.13, and where C_l depends only on ρ and $[A]^3(\infty)$. The inequality of statement i) now follows from definition (6.39a) of $\overline{Q}_n(t)$ and from inequalities (6.38c).

Next we shall use Theorem 5.14 with n_0 instead of L, l instead of n and with $A = A^* + K$. Since $n \ge 18$, $n_0 + 1 \le l \le n$, it follows from the definition of n_0 that $18 \le n_0 + 9 \le l + 8 \le 2n_0$. We note that $\partial^\mu A_\mu = 0$, according to Proposition 6.2 and the definition of the space $\mathcal{F}_n^M$. Since $[A]^{n_0+2}(t) \le C_{n_0}\ \chi^{(n_0+4)}$, according to (6.38c), and $n_0 + 4 \le n$, the hypothesis in Theorem 5.14, that A_Y is a continuous map from $\mathbb{R}^+$ to weighted L^∞ spaces, are satisfied. To prove also that the other hypothesis on A are satisfied, we estimate $S^{\rho,N}, 0 \le N \le n$. Since $\square A_\mu^* = \overline{\phi}^* \gamma_\mu \phi^*$ according to equation (6.2b) and substitution (6.30) it follows from Proposition 6.2 and Corollary 2.6 that

$$\|(\delta(t))^{3/2}\square A_Y^*(t)\|_{L^2(\mathbb{R}^3,\mathbb{R}^4)} \le C_{|Y|}\|u\|_{E^\rho_{N_0}}\|u\|_{E^\rho_{N_0+|Y|}}, \tag{6.40}$$

for $Y \in \Pi'$, where $C_{|Y|}$ depends only on ρ and $\|u\|_{E^\rho_{N_0}}$. The gauge conditions $\partial^\mu A^*_\mu = 0$ and $\partial^\mu K_\mu = 0$, the inequality $\||\nabla|^\varepsilon h\|_{L^2(\mathbb{R}^3,\mathbb{C})} \leq \|h\|^{1-\varepsilon}_{L^2}\||\nabla|h\|^\varepsilon_{L^2}$ for $0 \leq \varepsilon \leq 1$, equalities (5.115a) and (5.115b) and inequality (6.40) give that

$$S^{\rho,N}(t) \leq C_N\|u\|_{E^\rho_{N_0+N}} + C\|K\|_{\mathcal{F}^M_N}, \tag{6.41a}$$

for $1/2 < \rho < 1, N \geq 0, t \geq 0$, where C_N depends only on ρ and $\|u\|_{E^\rho_{N_0}}$ and where C depends only on ρ. It now follows from definition (6.37) of $\chi^{(N)}$ that

$$S^{\rho,N}(t) \leq C_N\chi^{(N)}, \quad S^\rho_{k,N}(t) \leq C_{k+N}\chi_{k,N}, \tag{6.41b}$$

for $1/2 < \rho < 1, N \geq 0, k \geq 0, t \geq 0$, where C_N and C_{k+N} are constants depending only on ρ and $\|u\|_{E^\rho_{N_0}}$. It follows from (6.41a), with $N = l$, that the rest of the hypothesis on A, in Theorem 5.14, are satisfied. The hypothesis concerning g in Theorem 5.14 are satisfied and so are the hypothesis on $R'_{n_0-1}, R^2_{n_0+1}$ and $R^\infty_{n_0-8}$.

In order to estimate $k^\infty_j(n_0,t)$, we begin by estimating $\overline{H}_j(t) = \overline{H}_j(\infty,t)$ defined in (5.169):

$$\overline{H}_j(t) = \sum_{n_1+n_2=j}(1+S^\rho_{10,n_1}(t))\big(R^2_{n_2+9}(t)+R^\infty_{n_2}(t)+R'_{n_2+7}(t)+Q^\infty_{n_2+8}(t)\big), \quad 0 \leq j, \tag{6.42}$$

where Q^∞_j, defined in Theorem 5.13, is given by

$$\begin{aligned} Q^\infty_j(t) = \sup_{s\geq t}\big(\wp^D_j(f(s))\big) + \sum_{\substack{n_1+n_2=n\\ n_2\leq j-1}}[A]_{3,n_1}(\infty) \\ \Big(\sup_{s\geq t}\big(\wp^D_{n_2}(f(s))\big) + \int_t^\infty(1+s)^{-3/2+\rho}\wp^D_{n_2}\big((1+\lambda_0(s))^{1/2}g(s)\big)ds\Big), \end{aligned} \tag{6.43}$$

$0 \leq j$ and where R'_j, R^2_j and R^∞_j are given in Theorem 5.8. It follows from (6.38c), (6.39a), (6.39b) and (6.43) that

$$\begin{aligned} Q^\infty_j(t) &\leq (1+t)^{-3/2+\rho}\Big(\sup_{s\geq t}\big((1+s)^{3/2-\rho}\wp^D_j(f(s))\big) + C_jQ'_j(t)\Big) \\ &\leq (1+t)^{-3/2+\rho}C_j\overline{Q}_j(t), \end{aligned} \tag{6.44}$$

for $0 \leq j \leq n$, where $C_j \geq 1$ depends only on ρ and $\|u\|_{E^\rho_{N_0}}$. Inequalities (6.41b) and (6.44) together with equality (6.42) give that

$$\begin{aligned} \overline{H}_j(t) \leq (1+t)^{-3/2+\rho}C_j\sum_{n_1+n_2=j}(1+\chi_{10,n_1}) \\ \Big(\overline{Q}_{n_2+8}(t) + \sup_{s\geq t}\big((1+s)^{3/2-\rho}\big(R^2_{n_2+9}(s)+R^\infty_{n_2}(s)+R'_{n_2+7}(s)\big)\big)\Big), \end{aligned}$$

$0 \le j \le n-10$, where C_j depends only on ρ and $\|u\|_{E^\rho_{N_0}}$. Definition (6.39c) of $R^{(1)}_j$, then gives that

$$\overline{H}_j(t) \le (1+t)^{-3/2+\rho} C_j \sum_{n_1+n_2=j} (1+\chi_{10,n_1}) R^{(1)}_{n_2}(t), \quad 0 \le j \le n-10, \tag{6.45}$$

where C_j depends only on ρ and $\|u\|_{E^\rho_{N_0}}$.

We can now estimate $\overline{k}_n(n_0,\infty,t)$, which we denote $\overline{k}_j(n_0,t)$ and which according to definition (5.173b) of $\overline{k}_j$, definition (5.171) of $\overline{\wp}_j$ and expression (6.42) of $\overline{H}_j$ reads

$$\begin{aligned}
&\overline{k}_j(n_0,t) \qquad (6.46)\\
&\quad = \sum_{\substack{n_1+n_2=j\\ 1\le n_1\le n_0\\ n_2\le n_0}} \Big([A]_{3,n_1}(t)(Q^\infty_{n_2}(t) + R^0_{n_2-1,1}(t)) + (1+t)^{1/2}[A]^{n_1+1}(t)\wp^D_{n_2}(g(t))\Big)\\
&\qquad + \sum_{\substack{n_1+n_2+n_3=j\\ n_1\le j-1\\ n_2\le n_0-1\\ n_3\le j-n_0}} S^{\rho,n_1}(t)(1+[A]_{2,n_2}(t))\overline{H}_{n_3}(t)\\
&\qquad + \sum_{\substack{Y_1,Y_2\in\Pi'\\ |Y_1|+|Y_2|=j\\ n_0\le|Y_1|\le j-1}} (1+t)^{2-\rho}\|\gamma^\mu G_{Y_1\mu}(t) g_{Y_2}(t)\|_D, \quad 0\le j\le n.
\end{aligned}$$

Since $G_\mu = A^*_\mu - \partial_\mu\vartheta(A^*) + M_\mu$, with $M_\mu = K_\mu - \partial_\mu\vartheta(K)$, it follows from inequality (5.116c) that

$$\begin{aligned}
&\|\gamma^\mu G_{Y_1\mu}(t) g_{Y_2}(t)\|_D \qquad (6.47)\\
&\quad \le C_{|Y_1|}[A^*]^{|Y_1|+1}(1+t)^{-3/2+\rho}\|g_{Y_2}(t)\|_D + C\|(\delta(t))^{-1}M_{Y_1}(t)\|_{L^2}\|\delta(t)g_{Y_2}(t)\|_{L^\infty},
\end{aligned}$$

$Y_1, Y_2 \in \Pi'$, where C and $C_{|Y|}$ depend only on ρ. Let

$$\Gamma_j(K,t) = \Big(\sum_{\substack{Y\in\Pi'\\ |Y|\le j}} \|(\delta(t))^{-1}M_Y(t)\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^4)}\Big)^{1/2} \tag{6.48}$$

for $j \ge 0$. It follows from inequality (5.124), with M_μ instead of G_μ, $\dot{K}_Y = K_{P_0Y}, b > 1/2$, and $a = 1/2$, that

$$\begin{aligned}
\Gamma_j(K,t) \le (1+t)^{-1/2} C \sup_{0\le s\le t} \Big(&(1+s)^{b-1}\wp^{M^0}_j\big((K(s),0)\big) \qquad (6.49a)\\
&+ (1+s)^b \wp^{M^1}_j\big((L(s),\dot{L}(s))\big) + (1+s)^{b-1} j C_j \wp^{M^1}_{j-1}\big((L(s),\dot{L}(s))\big)\Big), \quad j\ge 0,
\end{aligned}$$

where C and C_j depend only on b. With $b = 3/4$, we obtain that

$$\Gamma_j(K,t) \le (1+t)^{-1/2}\big(C\|K\|_{\mathcal{F}^M_j} + jC_j\|K\|_{\mathcal{F}^M_{j-1}}\big), \quad j\ge 0, \tag{6.49b}$$

where C and C_j are as in (6.49a). The first of inequalities (6.38a) and inequality (6.49b) give that

$$\begin{aligned}
&\sum_{\substack{Y_1,Y_2\in\Pi' \\ |Y_1|+|Y_2|=j \\ n_0\leq|Y_1|\leq j-1}} \|\gamma^\mu G_{Y_1\mu}(t)g_{Y_2}(t)\|_D \qquad (6.50a)\\
&\quad\leq (1+t)^{-3/2+\rho}C'_j \sum_{\substack{n_1+n_2=j \\ n_0\leq n_1\leq j-1}} \|u\|_{E^\rho_{N_0+n_1+1}}\wp^D_{n_2}(g(t))\\
&\qquad + (1+t)^{-1/2}C_j \sum_{\substack{Z\in\Pi' \\ n_1+|Z|=j \\ n_0\leq n_1\leq j-1}} \|K\|_{\mathcal{F}^M_{n_1}}\|\delta(t)g_Z(t)\|_{L^\infty}, \quad j\geq 0,
\end{aligned}$$

where C'_j depends only on ρ and $\|u\|_{E^\rho_{N_0}}$ and C_j is constant. We note that the right-hand side of this inequality vanishes if $0\leq j\leq n_0$. If $\overline{\chi}^{(j)} = \|u\|_{E^\rho_{N_0+j+1}} + \|K\|_{\mathcal{F}^M_j}$, it then follows from the definition of $R^\infty_j(t)$ and from inequality (6.50a) that

$$\begin{aligned}
&\sum_{\substack{Y_1,Y_2\in\Pi' \\ |Y_1|+|Y_2|=j \\ n_0\leq|Y_1|\leq j-1}} \|\gamma^\mu G_{Y_1\mu}(t)g_{Y_2}(t)\|_D \qquad (6.50b)\\
&\quad\leq C_j \sum_{\substack{n_1+n_2=j \\ n_1\leq j-1 \\ n_2\leq j-n_0}} \overline{\chi}^{(n_1)}\big((1+t)^{-3/2+\rho}\wp^D_{n_2}(g(t)) + (1+t)^{-1}R^\infty_{n_2}(t)\big), \quad j\geq 0,
\end{aligned}$$

where C_j depends only on ρ and $\|u\|_{E^\rho_{N_0}}$. Inequalities (6.38c), (6.41b), (6.45) and (6.50b), and equality (6.46) give

$$\begin{aligned}
&\overline{k}_j(n_0,t) \qquad (6.51)\\
&\quad\leq C_j \sum_{\substack{n_1+n_2=j \\ 1\leq n_1\leq n_0 \\ n_2\leq n_0}} \chi_{5,n_1}\big(Q^\infty_{n_2}(t) + R^0_{n_2-1,1}(t) + (1+t)^{1/2}\wp^D_{n_2}(g(t))\big)\\
&\qquad + (1+t)^{-3/2+\rho}C_j \sum_{\substack{n_1+n_2+n_3+n_4=j \\ n_1\leq j-1 \\ n_2\leq n_0-1 \\ n_3+n_4\leq j-n_0}} \chi^{(n_1)}(1+\chi_{4,n_2})(1+\chi_{10,n_3})R^{(1)}_{n_4}(t)\\
&\qquad + C_j \sum_{\substack{n_1+n_2=j \\ n_1\leq j-1 \\ n_2\leq j-n_0}} \overline{\chi}^{(n_1)}\big((1+t)^{-3/2+\rho}\wp^D_{n_2}(g(t)) + (1+t)^{-1}R^\infty_{n_2}(t)\big), \quad j\geq 0,
\end{aligned}$$

where C_j depends only on ρ and $\|u\|_{E^\rho_{N_0}}$. According to definition (6.39) of $R^{(1)}_j$ it follows from inequality (6.51) that

$$\begin{aligned}
&\overline{k}_j(n_0,t) \qquad\qquad (6.52)\\
&\quad\le (1+t)^{-1/2}C_j\Big(\sum_{\substack{n_1+n_2=j\\ 1\le n_1\le n_0\\ n_2\le n_0}} \chi_{5,n_1}(1+t)^{1/2}\big(Q^{\infty}_{n_2}(t)+R^0_{n_2-1,1}(t)+(1+t)^{1/2}\wp^D_{n_2}(g(t))\big)\\
&\qquad + (1+t)^{-1+\rho}\sum_{\substack{n_1+n_2+n_3+n_4=j\\ n_1\le j-1, n_2\le n_0-1\\ n_3+n_4\le j-n_0}} \overline{\chi}^{(n_1)}(1+\chi_{4,n_2})(1+\chi_{10,n_3})R^{(1)}_{n_4}(t)\Big), \quad j\ge 0,
\end{aligned}$$

where C_j depends only on ρ and $\|u\|_{E^{\rho}_{N_0}}$.

We can now estimate $k'_j(n_0,\infty,t)$ defined by (5.173a). With the notation $k'_j(n_0,t)=k'_j(n_0,\infty,t)$, it follows from definitions (5.171) and (5.173a) that

$$\begin{aligned}
k'_j(n_0,t) &= (1+t)^{-3/2+\rho}\sum_{\substack{n_1+n_2=j\\ 1\le n_1\le n_0\\ n_2\le n_0}} [A]^{n_1+1}(t)Q^{\infty}_{n_2}(t) \qquad\qquad (6.53)\\
&\quad + (1+t)^{-1/2}\sum_{\substack{n_1+n_2=j\\ n_0+1\le n_1\le j-1}} S^{\rho,n_1}\overline{H}_{n_2}(t) + \int_t^{\infty}(1+s)^{-2+\rho}\overline{k}_j(n_0,s)ds, \quad j\ge 0.
\end{aligned}$$

Inequalities (6.38c), (6.41b), (6.44), (6.45), (6.52) and (6.53) give that

$$\begin{aligned}
&k'_j(n_0,t)\\
&\quad\le (1+t)^{-3+2\rho}C_j\sum_{\substack{n_1+n_2=j\\ 1\le n_1\le n_0\\ n_2\le n_0}} \chi^{(n_1+3)}\overline{Q}_{n_2}(t)\\
&\qquad + (1+t)^{-2+\rho}C_j\sum_{\substack{n_1+n_2+n_3=j\\ n_1\le j-1\\ n_2+n_3\le j-n_0-1}} \chi^{(n_1)}(1+\chi_{10,n_2})R^{(1)}_{n_3}(t)\\
&\qquad + (1+t)^{-3/2+\rho}C_j\sup_{s\ge t}\Big(\sum_{\substack{n_1+n_2=j\\ 1\le n_1\le n_0\\ n_2\le n_0}} \chi_{5,n_1}(1+s)^{1/2}\\
&\qquad\qquad \big(Q^{\infty}_{n_2}(s)+R^0_{n_2-1,1}(s)+(1+s)^{1/2}\wp^D_{n_2}(g(s))\big)\\
&\qquad + \sum_{\substack{n_1+n_2+n_3+n_4=j\\ n_1\le j-1\\ n_2\le n_0-1\\ n_3+n_4\le j-n_0}} \overline{\chi}^{(n_1)}(1+\chi_{4,n_2})(1+\chi_{10,n_3})(1+s)^{-1+\rho}R^{(1)}_{n_4}(s)\Big), \quad j\ge 0,
\end{aligned}$$

where C_j depends only on ρ and $\|u\|_{E^{\rho}_{N_0}}$. It follows from this inequality, from inequality (6.44) for Q^{∞}_j and by regrouping the second and the fourth sum on the right-hand side

that

$$k'_j(n_0,t) \le (1+t)^{-3/2+\rho} C_j \Big(\sum_{\substack{n_1+n_2=j \\ 1\le n_1 \le n_0 \\ n_2 \le n_0}} \chi_{5,n_1} \big(\overline{Q}_{n_2}(t) + \sup_{s\ge t} \big((1+s)^{1/2} R^0_{n_2-1,1}(s) + (1+s)\wp^D_{n_2}(g(s))\big)\big) + \sum_{\substack{n_1+n_2+n_3+n_4=j \\ n_1\le j-1, n_2\le n_0-1 \\ n_3+n_4\le j-n_0}} \overline{\chi}^{(n_1)}(1+\chi_{4,n_2})(1+\chi_{10,n_3}) R^{(1)}_{n_4}(t)\Big), \quad j\ge 0, \tag{6.54}$$

where C_j depends only on ρ and $\|u\|_{E^\rho_{N_0}}$.

To estimate $k_j(n_0,t)$, where $k_j(n_0,t)=k_j(n_0,\infty,t)$ is defined in Theorem 5.14, let us introduce the notation

$$\begin{aligned} k''_j(n_0,t) &= (1+t)^{-1/2} S^{\rho,j}(t)\overline{H}_0(t) \\ &+ \int_t^\infty \Big((1+s)^{-2+\rho} S^{\rho,j}(s)(1+[A]^1(s))\overline{H}_1(s) + \big(\sum_{\substack{Y\in\Pi' \\ |Y|=j}} \|\gamma^\mu G_{Y\mu}(s) g(s)\|^2_D\big)^{1/2}\Big) ds, \end{aligned} \tag{6.55}$$

$j\ge 0$. It follows from inequalities (6.38c), (6.41a) and definition (6.55) of k''_j that

$$k''_j(n_0,t) \le (1+t)^{-1+\rho}\big(C\|K\|_{\mathcal{F}^M_j} + C_j\|u\|_{E^\rho_{N_0+j}}\big) \sup_{s\ge t}\big((1+C_1\chi^{(3)})\overline{H}_1(s)\big) + \int_t^\infty \big(\sum_{\substack{Y\in\Pi' \\ |Y|=j}} \|\gamma^\mu G_{Y\mu}(s) g(s)\|^2_D\big)^{1/2} ds, \quad j\ge 0, \tag{6.56}$$

where C depends only on ρ and C_j, $j\ge 0$ depends only on ρ and $\|u\|_{E^\rho_{N_0}}$. Here we have used that $-1/2 < -1+\rho$ for $1/2<\rho<1$. Inequalities (6.38a), (6.45), (6.47), (6.49b) and (6.56) give that

$$\begin{aligned} k''_j(n_0,t) &\le (1+t)^{-3/2+\rho} C_1 \big(\|K\|_{\mathcal{F}^M_j} + C_j\|u\|_{E^\rho_{N_0+j}}\big)(1+\chi^{(3)}) \sum_{n_1+n_2=1} (1+\chi_{10,n_1}) R^{(1)}_{n_2}(t) \\ &+ (1+t)^{-3/2+\rho} \sup_{s\ge t} \Big(C_j \|u\|_{E^\rho_{N_0+j}} (1+s)\|g(s)\|_D \\ &+ \big(C\|K\|_{\mathcal{F}^M_j} + jC_j\|K\|_{\mathcal{F}^M_{j-1}}\big)(1+s)^{1/2}\|\delta(s)g(s)\|_{L^\infty}\Big), \quad j\ge 0, \end{aligned} \tag{6.57}$$

where C depends only on ρ and $C_j, j\ge 0$, on ρ and $\|u\|_{E^\rho_{N_0}}$. By adding the terms, proportional to $\|u\|_{E^\rho_{N_0+j}}$ and $\|K\|_{\mathcal{F}^M_{j-1}}$, on the right-hand side of inequality (6.57) to the

second sum on the right-hand side of inequality (6.54), we obtain that

$$
\begin{aligned}
&k'_j(n_0,t)+k''_j(n_0,t) \qquad (6.58)\\
&\quad\le (1+t)^{-3/2+\rho}C\Big(\Big(\sup_{s\ge t}\big((1+s)^{1/2}\|\delta(s)g(s)\|_{L^\infty}\big)\\
&\quad + (1+\chi^{(3)})\sum_{n_1+n_2=1}(1+\chi_{10,n_1})R^{(1)}_{n_2}(t)\Big)\|K\|_{\mathcal{F}^M_j}\\
&\quad + C_j\sum_{\substack{n_1+n_2=j\\1\le n_1\le n_0\\n_2\le n_0}}\chi_{5,n_1}\big(\overline{Q}_{n_2}(t)+\sup_{s\ge t}\big((1+s)^{1/2}R^0_{n_2-1,1}(s)+(1+s)\wp^D_{n_2}(g(s))\big)\big)\\
&\quad + C_j\sum_{\substack{n_1+n_2+n_3+n_4=j\\n_1\le j-1,n_2\le n_0-1\\n_3+n_4\le j-n_0}}\overline{\chi}^{(n_1)}(1+\chi_{4,n_2})(1+\chi_{10,n_3})R^{(1)}_{n_4}(t)\Big), \quad j\ge n_0+1,
\end{aligned}
$$

where C and C_j depend only on ρ and $\|u\|_{E^\rho_{N_0}}$. Since

$$R^0_{j-1,1}(t)\le (1+t)^{1/2}\wp^D_{j-1}\big((1+\lambda_0(t))^{1/2}g(t)\big), \quad j\ge 1, \qquad (6.59)$$

it follows from inequality (6.54) that

$$k'_j(n_0,t)\le (1+t)^{-3/2+\rho}C_jh'_j(n_0,t), \quad j\ge 0, \qquad (6.60a)$$

and from inequality (6.58) that

$$k'_j(n_0,t)+k''_j(n_0,t)\le (1+t)^{-3/2}\big(C_jh'_j(n_0,t)+Ch''_j(t)\big), \quad j\ge n_0+1, \qquad (6.60b)$$

where C and C_j depend only on ρ and $\|u\|_{E^\rho_{N_0}}$. Using inequalities (6.38c), (6.59) and (6.60b), we obtain that

$$
\begin{aligned}
k^\infty_j(n_0,t) &\le (1+t)^{-3/2+\rho}C_j\Big(h'_j(n_0,t)+h''_j(t) \qquad (6.61)\\
&\qquad + \chi^{(5)}\sup_{s\ge t}\big((1+s)\wp^D_{j-1}\big((1+\lambda_0(s))^{1/2}g(s)\big)\big)\Big)\\
&= (1+t)^{-3/2+\rho}C_jh^\infty_j(n_0,t), \quad j\ge n_0+1,
\end{aligned}
$$

where $h^\infty_j(n_0,t)$ is defined in statement i) of the proposition and where C_j depends only on ρ and $\|u\|_{E^\rho_{N_0}}$. Now, $\chi^{(j)}<\infty$ and $\overline{\chi}^{(j)}<\infty$ for $0\le j\le n$, since $u\in E^{\circ\rho}_\infty$ and $K\in\mathcal{F}^M_n$ according to the hypothesis. It follows that $\sup_{t\ge0}(Q'_j(t))<\infty$ for $0\le j\le n-5$ by using expression (6.39b) and that $\sup_{t\ge 0}\big((1+t)^{3/2-\rho}\wp^D_j(f(t))\big)<\infty$ for $0\le j\le n$ and $\sup_{t\ge0}\big((1+t)\wp^D_j\big((1+\lambda_0(t))^{1/2}g(t)\big)\big)<\infty$ for $0\le j\le n-1$. Definition (6.39a) then gives that $\sup_{t\ge0}(\overline{Q}_j(t))<\infty$ for $0\le j\le n-5$. Definition (6.39c) and the hypothesis on $R'_{n_0-1}, R^2_{n_0+1}$ and $R^\infty_{n_0}$, now give that $\sup_{t\ge0}(R^{(1)}_j(t))<\infty$ for $0\le j\le n_0-8$, since

$n_0 \leq n-5$ for $n \geq 19$. The function $t \mapsto h'_j(n_0,t)$ is uniformly bounded on $\mathbb{R}^+$ for $0 \leq j \leq n$, since in its definition (6.39d) $n_0 \leq n-5, n-n_0 \leq n_0-9$. The function h''_j, defined in (6.39e) is also uniformly bounded on $\mathbb{R}^+$ for $0 \leq j \leq n$. This proves, together with inequalities (6.60a), (6.60b) and (6.61) the claimed properties of the functions $k'_j(n_0,\cdot)$, k''_j, $k^\infty_j(n_0,\cdot)$, $h'_j(n_0,\cdot)$, h''_j, $h^\infty_j(n_0,\cdot)$. The estimates in statement ii) follows from inequality (6.61), Theorem 5.14 and inequality (6.38c). Statement iii) follows from inequalities (5.170) and (6.45). Statement iv) follows from inequalities (5.176), (5.177), (6.38c) and from the already proved inequalities for $k'_l(n_0,t)$ and $k''_l(t)$ in terms of $h'_l(n_0,t)$ and $h''_l(t)$. This proves the proposition.

In order to use Proposition 6.4 for the particular case $g = \gamma^\mu(L_\mu - \partial_\mu\vartheta(L))\phi^*$, $L \in \mathcal{F}^M_n$, we group together *preliminary estimates* of R'_j, R^∞_j, R^2_j defined in Theorem 5.8.

Lemma 6.5. *Let $1/2 < \rho < 1, n \geq 5, K \in \mathcal{F}^M_n, L \in \mathcal{F}^M_n$, $r_Y \in C^0(\mathbb{R}^+, D), r_Y = \xi^D_Y r$ for $Y \in \Pi'$ and let*

$$\sup_{t\geq 0}\big(\wp^D_j((1+\lambda_1(t))^{k/2}r(t)) + H_l(r,t) + u_l(t)\big) < \infty$$

for $j+k=l, l\geq 0$, where

$$H_l(r,t) = \sum_{\substack{Y\in\Pi'\\ k+|Y|\leq l}} \|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2} r_Y(t)\|_{L^\infty}, \quad l \geq 0$$

and

$$u_l(t) = H_{l+1}(r,t) + \sum_{\substack{Y\in\Pi'\\ k+|Y|\leq l}} \|(\delta(t))^{3-\rho}(1+\lambda_1(t))^{k/2}((m+i\gamma^\mu\partial_\mu)r_Y)(t)\|_{L^\infty}, \quad l \geq 0.$$

Let $A = A^* + K$ and let $G_\mu = A_\mu - \partial_\mu\vartheta(A)$. If

$$g = \gamma^\mu(L_\mu - \partial_\mu\vartheta(L))r,$$

then $g_Y \in C^0(\mathbb{R}^+, D)$, $\|g_Y(t)\|_D \to 0$ when $t \to \infty$, $(m - i\gamma^\mu\partial_\mu + \gamma^\mu G_\mu)g_Y \in L^1(\mathbb{R}^+, D)$ for $Y \in \Pi'$, $|Y| \leq n$ and:

i) $$R'_l(t) \leq (1+t)^{-3/2+\rho} C'_l\Big(\sum_{n_1+n_2+n_3=l}(1+\|u\|_{E^\rho_{N_0+n_1+1}})\|L\|_{\mathcal{F}^M_{n_2}} u_{n_3}(t) + \sum_{n_1+n_2+n_3=l} \min\big(\|K\|_{\mathcal{F}^M_{n_1+3}}\|L\|_{\mathcal{F}^M_{n_2}}, \|K\|_{\mathcal{F}^M_{n_1}}\|L\|_{\mathcal{F}^M_{n_2+3}}\big)u_{n_3}(t)\Big), \quad 0 \leq l \leq n,$$

ii) $$R^\infty_l(t) \leq (1+t)^{-1} C_l \sum_{n_1+n_2=l} \|L\|_{\mathcal{F}^M_{n_1+3}} H_{n_2}(r,t), \quad 0 \leq l \leq n-3,$$

iii) $$\wp^D_l\big((\delta(t))^{1/2}(1+\lambda_1(t))^{k/2}g(t)\big) \leq (1+t)^{-1/2} \sum_{n_1+n_2=l} C_{n_1,n_2}\|L\|_{\mathcal{F}^M_{n_1}} H_{n_2+k}(r,t), \quad 0 \leq l \leq n, k \geq 0,$$

iv) $$\wp^D_{l,i}(f(t)) \leq (1+t)^{-3/2+\rho}(1+\chi^{(3)}) \Big(C^{(i)}\|L\|_{\mathcal{F}^M_l}\sup_{s\geq t}(u_1(s)) + C_l \sum_{\substack{n_1+n_2=l\\ n_1\leq l-1}} \|L\|_{\mathcal{F}^M_{n_1}}\sup_{s\geq t}(u_{n_2+1}(s))\Big), \quad 0 \leq l \leq n,$$

$0 \leq i \leq n$, where $C^{(i)} = 0$ for $1 \leq i \leq n$ and $C^{(0)} = C_{l,0} = C_{0,l} = C$. Here C, C_l and C_{n_1,n_2} depend only on ρ, and C'_l depends only on ρ and $\|u\|_{E^\rho_{N_0}}$.

Proof. Let $B_\mu = -\partial_\mu \vartheta(L), B_{Y,\mu} = (B_Y)_\mu = (\xi^M_Y B)_\mu$ for $Y \in \Pi', 0 \leq \mu \leq 3$. Since, by covariance, $B_{Y,\mu} = -\partial_\mu \vartheta(L_Y)$ for $Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$, it follows from equality (4.83), using that $\partial_\mu L^\mu = 0$, that

$$|B_{ZY}(y)| \leq C \int_0^1 s^{|Z|} \Big(\sum_{0 \leq \mu < \nu \leq 3} |L_{ZM_{\mu\nu}Y}(sy)| + |L_{ZY}(sy)| \Big) ds, \tag{6.62}$$

for $Z \in \Pi' \cap U(\mathbb{R}^4), Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$ and $y \in \mathbb{R}^+ \times \mathbb{R}^3$. Let

$$I_l(t,x) = \int_0^1 s^l (1 + s(t + |x|))^{-1} (1 + st)^{-1+\rho} (1 + s\big|t - |x|\big|)^{-1/2} ds, \quad l \in \mathbb{N}. \tag{6.63}$$

It follows from Lemma 6.3 and inequality (6.62) that

$$|B_{ZY}(t,x)| \leq C \|L\|_{\mathcal{F}^M_{|Z|+|Y|+3}} I_{|Z|}(t,x), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \tag{6.64}$$

for $Z \in \Pi' \cap U(\mathbb{R}^4), Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$, where C depends only on ρ. To estimate $I_0(t,x)$ we observe that

$$\begin{aligned}(1 + s(t + |x|))^{-1} (1 + st)^{-1+\rho} (1 + s\big|t - |x|\big|)^{-1/2} \\ \leq C\big(1 + s((\delta(t))(x)\big)^{-2+\rho}, \quad t \geq 0, x \in \mathbb{R}^3,\end{aligned}$$

where C depends only on ρ. Integration of this inequality gives, with a new constant, that

$$I_0(t,x) \leq C((\delta(t))(x))^{-1}, \quad t \geq 0, x \in \mathbb{R}^3, \tag{6.65a}$$

where C depends only on ρ. For $l \geq 1$, proceeding as in the end of the proof of Lemma 4.4, it follows that

$$\begin{aligned}I_l(t,x) \leq C((\delta(t))(x))^{-1} (1+t)^{-1+\rho} (1 + \big|t - |x|\big|)^{-1/2} \\ \int_0^1 \frac{s((\delta(t))(x))}{1 + s((\delta(t))(x))} \Big(\frac{s(1+t)}{1+s(1+t)}\Big)^{1-\rho} \Big(\frac{s(1+\big|t-|x|\big|)}{1+s(1+\big|t-|x|\big|)}\Big)^{1/2} s^{l-5/2+\rho} ds.\end{aligned}$$

Since $l - 5/2 + \rho > -1$ for $l \geq 1$ and $1/2 < \rho < 1$, we obtain that

$$I_l(t,x) \leq C((\delta(t))(x))^{-1} (1+t)^{-1+\rho} (1 + \big|t - |x|\big|)^{-1/2}, \quad l \geq 1, \tag{6.65b}$$

$t \geq 0, x \in \mathbb{R}^3$, where C depends only on ρ. It follows from inequalities (6.64), (6.65a) and (6.65b) that

$$|B_Y(t,x)| \leq C((\delta(t))(x))^{-1} \|L\|_{\mathcal{F}^M_{|Y|+3}}, \quad Y \in \Pi' \tag{6.66a}$$

and that

$$|B_Y(t,x)| \leq C((\delta(t))(x))^{-1}(1+t)^{-1+\rho}(1+\big|t-|x|\big|)^{-1/2}\|L\|_{\mathcal{F}^M_{|Y|+3}}, \tag{6.66b}$$

$Y \in \boldsymbol{\sigma}^1, 1/2 < \rho < 1$, where C depends only on ρ.

Let $A = A^* + K$, $G_\mu = A_\mu - \partial_\mu \vartheta(A)$ and $F_\mu = L_\mu - \partial_\mu \vartheta(L)$. To estimate $\wp^D_l\big(\delta(t)(1+\lambda_1(t))^{k/2}g'(t)\big)$, $l \geq 0$, $k \geq 0$, where $g' = (2m)^{-1}(m - i\gamma^\mu\partial_\mu + \gamma^\mu G_\mu)g$, we use equality (5.7a), which gives that

$$\begin{aligned} g'_Y &= \xi^D_Y g' \\ &= (2m)^{-1}\sum^{Y}_{Y_1,Y_2}\Big(\gamma^\nu F_{Y_1\nu}\xi^D_{Y_2}\big((m+i\gamma^\mu\partial_\mu - \gamma^\mu G_\mu)r\big) \\ &\qquad - 2iF^\mu_{Y_1}\partial_\mu r_{Y_2} + i(\partial_\mu B^\mu_{Y_1})r_{Y_2} \\ &\qquad - \frac{i}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\big((\partial_\mu L_{Y_1\nu}) - (\partial_\nu L_{Y_1\mu})\big)r_{Y_2}\Big) \\ &\qquad + m^{-1}\sum^{Y}_{Y_1,Y_2,Y_3} G_{Y_1\mu}F^\mu_{Y_2}r_{Y_3}, \quad Y \in \Pi', \end{aligned} \tag{6.67}$$

where $G_Y = \xi^M_Y G, F_Y = \xi^M_Y F$ for $Y \in \Pi'$ and where we have used that $\partial_\mu L^\mu = 0$ and $\partial_\mu A^\mu = 0$, according to the definition of $\mathcal{F}^M_n$ and Proposition 6.2. It follows that

$$\begin{aligned} g'_Y = (2m)^{-1}&\sum^{Y}_{Y_1,Y_2}\Big(\gamma^\nu F_{Y_1\nu}(m+i\gamma^\mu\partial_\mu)r_{Y_2} - 2iF^\mu_{Y_1}\partial_\mu r_{Y_2} + i(\partial_\mu B^\mu_{Y_1})r_{Y_2} \\ &- \frac{i}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)((\partial_\mu L_{Y_1\nu}) - (\partial_\nu L_{Y_1\mu}))r_{Y_2}\Big) \\ &+ (2m)^{-1}\sum^{Y}_{Y_1,Y_2,Y_3}\big(G_{Y_1\mu}F^\mu_{Y_2}r_{Y_3} + \frac{1}{2}G_{Y_1\mu}F_{Y_2\nu}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)r_{Y_3}\big), \quad Y \in \Pi'. \end{aligned} \tag{6.68}$$

Inequality (5.130a), with $\varepsilon = 1/2, b = 5/2 - \rho$ and $a = 1$, gives that

$$\Big(\sum_{\substack{Y\in\Pi' \\ |Y|\leq j}} \|(\delta(t))^{-1/2}\partial_\mu B^\mu_Y(t)\|^2_{L^2}\Big)^{1/2} \leq (1+t)^{-1}\big(C\|L\|_{\mathcal{F}^M_j} + jC_j\|L\|_{\mathcal{F}^M_{j-1}}\big), \quad j \geq 0, \tag{6.69}$$

where C and C_j depend only on ρ. Let $Q_Y(y) = y^\mu F_{Y\mu}(y)$, $Y \in \Pi', y \in \mathbb{R}^+ \times \mathbb{R}^3$. It follows from inequality (5.135) and the definition of $\|\cdot\|_{\mathcal{F}^M_j}$ that

$$\sum_{\substack{Y\in\Pi' \\ |Y|\leq j}} \|(\delta(t))^{-1}Q_Y(t)\|_{L^2(\mathbb{R}^3,\mathbb{R})} \leq jC_j(1+t)^{-1/2}\|L\|_{\mathcal{F}^M_j}, \quad j \geq 0, \tag{6.70}$$

where C_j depends only on ρ. Inequalities (5.7d) and (6.70) give that

$$\sum_{Y_1,Y_2}^{Y} \|\delta(t)(1+\lambda_1(t))^{k/2} F^{\mu}_{Y_1}(t)\partial_\mu r_Y(t)\|_D \\ \leq C_{|Y|}(1+t)^{-1} \sum_{\substack{Z\in\Pi' \\ n_1+n_2=|Y| \\ |Z|\leq n_2+1}} \|L\|_{\mathcal{F}^M_{n_1}} \|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2} r_Z(t)\|_{L^\infty}, \tag{6.71}$$

$Y \in \Pi'$, where $C_{|Y|}$ depends only on ρ. Since $A = A^* + K$, it follows from inequalities (5.116c), (6.38a), (6.49b) and (6.66a) that

$$\sum_{Y_1,Y_2,Y_3}^{Y} \|\delta(t)(1+\lambda_1(t))^{k/2} G_{Y_1\mu}(t) F_{Y_2\nu}(t) r_{Y_3}(t)\|_D \\ \leq C'_{|Y|}(1+t)^{-3/2+\rho} \sum_{Y_1,Y_2,Y_3}^{Y} \|u\|_{E^\rho_{N_0+|Y_1|+1}} \|L\|_{\mathcal{F}^M_{|Y_2|}} \|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2} r_{Y_3}(t)\|_{L^\infty} \\ + C_{|Y|}(1+t)^{-1} \sum_{Y_1,Y_2,Y_3}^{Y} \min\left(\|K\|_{\mathcal{F}^M_{|Y_1|+3}} \|L\|_{\mathcal{F}^M_{|Y_2|}}, \|K\|_{\mathcal{F}^M_{|Y_1|}} \|L\|_{\mathcal{F}^M_{|Y_2|+3}}\right) \\ \|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2} r_{Y_3}(t)\|_{L^\infty}, \tag{6.72}$$

$Y \in \Pi'$, where $C_{|Y|}$ depends only on ρ and $C'_{|Y|}$ depends only on $\|u\|_{E^\rho_{N_0}}$ and ρ. Inequalities (6.49b), (6.69), (6.71) and (6.72) give together with equality (6.68) that

$$\|\delta(t)(1+\lambda_1(t))^{k/2} g'_Y(t)\|_D \\ \leq (1+t)^{-3/2+\rho} C_{|Y|} \sum_{Y_1,Y_2}^{Y} \|L\|_{\mathcal{F}^M_{|Y_1|}} \Big(\|(\delta(t))^{3-\rho}(1+\lambda_1(t))^{k/2}(m+i\gamma^\mu\partial_\mu) r_Y(t)\|_{L^\infty} \\ + \sum_{\substack{Z\in\Pi' \\ |Z|\leq|Y_2|+1}} \|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2} r_Z(t)\|_{L^\infty}\Big) \\ + (1+t)^{-1} C_{|Y|} \sum_{Y_1,Y_2,Y_3}^{Y} \min\left(\|K\|_{\mathcal{F}^M_{|Y_1|+3}} \|L\|_{\mathcal{F}^M_{|Y_2|}}, \|K\|_{\mathcal{F}^M_{|Y_1|}} \|L\|_{\mathcal{F}^M_{|Y_2|+3}}\right) \\ \|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2} r_{Y_3}(t)\|_{L^\infty} \\ + (1+t)^{-3/2+\rho} C'_{|Y|} \sum_{Y_1,Y_2,Y_3}^{Y} \|u\|_{E^\rho_{N_0+|Y_1|+1}} \|L\|_{\mathcal{F}^M_{|Y_2|}} \|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2} r_{Y_3}(t)\|_{L^\infty}, \tag{6.73}$$

$Y \in \Pi'$, where $C_{|Y|}$ only depends ρ and $C'_{|Y|}$ only on ρ and $\|u\|_{E^\rho_{N_0}}$. The estimate of $R'_l(t)$ in statement i) of the lemma follows from the definition of $u_j(t)$, the definition of R'_j in Theorem 5.8 and from inequality (6.73). If on the right-hand side of equality (6.68) we restrict the domain of summation in the last sum over Y_1, Y_2, Y_3 to $Y_1 = \mathbb{I}$, then the right-hand side of this equality is equal to $(2m)^{-1}(m - i\gamma^\mu\partial_\mu + \gamma^\mu G_\mu) g_Y$. The method of

the proof of inequality (6.73) then gives also that $\|\delta(t)((m - i\gamma^\mu\partial_\mu + \gamma^\mu G_\mu)g_Y)(t)\|_D$ is majorized by the right-hand side of inequality (6.73). This shows that

$$(m - i\gamma^\mu\partial_\mu + \gamma^\mu G_\mu)g_Y \in L^1(\mathbb{R}^+, D), \quad Y \in \Pi', |Y| \leq n.$$

Since

$$g_Y = \sum_{Y_1,Y_2}^{Y} \gamma^\mu F_{Y_1\mu} r_{Y_2}, \quad Y \in \Pi',$$

it follows from Lemma 6.3 and inequality (6.66a) that

$$\begin{aligned} &\|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2} g_Y(t)\|_{L^\infty} \\ &\quad \leq C_{|Y|}(1+t)^{-1} \sum_{\substack{Z\in\Pi' \\ n_1+|Z|=|Y|}} \|L\|_{\mathcal{F}^M_{n_1+3}} \|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2} r_Z(t)\|_{L^\infty}, \end{aligned} \tag{6.74}$$

$Y \in \Pi', |Y| \leq n-3$, where $C_{|Y|}$ depends only on ρ. This inequality and the definition of R^∞_j in Theorem 5.8 prove statement ii) of the lemma. According to inequality (6.49b) and the definition of $\mathcal{F}^M_j$ we obtain that

$$\begin{aligned} &\wp^D_j\big((\delta(t))^{1/2}(1+\lambda_1(t))^{k/2} g(t)\big) \\ &\quad \leq (1+t)^{-1/2} \sum_{\substack{Z\in\Pi' \\ n_1+|Z|\leq j}} C_{n_1,|Z|} \|L\|_{\mathcal{F}^M_{n_1}} \|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2} r_Z(t)\|_{L^\infty}, \quad 0 \leq j \leq n, \end{aligned} \tag{6.75}$$

where C_{n_1,n_2} depends only on ρ. This inequality proves statement iii) of the lemma and it also proves that $\|g_Y(t)\|_D \to 0$, when $t \to \infty$ for $Y \in \Pi', |Y| \leq n$.

To prove statement iv) of the lemma we shall use statement i) of Proposition 5.16, with $t_0 = \infty, a_\mu = L_\mu, \rho' = 0, 1/2 < \rho < 1, \eta \in]1/2, \rho[$ and $\varepsilon = 1/2$. It follows from inequality (6.41a) and from definitions (5.115a) and (5.115b) of $S^{\rho,j}$ that $(A_Y, A_{P_0 Y}) \in C^0(\mathbb{R}^+, M^1)$ for $Y \in \Pi', |Y| \leq n$. Since, $\partial_\mu A^\mu = 0$ and $n \geq 2$, the hypothesis on A in Proposition 5.16 are satisfied. Due to the hypothesis on r and since $L \in \mathcal{F}^M_n$, the hypothesis of statement i) of Proposition 5.16 are satisfied. To use the estimate of $\wp^D_l(f(t))$ in that statement for $0 \leq l \leq n$, we first estimate $\theta^D_j(t)$ and $\tau^M_j(\infty, t)$, which we denote by $\tau^M_j(t)$. Here $\tau^M_j(\infty, t)$ is defined in (5.186) and the sequel. Definitions (6.32b) and (6.32c) of $\|\cdot\|_{\mathcal{F}^M_j}$ show that, if $L \in \mathcal{F}^M_j$, then

$$\tau^M_j(t) \leq (1+t)^{-3/2+\rho} C \|L\|_{\mathcal{F}^M_j}, \quad j \geq 0, \tag{6.76}$$

where C depends only on ρ. It follows from inequality (6.38c) that

$$\begin{aligned} \theta^D_j(t) \leq &\sum_{\substack{Y\in\Pi' \\ |Y|\leq j+1}} \sup_{s\geq t} \big((1+C_1\chi^{(3)})\|(\delta(s))^{3/2} r_Y(s)\|_{L^\infty}\big) \\ &+ \sum_{\substack{Y\in\Pi' \\ |Y|\leq j}} \sup_{s\geq t} \big(\|(\delta(s))^{3-\rho}((i\gamma^\mu\partial_\mu + m) r_Y)(s)\|_{L^\infty}\big) \\ \leq &\, C'(1+\chi^{(3)}) \sup_{s\geq 0} u_{j+1}(s), \quad j \geq 0, \end{aligned} \tag{6.77}$$

where C_1 and C' depend only on ρ and $\|u\|_{E^\rho_{N_0}}$. Inequalities (6.76) and (6.77) and the estimate in statement i) of Proposition 5.16 give that

$$\wp^D_{l,j}(f(t)) \leq (1+\chi^{(3)})(1+t)^{-3/2+\rho}\Big(C^{(j)}\|L\|_{\mathcal{F}^M_l}\sup_{s\geq t}(u_1(s)) + C_l \sum_{\substack{n_1+n_2=l\\ n_1\leq l-1}} \|L\|_{\mathcal{F}^M_{n_1}} \sup_{s\geq t}(u_{n_2+1}(s))\Big), \quad 0\leq l\leq n, \tag{6.78}$$

where $C^{(0)}$ and C_l depend only on ρ and $\|u\|_{E^\rho_{N_0}}$, and $C^{(j)}=0$ for $1\leq j\leq n$. This proves the proposition.

We shall prove a result, analog to Lemma 6.5, for the case where $g=\gamma^\mu(L_\mu-\partial_\mu\vartheta(L))\Phi$, $L\in\mathcal{F}^M_n$ and $\Phi\in\mathcal{F}^D_n$.

Lemma 6.6. *Let $1/2<\rho<1$, $n\geq 5$, $K\in\mathcal{F}^M_n$, $L\in\mathcal{F}^M_n$, $\Phi\in\mathcal{F}^D_n$ and let*

$$H_l(\Phi,t)=\sum_{\substack{Y\in\Pi'\\ k+|Y|\leq l}} \|\delta(t)^{3/2}(1+\lambda_1(t))^{k/2}\Phi_Y(t)\|_{L^\infty},$$

$$u_l(\Phi,t)=H_{l+1}(\Phi,t)+\sum_{\substack{Y\in\Pi'\\ k+|Y|\leq l}} \|\delta(t)^{3-\rho}(1+\lambda_1(t))^{k/2}\big(\xi^M_Y(i\gamma^\mu\partial_\mu+m-\gamma^\mu G_\mu)\Phi\big)(t)\|_{L^\infty},$$

$$U_{l,k}(\Phi,t)=\wp^D_{l+1}\big((1+\lambda_1(t))^{k/2}\Phi(t)\big)+\wp^D_l\big((1+\lambda_0(t))^{1/2}(1+\lambda_1(t))^{k/2}\Phi(t)\big)+\overline{U}_{l,k}(\Phi,t),$$

$$\overline{U}_{l,k}(\Phi,t)=\wp^D_l\big(\delta(t)^{3/2-\rho}(1+\lambda_1(t))^{k/2}\big((i\gamma^\mu\partial_\mu+m-\gamma^\mu G_\mu)\Phi\big)(t)\big)$$

and

$$U_l(\Phi,t)=\sum_{i+j=l} U_{i,j}(\Phi,t).$$

Then:

i) $$R'_l(t)\leq(1+t)^{-3/2+\rho}\Big(C'_l\sum_{n_1+n_2+n_3=l}(1+\|u\|_{E^\rho_{N_0+n_1+1}}) \min\big(\|L\|_{\mathcal{F}^M_{n_2}}u_{n_3}(\Phi,t),\|L\|_{\mathcal{F}^M_{n_2+4}}U_{n_3}(\Phi,t)\big) + C_l\sum_{n_1+n_2+n_3=l}\min\big(\|K\|_{\mathcal{F}^M_{n_1}}\|L\|_{\mathcal{F}^M_{n_2+3}}u_{n_3}(\Phi,t), \|K\|_{\mathcal{F}^M_{n_1+3}}\|L\|_{\mathcal{F}_{n_2}}u_{n_3}(\Phi,t),\|K\|_{\mathcal{F}^M_{n_1+3}}\|L\|_{\mathcal{F}^M_{n_2+3}}U_{n_3}(\Phi,t)\big)\Big),\quad 0\leq l,$$

ii) $$R^\infty_l(t)\leq(1+t)^{-1}C_l\sum_{n_1+n_2=l}\|L\|_{\mathcal{F}^M_{n_1+3}}H_{n_2}(\Phi,t),\quad 0\leq l,$$

iii) $\wp_l^D\big(\delta(t)(1+\lambda_1(t))^{k/2}g(t)\big) \leq \sum_{n_1+n_2=l} C_{n_1,n_2} \min\big(\|L\|_{\mathcal{F}^M_{n_1}} H_{n_2+k+1}(\Phi,t),$

$\|L\|_{\mathcal{F}^M_{n_1+3}} \wp^D_{n_2}\big((1+\lambda_1(t))^{k/2}\Phi(t)\big)\big), \quad 0 \leq l,$

iv) $\wp_l^D(f(t)) + (1+t)\wp_l^D(g(t)) \leq (1+t)^{-3/2+\rho}$

$\sum_{n_1+n_2=l} C_{n_1,n_2} \min\big(\|L\|_{\mathcal{F}^M_{n_1+3}} \|\Phi\|_{\mathcal{F}^D_{n_2}}, \|L\|_{\mathcal{F}^M_{n_1}} \sup_{s\geq t}\big((1+s)^{3/2-\rho}H_{n_2}(\Phi,s)\big)\big)$

and

$\wp^D_{l,i}(f(t)) \leq (1+t)^{-3/2+\rho} \sum_{n_1+n_2=l-1} C'_{n_1,n_2} \sup_{s\geq t}$

$\min\big(\|L\|_{\mathcal{F}^M_{n_1}}\big(u_{n_2}(\Phi,s) + (1+s)^{3/2-\rho}H_{n_2+1}(\Phi,s)\big), \|L\|_{\mathcal{F}^M_{n_1+3}}\big(\|\Phi\|_{\mathcal{F}^M_{n_2+1}} + \overline{U}_{n_2,0}(\Phi,s)\big)\big),$

$l \geq 0, 1 \leq i \leq l$. Here $C_{l,0} = C_{0,l} = C'_{l-1,0} = C'_{0,l-1} = C$, C_{n_1,n_2}, C'_{n_1,n_2} depend only on ρ and C'_l depends only on ρ and $\|u\|_{E^\rho_{N_0}}$.

Proof. Since

$$g_Y = \sum_{Y_1,Y_2}^{Y} \gamma^\mu F_{Y_1\mu}\Phi_{Y_2}, \quad Y \in \Pi'$$

where $F_\mu = L_\mu - \partial_\mu\vartheta(L), (\xi^M_Y F)_\mu = F_{Y\mu}$, and since $\delta(t) \leq C(1+\lambda_1(t)+t)$ where C is independent of t, it follows that

$$\begin{aligned}&\|\delta(t)(1+\lambda_1(t))^{k/2}g_Y(t)\|_D \\ &\leq C\sum_{Y_1,Y_2}^{Y} \min\big(\|\delta(t)F_{Y_1}(t)\|_{L^\infty}\|(1+\lambda_1(t))^{k/2}\Phi_{Y_2}(t)\|_D, \\ &\qquad (1+t)^{1/2}\|\delta(t)^{-1}F_{Y_1}(t)\|_{L^2}H_{|Y_2|+k+1}(\Phi,t)\big),\end{aligned}$$

where C is independent of $t \geq 0$, $Y \in \Pi'$, L and Φ. Inequality (6.49b), Lemma 6.3 and inequality (6.66a) then give that

$$\begin{aligned}&\wp_l^D\big(\delta(t)(1+\lambda_1(t))^{k/2}(g(t)\big) \\ &\leq \sum_{n_1+n_2=l} C_{n_1,n_2}\min\big(\|L\|_{\mathcal{F}^M_{n_1+3}}\wp^D_{n_2}((1+\lambda_1(t))^{k/2}\Phi(t)), \|L\|_{\mathcal{F}^M_{n_1}}H_{n_2+k+1}(\Phi,t)\big),\end{aligned}$$

where $C_{l,0} = C_{0,l} = C$ and C_{n_1,n_2} depends only on ρ. This proves statement iii) of the lemma. Statement iv) follows from statement iii) and the definition of the spaces $\mathcal{F}^D_j$ and by using (5.7b′) when $i \geq 1$. Statement ii) follows as in the proof of Lemma 6.5. To prove statement i) let $\overline{g} = (i\gamma^\mu\partial_\mu + m - \gamma^\mu G_\mu)\Phi$ and $\overline{g}_Y = \xi^D_Y\overline{g}$, $g'_Y = (2m)^{-1}\xi^D_Y(m - i\gamma^\mu\partial_\mu + \gamma^\mu G_\mu)g$, for $Y \in \Pi'$. It follows from equality (6.67) that

$$\begin{aligned}g'_Y = (2m)^{-1}&\sum_{Y_1,Y_2}^{Y}\Big(\gamma^\nu F_{Y_1\nu}\overline{g}_{Y_2} - 2iF^\mu_{Y_1}\partial_\mu\Phi_{Y_2} + i(\partial_\mu B^\mu_{Y_1})\Phi_{Y_2} \\ &- \frac{i}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\big((\partial_\mu L_{Y_1\nu}) - (\partial_\nu L_{Y_1\mu})\big)\Phi_{Y_2}\Big) \\ &+ m^{-1}\sum_{Y_1,Y_2,Y_3}^{Y} G_{Y_1\mu}F^\mu_{Y_2}\Phi_{Y_3}, \quad Y \in \Pi',\end{aligned}$$

where $B_\mu = -\partial_\mu \vartheta(L)$. We denote S_{Y_1,Y_2} the terms of the first sum and T_{Y_1,Y_2,Y_3} the terms of the second sum of this expression. It follows by the same argument as in the proof of Lemma 6.5 led from (6.68) to (6.73) that

$$\|\delta(t)(1+\lambda_1(t))^{k/2} S_{Y_1,Y_2}(t)\|_D \leq (1+t)^{-3/2+\rho} a_{|Y|} \|L\|_{\mathcal{F}^M_{|Y_1|}} u_{|Y_2|+k}(\Phi, t)$$

and that

$$\begin{aligned}
&\|\delta(t)(1+\lambda_1(t))^{k/2} T_{Y_1,Y_2,Y_3}(t)\|_D \\
&\quad \leq (1+t)^{-3/2+\rho} b'_{|Y|} \|u\|_{E^\rho_{N_0+|Y_1|+1}} \|L\|_{\mathcal{F}^M_{|Y_2|}} H_{|Y_3|+k}(\Phi, t) \\
&\qquad + (1+t)^{-1} b_{|Y|} \min\bigl(\|K\|_{\mathcal{F}^M_{|Y_1|}} \|L\|_{\mathcal{F}^M_{|Y_2|+3}}, \|K\|_{\mathcal{F}^M_{|Y_1|+3}} \|L\|_{\mathcal{F}^M_{|Y_2|}}\bigr) H_{|Y_3|+k}(\Phi, t),
\end{aligned}$$

where $a_{|Y|}$ and $b_{|Y|}$ depend only on ρ and $b'_{|Y|}$ depends only on ρ and $\|u\|_{E^\rho_{N_0}}$. Thus, to prove statement i) of the lemma we only have to show that the last argument of the minima functions in the estimate of R'_l in statement i) majorizes S_{Y_1,Y_2} and T_{Y_1,Y_2,Y_3}. It follows from inequality (5.7d) and equality (5.133) that

$$\begin{aligned}
&\|\delta(t)(1+\lambda_1(t))^{k/2} F^\mu_{Y_1} \partial_\mu \Phi_{Y_2}(t)\|_D \\
&\quad \leq C_{Y_1} \Bigl(\sum_{\substack{Z \in \Pi' \\ |Z| \leq |Y_1|}} \|F_Z(t)\|_{L^\infty} \Bigr) \wp^D_{|Y_2|+1}\bigl((1+\lambda_1(t))^{k/2} \Phi(t)\bigr)
\end{aligned}$$

and it then follows that

$$\begin{aligned}
&\|\delta(t)(1+\lambda_1(t))^{k/2} S_{Y_1,Y_2}\|_D \\
&\quad \leq C_Y \Bigl(\|\delta(t) F_{Y_1}(t)\|_{L^\infty} \|(1+\lambda_1(t))^{k/2} \overline{g}_{Y_2}(t)\|_D \\
&\qquad + \sum_{\substack{Z \in \Pi' \\ |Z| \leq |Y_1|}} \|F_Z(t)\|_{L^\infty} \wp^D_{|Y_2|+1}\bigl((1+\lambda_1(t))^{k/2} \Phi(t)\bigr) \\
&\qquad + \Bigl(\|\delta(t)(1+\lambda_0(t))^{-1/2} \partial_\mu B^\mu_{Y_1}(t)\|_{L^\infty} + \sum_{\mu,\nu} \|\delta(t)(1+\lambda_0(t))^{-1/2} \partial_\mu L_{Y_1 \nu}(t)\|_{L^\infty} \Bigr) \\
&\qquad\quad \wp^D_{|Y_2|}\bigl((1+\lambda_0(t))^{1/2}(1+\lambda_1(t))^{k/2} \Phi(t)\bigr) \Bigr),
\end{aligned}$$

where λ_0 is given in Theorem 5.5. It now follows from inequalities (5.33), (6.66a), (6.66b) and from Lemma 6.3 that

$$\begin{aligned}
&\|\delta(t)(1+\lambda_1(t))^{k/2} S_{Y_1,Y_2}(t)\|_D \\
&\quad \leq C_{|Y|} \Bigl(\|L\|_{\mathcal{F}^M_{|Y_1|+3}} \|(1+\lambda_1(t))^{k/2} \overline{g}_{Y_2}(t)\|_D \\
&\qquad + (1+t)^{-1} \|L\|_{\mathcal{F}^M_{|Y_1|+3}} \wp^D_{|Y_2|+1}\bigl((1+\lambda_1(t))^{k/2} \Phi(t)\bigr) \\
&\qquad + (1+t)^{-3/2+\rho} \|L\|_{\mathcal{F}^M_{|Y_1|+4}} \wp^D_{|Y_2|}\bigl((1+\lambda_0(t))^{1/2}(1+\lambda_1(t))^{k/2} \Phi(t)\bigr) \Bigr).
\end{aligned}$$

This gives that

$$\|\delta(t)(1+\lambda_1(t))^{k/2} S_{Y_1,Y_2}(t)\|_D \leq C_{|Y|}(1+t)^{-3/2+\rho}\|L\|_{\mathcal{F}^M_{|Y_1|+4}} U_{|Y_2|,k}(\Phi,t),$$

where $C_{|Y|}$ depends only on ρ. Proceeding as in the proof of Lemma 6.5, we obtain that

$$\begin{aligned}&\|\delta(t)(1+\lambda_1(t))^{k/2} T_{Y_1,Y_2,Y_3}(t)\|_D \\ &\quad\leq C'_{|Y_1|}\|u\|_{E^\rho_{N_0+|Y_1|+1}}\|\delta(t)^{\rho-1/2}(1+\lambda_1(t))^{k/2} F_{Y_2}(t)\Phi_{Y_3}(t)\|_D \\ &\qquad + C\|K\|_{\mathcal{F}^M_{|Y_1|+3}}\|F_{Y_2}(t)\|_{L^\infty}\wp^D_{|Y_3|}\big((1+\lambda_1(t))^{k/2}\Phi(t)\big).\end{aligned}$$

Lemma 6.3 and inequality (6.66a) then give, changing constants, that

$$\begin{aligned}&\|\delta(t)(1+\lambda_1(t))^{k/2} T_{Y_1,Y_2,Y_3}(t)\|_D \\ &\quad\leq (1+t)^{-3/2+\rho}\big(C'_{|Y_1|}\|u\|_{E^\rho_{N_0+|Y_1|+1}} + C(1+t)^{-\rho+1/2}\|K\|_{\mathcal{F}^M_{|Y_1|+3}}\big) \\ &\qquad \|L\|_{\mathcal{F}^M_{|Y_2|+3}}\wp^D_{|Y_3|}\big((1+\lambda_1(t))^{k/2}\Phi(t)\big),\end{aligned}$$

where $C'_{|Y_1|}$ depends only on ρ and $\|u\|_{E^\rho_{N_0}}$ and C depends only on ρ. This proves statement i) of the lemma and therefore the lemma.

We now return to the *study of the map* $\mathcal{N}$ introduced after equation (6.33).

Proposition 6.7. *If $1/2<\rho<1$ and $n\geq 50$, then $\mathcal{N}$ is a map from $E^{\circ\rho}_\infty \times \mathcal{F}^M_n$ to $\mathcal{F}^M_n$,*

$$\|\mathcal{N}(u,K)\|_{\mathcal{F}^M_n} \leq C_n\|u\|_{E^\rho_{N_0+n+3}}(\|u\|_{E^\rho_{N_0+n}} + \|K\|_{\mathcal{F}^M_n})$$

and

$$\|\mathcal{N}(u,K^{(1)}) - \mathcal{N}(u,K^{(2)})\|_{\mathcal{F}^M_n} \leq C'_n\|u\|_{E^\rho_{N_0+n+3}}\|K^{(1)}-K^{(2)}\|_{\mathcal{F}^M_n},$$

for $u\in E^{\circ\rho}_\infty, K, K^{(1)}, K^{(2)} \in \mathcal{F}^M_n$, where C_n depends only on $\rho, \|u\|_{E^\rho_{N_0+n+2}}$ and $\|K\|_{\mathcal{F}^M_n}$, and C'_n depends only on $\rho, \|u\|_{E^\rho_{N_0+n+2}}$ and $\|K^{(i)}\|_{\mathcal{F}^M_n}, i=1,2$.

Proof. Let $u\in E^{\circ\rho}_\infty, K\in\mathcal{F}^M_n, L\in\mathcal{F}^M_n$, $A=A^*+K$ and let $G_\mu = A_\mu - \partial_\mu\vartheta(A)$ for $0\leq\mu\leq 3$.

We first consider the equation

$$(i\gamma^\mu\partial_\mu + m - \gamma^\mu G_\mu)\Psi = \gamma^\mu\big(L_\mu - (\partial_\mu\vartheta(L))\big)r, \tag{6.79}$$

with unknown $\Psi\in\mathcal{F}^M_n$, where $r=(D^j\phi^*)(u;v_1,\ldots,v_j)$ for some $j\geq 0$ and $v_1,\ldots,v_j\in E^{\circ\rho}_\infty$. (For this proof we only need the case $j=0$ and we consider $j\geq 0$ only to prepare later proofs). It follows from Proposition 6.2 that

$$H_l(r,t) \leq C_{l,j}\big(R^j_{N_0,l}(v_1,\ldots,v_j) + \|u\|_{E^\rho_{N_0+l}}\|v_1\|_{E^\rho_{N_0}}\cdots\|v_j\|_{E^\rho_{N_0}}\big), \tag{6.80}$$

$l \geq 0$, where $H_l(r,t)$ is defined in Lemma 6.5 and $C_{l,j}$ depends only on ρ and $\|u\|_{E^\rho_{N_0}}$. Equations (6.2a) and substitution (6.30) give that

$$(i\gamma^\mu \partial_\mu + m)\phi^* = \gamma^\mu (A^*_{0,\mu} + B^*_{0,\mu})\phi^*.$$

Derivation of this equation j times in u, inequality (5.116c), Proposition 6.2 and the convexity property of the seminorms $\|\cdot\|_{E^\rho_N}$ in Corollary 2.6 show together with inequality (6.80) that

$$u_l(t) \leq C_{l,j}\big(R^j_{N_0,l+1}(v_1,\ldots,v_j) + \|u\|_{E^\rho_{N_0+l+1}} \|v_1\|_{E^\rho_{N_0}} \cdots \|v_j\|_{E^\rho_{N_0}}\big), \tag{6.81}$$

$l \geq 0$, where $u_l(t)$ is defined in Lemma 6.5 and where $C_{l,j}$ depends only on ρ and $\|u\|_{E^\rho_{N_0}}$. It follows from Proposition 6.2 that

$$\begin{aligned} &\sum_{i+k\leq l} \wp^D_i\big((1+\lambda_1(t))^{k/2} r(t)\big) \\ &\qquad \leq C_{l,j}\big(R^j_{N_0,l}(v_1,\ldots,v_j) + \|u\|_{E^\rho_{N_0+l}} \|v_1\|_{E^\rho_{N_0}} \cdots \|v_l\|_{E^\rho_{N_0}}\big), \end{aligned} \tag{6.82}$$

$l \geq 0$, where $C_{l,j}$ depends only on ρ and $\|u\|_{E^\rho_{N_0}}$. According to inequalities (6.80), (6.81) and (6.82), the hypotheses of Lemma 6.5 are satisfied, and it follows from this lemma that, if $g = \gamma^\mu(L_\mu - \partial_\mu \vartheta(L))$, $g_Y = \xi^D_Y g$ for $Y \in \Pi'$, then $g_Y \in C^0(\mathbb{R}^+, D)$, $\|g_Y(t)\|_D \to 0$ when $t \to \infty$ and $(m - i\gamma^\mu\partial_\mu + \gamma^\mu G_\mu) g_Y \in L^1(\mathbb{R}^+, D)$ for $Y \in \Pi'$, $|Y| \leq n$. Statement i)–iv) of Lemma 6.5 and the convexity property of the seminorms $\|\cdot\|_{E^\rho_N}$ give, with the notation

$$\tau^j_{N_0,l} = R^j_{N_0,l}(v_1,\ldots,v_j) + \|u\|_{E^\rho_{N_0+l}} \|v_1\|_{E^\rho_{N_0}} \cdots \|v_j\|_{E^\rho_{N_0}},$$

that

$$\begin{aligned} R'_l(t) \leq (1+t)^{-3/2+\rho} C_{l,j}\Big(&\sum_{n_1+n_2=l} \tau^j_{N_0,n_1+2} \|L\|_{\mathcal{F}_{n_2}} \\ &+ \sum_{\substack{n_1+n_2+n_3=l \\ n_2\leq n_3}} \tau^j_{N_0,n_1+1}\big(\|K\|_{\mathcal{F}^M_{n_2+3}} \|L\|_{\mathcal{F}^M_{n_3}} + \|L\|_{\mathcal{F}^M_{n_2+3}} \|K\|_{\mathcal{F}^M_{n_3}}\big)\Big), \quad 0 \leq l \leq n, \end{aligned} \tag{6.83a}$$

$$R^\infty_l(t) \leq (1+t)^{-1} C_{l,j} \sum_{n_1+n_2=l} \tau^j_{N_0,n_1} \|L\|_{\mathcal{F}^M_{n_2+3}}, \quad 0 \leq l \leq n-3, \tag{6.83b}$$

$$\begin{aligned} &\wp^D_l\big((1+\lambda_1(t))^{k/2} g(t)\big) \\ &\qquad \leq (1+t)^{-1} C_{l,k,j} \sum_{n_1+n_2=l} \tau^j_{N_0,n_1+k} \|L\|_{\mathcal{F}^M_{n_2}}, \quad 0 \leq l \leq n, k \geq 0, \end{aligned} \tag{6.83c}$$

and

$$\wp_l^D(f(t)) \le (1+t)^{-3/2+\rho} C_j (1+\chi^{(3)}) \Big(\tau^j_{N_0,2}\|L\|_{\mathcal{F}^M_l} + C_{l,j} \sum_{\substack{n_1+n_2=l\\ n_2\le l-1}} \tau^j_{N_0,n_1+2}\|L\|_{\mathcal{F}^M_{n_2}}\Big), \quad 0\le l\le n, \tag{6.83d}$$

where $C_j, C_{l,j}$ and $C_{l,k,j}$ depend only on ρ and $\|u\|_{E^\rho_{N_0}}$. This proves that the hypothesis of Proposition 6.4 are satisfied and therefore we can conclude that equation (6.79) has a unique solution $\Psi \in \mathcal{F}^D_n$, satisfying the inequalities of statement i)–iv) of Proposition 6.4.

Proposition 6.2 shows that $\Delta^{*M} \in \mathcal{F}^M_l$ for $l \ge 0$. Let $K^{(1)}, K^{(2)} \in \mathcal{F}^M_n$ and let $\Phi^{(i)} \in \mathcal{F}^M_n$ be the solution of equation (6.79) when $L = K^{(i)} + \Delta^{*M}, K = K^{(i)}$ and $r = \phi^*$, i.e. the solution of equation (6.31b) with $K = K^{(i)}$. According to expression (6.33) let

$$K'^{(i)}_\mu(t) = -\int_t^\infty |\nabla|^{-1}\sin(|\nabla|(t-s))(\overline{\Phi}^{(i)}\gamma_\mu\Phi^{(i)} + \overline{\Phi}^{(i)}\gamma_\mu\phi^* + \overline{\phi}^*\gamma_\mu\Phi^{(i)})(s)ds, \tag{6.84}$$

for $0\le\mu\le 3, t\ge 0, i=1,2$. It follows from equation (6.2a), with $\phi_1 = \phi^*$, from equation (6.79) and from expression (6.84) that $\partial^\mu K'^{(i)}_\mu = 0$. It follows from expression (6.84) that

$$\begin{aligned}(K'^{(1)}_Y &- K'^{(2)}_Y)_\mu(t)\\ &= -\int_t^\infty |\nabla|^{-1}\sin(|\nabla|(t-s))\\ &\quad \sum_{Y_1,Y_2}^{Y}\big((\overline{\Phi}^{(1)}_{Y_1} + \overline{\phi}^*_{Y_1})\gamma_\mu(\Phi^{(1)}_{Y_2} - \Phi^{(2)}_{Y_2}) + (\overline{\Phi}^{(1)}_{Y_1} - \overline{\Phi}^{(2)}_{Y_1})\gamma_\mu(\Phi^{(2)}_{Y_2} + \phi^*_{Y_2})\big)(s)ds,\end{aligned} \tag{6.85}$$

for $Y \in \Pi'$. Using the notation

$$H_l(\Phi,t) = \sum_{\substack{Y\in\Pi'\\ k+|Y|\le l}} \|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2}\Phi_Y(t)\|_{L^\infty}, \tag{6.86}$$

for $l\ge 0, N_Y = K'^{(1)}_Y - K'^{(2)}_Y, \dot N_Y = N_{P_0 Y}$, equality (6.85) gives that

$$\begin{aligned}&(1+t)^{1-\rho}\wp_l^{M^0}(N(t),\dot N(t)) + (1+t)^{2-\rho}\wp_l^{M^1}(N(t),\dot N(t))\\ &\le \sup_{s\ge t}\Big((1+s)^{3/2-\rho}\Big(C\big(H_0(\Phi^{(1)},s) + H_0(\Phi^{(2)},s) + H_0(\phi^*,s)\big)\wp_l^D(\Phi^{(1)}(s) - \Phi^{(2)}(s))\\ &\quad + C\big(\wp_l^D(\Phi^{(1)}(s)) + \wp_l^D(\Phi^{(2)}(s))\big)H_0(\Phi^{(1)} - \Phi^{(2)}, s)\\ &\quad + C_l \sum_{\substack{n_1+n_2=l\\ n_1\le n_2\le l-1}} \Big(\big(H_{n_1}(\Phi^{(1)},s) + H_{n_1}(\Phi^{(2)},s)\big)\wp^D_{n_2}(\Phi^{(1)}(s) - \Phi^{(2)}(s))\\ &\qquad + H_{n_1}(\Phi^{(1)} - \Phi^{(2)},s)\big(\wp^D_{n_2}(\Phi^{(1)}(s)) + \wp^D_{n_2}(\Phi^{(2)}(s))\big)\Big)\\ &\quad + C_l \sum_{\substack{n_1+n_2=l\\ n_2\le l-1}} H_{n_1}(\phi^*,s)\wp^D_{n_2}(\Phi^{(1)}(s) - \Phi^{(2)}(s))\Big)\Big), \quad 0\le l\le n, t\ge 0,\end{aligned} \tag{6.87}$$

where C and C_l are independent of t, ϕ^* and $\Phi^{(i)}$. Using that there are two positive real numbers C and C' such that $(\delta(t))(x) \leq C(1+t+(\lambda_1(t))(x)) \leq C''(\delta(t))(x)$ and that $(\delta(t))(x) = 1+t+|x|$ we obtain by Hölder inequality and with a new constant:

$$\begin{aligned}
&(1+t)^{1-\rho}\|\delta(t)f_1(t)f_2(t)\|_{L^{6/5}} \\
&\quad\leq (1+t)^{1-\rho}\|f_1(t)\|_{L^2}\|\delta(t)f_2(t)\|_{L^3} \\
&\quad\leq (1+t)^{3/2-\rho}C\|f_1(t)\|_{L^2}\|f_2(t)\|_{L^2}^{2/3}\|(\delta(t))^{3/2}(1+\lambda_1(t))^{3/2}f_2(t)\|_{L^\infty}^{1/3} \\
&\quad\leq (1+t)^{3/2-\rho}C\|f_1(t)\|_{L^2}\big(\|f_2(t)\|_{L^2}+\|(\delta(t))^{3/2}(1+\lambda_1(t))^{3/2}f_2(t)\|_{L^\infty}\big).
\end{aligned} \tag{6.88}$$

This inequality and the fact that $\square K'^{(i)}_\mu = \overline{\Phi}^{(i)}\gamma_\mu\Phi^{(i)} + \overline{\Phi}^{(i)}\gamma_\mu\phi^* + \overline{\phi}^*\gamma^\mu\Phi^{(i)}$ give, similarly as inequality (6.87) was obtained, that

$$\begin{aligned}
&\sum_{\substack{Y\in\Pi' \\ |Y|\leq l}} ((1+t)^{2-\rho}\|\delta(t)\square N_Y(t)\|_{L^2} + (1+t)^{1-\rho}\|\delta(t)\square N_Y(t)\|_{L^{6/5}} \\
&\quad\leq \sup_{s\geq t}\Big((1+s)^{3/2-\rho}\Big(C\big(H_3(\Phi^{(1)},s)+H_3(\Phi^{(2)},s)+H_3(\phi^*,s) \\
&\qquad + \wp_0^D(\Phi^{(1)}(s)) + \wp_0^D(\Phi^{(2)}(s)) + \wp_0^D(\phi^*(s))\big)\wp_l^D(\Phi^{(1)}(s)-\Phi^{(2)}(s)) \\
&\qquad + C\big(\wp_l^D(\Phi^{(1)}(s)) + \wp_l^D(\Phi^{(2)}(s))\big)\big(\wp_0^D(\Phi^{(1)}(s)-\Phi^{(2)}(s)) + H_3(\Phi^{(1)}-\Phi^{(2)},s)\big) \\
&\qquad + C_l \sum_{\substack{n_1+n_2=l \\ n_1\leq n_2\leq l-1}} \Big(\big(H_{n_1+3}(\Phi^{(1)},s)+H_{n_1+3}(\Phi^{(2)},s)+H_{n_1+3}(\phi^*,s) \\
&\qquad + \wp_{n_1}^D(\Phi^{(1)}(s)) + \wp_{n_1}^D(\Phi^{(2)}(s)) + \wp_{n_1}^D(\phi^*(s))\big)\wp_{n_2}^D(\Phi^{(1)}(s)-\Phi^{(2)}(s)) \\
&\qquad + \big(H_{n_1+3}(\Phi^{(1)}-\Phi^{(2)},s) + \wp_{n_1}^D(\Phi^{(1)}(s)-\Phi^{(2)}(s))\big)\big(\wp_{n_2}^D(\Phi^{(1)}(s)) + \wp_{n_2}^D(\Phi^{(1)}(s))\big)\Big) \\
&\qquad + C_l \sum_{\substack{n_1+n_2=l \\ n_2\leq l-1}} \big(H_{n_1+3}(\phi^*,s) + \wp_{n_1}^D(\phi^*(s))\big)\wp_{n_2}^D(\Phi^{(1)}(s)-\Phi^{(2)}(s))\Big)\Big),
\end{aligned} \tag{6.89}$$

$0 \leq l \leq n, t \geq 0$, where C and C_l are independent of t, ϕ^* and $\Phi^{(i)}$. It follows from inequalities (6.87) and (6.89) that

$$\begin{aligned}
&\|K'^{(1)} - K'^{(2)}\|_{\mathcal{F}_l^M} \\
&\quad\leq \sup_{t\geq 0}\Big(C\big(H_3(\phi^*,t) + \wp_0^D(\phi^*(t)) + \sum_{i=1}^{2}\big(H_3(\Phi^{(i)},t) + \wp_0^D(\Phi^{(i)}(t))\big)\big)\|\Phi^{(1)}-\Phi^{(2)}\|_{\mathcal{F}_l^D} \\
&\qquad + C\big(\wp_0^D(\Phi^{(1)}(t)-\Phi^{(2)}(t)) + H_3(\Phi^{(1)}-\Phi^{(2)},t)\big)\big(\|\Phi^{(1)}\|_{\mathcal{F}_l^D} + \|\Phi^{(2)}\|_{\mathcal{F}_l^D}\big) \\
&\qquad + C_l \sum_{\substack{n_1+n_2=l \\ n_1\leq n_2\leq l-1}} \Big(\big(H_{n_1+3}(\phi^*,t) + \wp_{n_1}^D(\phi^*(t)) \\
&\qquad\quad + \sum_{i=1}^{2}\big(H_{n_1+3}(\Phi^{(i)},t) + \wp_{n_1}^D(\Phi^{(i)}(t))\big)\big)\|\Phi^{(1)}-\Phi^{(2)}\|_{\mathcal{F}_{n_2}^D}
\end{aligned} \tag{6.90}$$

$$+ \big(H_{n_1+3}(\Phi^{(1)} - \Phi^{(2)}, t) + \wp^D_{n_1}(\Phi^{(1)}(t) - \Phi^{(2)}(t))\big)\big(\|\Phi^{(1)}\|_{\mathcal{F}^D_{n_2}} + \|\Phi^{(2)}\|_{\mathcal{F}^D_{n_2}}\big)\Big)$$
$$+ C_l \sum_{\substack{n_1+n_2=l \\ n_2 \leq l-1}} \Big(H_{n_1+3}(\phi^*, t) + \wp^D_{n_1}(\phi^*(t))\Big)\|\Phi^{(1)} - \Phi^{(2)}\|_{\mathcal{F}^D_{n_2}}\Big), \quad 0 \leq l \leq n, t \geq 0,$$

where C and C_l are independent of t, ϕ^* and $\Phi^{(i)}$.

Let us denote by $R'_l(t)$ (resp. $R^\infty_l(t), R^2_l(t), \wp^D_l(f(t))$) on the left-hand side of inequality (6.83a) (resp. (6.83b), (6.83c), (6.83d)) by $R'_l(K,L,t)$ (resp. $R^\infty_l(L,t)$, $(\wp^D_l((1+\lambda_1(t))^{k/2}g(L,t))$, $\wp^D_l(f(K,L,t)))$ to indicate the dependence of these quantities on $K, L \in \mathcal{F}^M_n$. It follows from inequalities (6.83a)–(6.83d), in the case $j = 0$, that

$$\sup_{t\geq 0}\big((1+t)^{3/2-\rho}\big(R'_l(K,L,t) + \wp^D_l(f(K,L,t))\big)\big) \tag{6.91a}$$
$$\leq C_l\big(1 + \|u\|_{E^\rho_{N_0+l_0}} + \|K\|_{\mathcal{F}^M_{l_0}}\big)\|u\|_{E^\rho_{N_0+l+2}}\|M\|_{\mathcal{F}^M_l},$$

$0 \leq l \leq n$, where l_0 is the integer part of $l/2+3$ and where C_l depends only on ρ and $\|u\|_{E^\rho_{N_0}}$, it follows that

$$\sup_{t\geq 0}((1+t)R^\infty_l(L,t)) \leq C_l\|u\|_{E^\rho_{N_0+l}}\|L\|_{\mathcal{F}^M_{l+3}}, \quad 0 \leq l \leq n-3, \tag{6.91b}$$

where C_l depends only on ρ and $\|u\|_{E^\rho_{N_0}}$ and it follows that

$$\sup_{t\geq 0}\big((1+t)\wp^D_l\big((1+\lambda_1(t))^{k/2}g(L,t)\big)\big) \leq C_{l,k}\|u\|_{E^\rho_{N_0+l+k}}\|L\|_{\mathcal{F}^M_l}, \tag{6.91c}$$

$0 \leq l \leq n, k \geq 0$, where $C_{l,k}$ depends only on ρ and $\|u\|_{E^\rho_{N_0}}$. Introducing also, for the left-hand side of inequalities (6.39a)–(6.39e) the notation $\overline{Q}_n(K,L,t)$, $Q'_n(K,L,t)$, $R^{(1)}_n(K,L,t)$, $h'_n(K,L,n_0,t)$ and $h''_n(K,L,t)$ to stress the dependence on K, L and n_0, it follows from inequalities (6.91a), (6.91b) and (6.91c) that, if n_0 is the integer part of $n/2+5$, then

$$\overline{Q}_l(K,L,t) \tag{6.92a}$$
$$\leq C_l\big(1 + \|u\|_{E^\rho_{N_0+l+5}} + \|K\|_{\mathcal{F}^M_{l+5}}\big)^2\|u\|_{E^\rho_{N_0+l+2}}\|L\|_{\mathcal{F}^M_l}, \quad 0 \leq l \leq n-5,$$

$$R^{(1)}_l(K,L,t) \tag{6.92b}$$
$$\leq C_l\big(1 + \|u\|_{E^\rho_{N_0+l+13}} + \|K\|_{\mathcal{F}^M_{l+13}}\big)^2\|u\|_{E^\rho_{N_0+l+10}}\|L\|_{\mathcal{F}^M_{l+9}}, \quad 0 \leq l \leq n-13,$$

$$h'_l(K,L,n_0,t) + h''_l(K,L,t) \tag{6.92c}$$
$$\leq C'_l\big(\|u\|_{E^\rho_{N_0+n_0+5}} + \|K\|_{\mathcal{F}^M_{n_0+5}} + \|u\|_{E^\rho_{N_0+l}} + \|K\|_{\mathcal{F}^M_l}\big)\|u\|_{E^\rho_{N_0+n_0+2}}\|L\|_{\mathcal{F}^M_{n_0}}, \ 0 \leq l \leq n,$$

and

$$h^\infty_l(K,L,n_0,t) \leq C'_l\big(\|u\|_{E^\rho_{N_0+n_0+2}}\|L\|_{\mathcal{F}^M_{n_0}} + \|u\|_{E^\rho_{N_0+l}}\|L\|_{\mathcal{F}^M_{l-1}}\big) \tag{6.92d}$$
$$\big(\|u\|_{E^\rho_{N_0+n_0+5}} + \|K\|_{\mathcal{F}^M_{n_0+5}} + \|u\|_{E^\rho_{N_0+l}} + \|K\|_{\mathcal{F}^M_l}\big), \quad 0 \leq l \leq n,$$

where C_l depends only on ρ and $\|u\|_{E^\rho_{N_0}}$, where C'_l depends only on ρ, $\|u\|_{E^\rho_{N_0+n_0+5}}$ and $\|K\|_{\mathcal{F}^M_{n_0+5}}$, and where h^∞_l is defined in statement ii) of Proposition 6.4. In deducing (6.92c) we have used that $n-n_0 \le n_0-9$. It follows from statements ii) and iii) of Proposition 6.4 and from inequalities (6.91a), (6.92b) and (6.92d) that, if $r=\phi^*$ then the solution of equation (6.79) satisfies

$$\|\Psi\|_{\mathcal{F}^D_n} \le C_n \|u\|_{E^\rho_{N_0+n+2}} \|L\|_{\mathcal{F}^M_n} \tag{6.93a}$$

and

$$H_l(\Psi,t) \le (1+t)^{-3/2+\rho} C_l \|u\|_{E^\rho_{N_0+l+10}} \|L\|_{\mathcal{F}^M_{l+9}}, \tag{6.93b}$$

$0 \le l \le n-13$, where C_n and C_l depend only on $\|u\|_{E^\rho_{N_0+n}}$ and $\|K\|_{\mathcal{F}^M_n}$, and where we have used that $n-n_0 \le n_0-9$.

It follows from inequalities (6.93a) and (6.93b), with $\Psi=\Phi^{(i)}$, $K=K^{(i)}$, $L=K^{(i)}+\Delta^{*M}$ and from Proposition 6.2 that

$$\begin{aligned}\|\Phi^{(i)}\|_{\mathcal{F}^D_n} &\le C^{(i)}_n \|u\|_{E^\rho_{N_0+n+2}} \|K^{(i)}+\Delta^{*M}\|_{\mathcal{F}^M_n} \\ &\le C'^{(i)}_n \|u\|_{E^\rho_{N_0+n+2}} (\|u\|_{E^\rho_{N_0+n}} + \|K^{(i)}\|_{\mathcal{F}^M_n})\end{aligned} \tag{6.94a}$$

and that

$$\begin{aligned}H_l(\Phi^{(i)},t) &\le (1+t)^{-3/2+\rho} C^{(i)}_l \|u\|_{E^\rho_{N_0+l+10}} \|K^{(i)}+\Delta^{*M}\|_{\mathcal{F}^M_{l+9}} \\ &\le (1+t)^{-3/2+\rho} C'^{(i)}_l \|u\|_{E^\rho_{N_0+l+10}} (\|u\|_{E^\rho_{N_0+l+9}} + \|K^{(i)}\|_{\mathcal{F}^M_{l+9}}),\end{aligned} \tag{6.94b}$$

$0 \le l \le n-13$, where $C^{(i)}_l, C'^{(i)}_l, C^{(i)}_n$ and $C'^{(i)}_n$ depend only on ρ, $\|u\|_{E^\rho_{N_0+n}}$ and $\|K\|_{\mathcal{F}^M_n}$. Using that $n \ge 32$, it follows from inequalities (6.90), (6.94a) and (6.94b) that

$$\begin{aligned}\|K'^{(1)}-K'^{(2)}\|_{\mathcal{F}^M_n} &\le C_n \|u\|_{E^\rho_{N_0+n+3}} \|\Phi^{(1)}-\Phi^{(2)}\|_{\mathcal{F}^D_n} \\ &\quad + C_n \|u\|_{E^\rho_{N_0+n+2}} \sum_{i \le n/2} \big(H_{i+3}(\Phi^{(1)}-\Phi^{(2)},t) + \wp^D_i(\Phi^{(1)}(t)-\Phi^{(2)}(t))\big),\end{aligned} \tag{6.95}$$

where C_n depends only on $\rho, \|u\|_{E^\rho_{N_0+n+2}}$ and $\|K^{(i)}\|_{\mathcal{F}^M_n}, i=1,2$.

In particular if $K^{(2)}=-\Delta^{*M}$ then $\Phi^{(2)}=0$ and $K'^{(2)}=0$. If moreover $K^{(1)}=K$, $\Phi=\Phi^{(1)}$ and $K'=K'^{(1)}$, then $K'=\mathcal{N}(u,K)$. It therefore follows from inequalities (6.94a), (6.94b) and (6.95) and from Proposition 6.2 that

$$\|\mathcal{N}(u,K)\|_{\mathcal{F}^M_N} \le C_n \|u\|_{N_0+n+3} (\|u\|_{E^\rho_{N_0+n}} + \|K\|_{\mathcal{F}^M_n}), \tag{6.96}$$

where C_n depends only on ρ, $\|u\|_{E^\rho_{N_0+n+2}}$ and $\|K\|_{\mathcal{F}^M_n}$. This proves that $\mathcal{N}$ is a map from $E^{\circ\rho}_\infty \times \mathcal{F}^M_n$ to $\mathcal{F}^M_n$.

To estimate $\|\Phi^{(1)}-\Phi^{(2)}\|_{\mathcal{F}^D_n}$, we note that according to the definition of $\Phi^{(i)}$, it follows from equation (6.79) that

$$\begin{aligned}&\big(i\gamma^\mu\partial_\mu+m-\gamma^\mu(A^*_\mu+K^{(1)}_\mu-\partial_\mu\vartheta(A^*+K^{(1)}))\big)(\Phi^{(1)}-\Phi^{(2)})\\&\quad=\gamma^\mu\big(K^{(1)}_\mu-K^{(2)}_\mu-\partial_\mu\vartheta(K^{(1)}-K^{(2)})\big)(\phi^*+\Phi^{(2)}),\end{aligned}\tag{6.97}$$

where $\Phi^{(i)}\in\mathcal{F}^D_n, K^{(i)}\in\mathcal{F}^M_n$ for $i=1$ and $i=2$. Let Δ be the unique solution in $\mathcal{F}^D_n$ of the equation

$$\begin{aligned}&\big(i\gamma^\mu\partial_\mu+m-\gamma^\mu(A^*_\mu+K^{(1)}_\mu-\partial_\mu\vartheta(A^*+K^{(1)}))\big)\Delta\\&\quad=\gamma^\mu\big(K^{(1)}_\mu-K^{(2)}_\mu-\partial_\mu\vartheta(K^{(1)}-K^{(2)})\big)\phi^*.\end{aligned}\tag{6.98}$$

According to inequalities (6.93a) and (6.93b) we obtain that

$$\|\Delta\|_{\mathcal{F}^D_n}\le C_n\|u\|_{E^\rho_{N_0+n+2}}\|K^{(1)}-K^{(2)}\|_{\mathcal{F}^M_n}\tag{6.99a}$$

and

$$H_l(\Delta,t)\le(1+t)^{-3/2+\rho}C_l\|u\|_{E^\rho_{N_0+l+10}}\|K^{(1)}-K^{(2)}\|_{\mathcal{F}^M_n},\tag{6.99b}$$

$0\le l\le n-13$, where C_l depends only on ρ, $\|u\|_{E^\rho_{N_0+n}}$ and $\|K^{(1)}\|_{\mathcal{F}^M_n}$. We next prove that the equation

$$\begin{aligned}&\big(i\gamma^\mu\partial_\mu+m-\gamma^\mu(A^*_\mu+K^{(1)}_\mu-\partial_\mu\vartheta(A^*+K^{(1)}))\big)\Delta'\\&\quad=\gamma^\mu\big(K^{(1)}_\mu-K^{(2)}_\mu-\partial_\mu\vartheta(K^{(1)}-K^{(2)})\big)\Phi^{(2)}\end{aligned}\tag{6.100}$$

has a unique solution $\Delta'\in\mathcal{F}^D_n$. To do this we first use Lemma 6.6. Let

$$u_l(\Phi^{(2)},t)=H_l(\Phi^{(2)},t)+\sum_{\substack{Y\in\Pi'\\k+|Y|\le l+1}}\|(\delta(t))^{3-\rho}(1+\lambda_1(t))^{k/2}((m+i\gamma^\mu\partial_\mu)\Phi^{(2)}_Y)(t)\|_{L^\infty}.$$

Since

$$\begin{aligned}(i\gamma^\mu\partial_\mu+m)\Phi^{(2)}&=\gamma^\mu\big(A^*_\mu+K^{(2)}_\mu-\partial_\mu\vartheta(A^*+K^{(2)})\big)\Phi^{(2)}\\&\quad+\gamma^\mu\big(K^{(2)}_\mu+\Delta^{*M}_\mu-\partial_\mu\vartheta(K^{(2)}+\Delta^{*M})\big)\phi^*,\end{aligned}$$

it follows from inequalities (5.116c), (6.38a), from Lemma 6.3 and from inequality (6.66a) that

$$\begin{aligned}u_l(\Phi^{(2)},t)&\le H_l(\Phi^{(2)},t)+C_l(\|u\|_{E^\rho_{l+4}}+\|K^{(2)}\|_{\mathcal{F}^M_{l+4}})H_{l+1}(\Phi^{(2)},t)\\&\quad+(1+t)^{1/2-\rho}C_l\|K^{(2)}+\Delta^{*M}\|_{\mathcal{F}^M_{l+4}}H_{l+1}(\phi^*,t),\end{aligned}$$

where C_l depends only on ρ and $\|u\|_{E^\rho_{N_0}}$. It then follows from Proposition 6.2 and inequality (6.94b) that

$$\begin{aligned} u_l(\Phi^{(2)},t) \leq (1+t)^{-3/2+\rho} C'_l \|u\|_{E^\rho_{N_0+11+l}} \|K^{(2)} + \Delta^{*M}\|_{\mathcal{F}^M_{10+l}} \\ + (1+t)^{(1/2-\rho)} C_l \|u\|_{E^\rho_{N_0+l+1}} \|K^{(2)} + \Delta^{*M}\|_{\mathcal{F}^M_{l+4}}, \quad 0 \leq l \leq n-14, \end{aligned} \tag{6.101a}$$

where C_l depends only on ρ and $\|u\|_{E^\rho_{N_0}}$ and C'_l depends only on ρ, $\|u\|_{E^\rho_{N_0+n}}$ and $\|K^{(2)}\|_{\mathcal{F}^M_n}$. Proposition 6.2, inequality (6.101a) and the inequality $-3/2+\rho < 1/2-\rho$ give that

$$u_l(\Phi^{(2)},t) \leq (1+t)^{1/2-\rho} C_l \|u\|_{E^\rho_{N_0+l+11}} (\|u\|_{E^\rho_{N_0+l+10}} + \|K^{(2)}\|_{\mathcal{F}^M_{l+10}}), \tag{6.101b}$$

for $0 \leq l \leq n-14$, where C_l depends only on ρ, $\|u\|_{E^\rho_{N_0+n}}$ and $\|K^{(2)}\|_{\mathcal{F}^M_n}$. This inequality and inequality (6.94b) show that the hypothesis of Lemma 6.6 are satisfied for $p_1 = n-14$, $p_2 = n-13$, since $n/2 \leq p_2$. We can therefore conclude that $g_Y \in C^0(\mathbb{R}^+, D) \cap L^1(\mathbb{R}^+, D)$, $Y \in \Pi'$, $|Y| \leq n$, where g is given by the right-hand side of equation (6.100), and by using (6.94a) and (6.94b) that

$$R'_l(t) \leq (1+t)^{-3/2+\rho} C_l \|u\|_{E^\rho_{N_0+l+11}} \|K^{(1)} - K^{(2)}\|_{\mathcal{F}^M_{n-14}}, \tag{6.102a}$$

for $0 \leq l \leq n-14$,

$$R^\infty_l(t) \leq (1+t)^{-5/2+\rho} C_l \|u\|_{E^\rho_{N_0+l+10}} \|K^{(1)} - K^{(2)}\|_{\mathcal{F}^M_{l+3}}, \tag{6.102b}$$

for $0 \leq l \leq n-13$,

$$\wp^D_l\big((1+\lambda_1(t))^{k/2} g(t)\big) \leq (1+t)^{-5/2+\rho} C_l \|u\|_{E^\rho_{N_0+l+k+10}} \|K^{(1)} - K^{(2)}\|_{\mathcal{F}^M_l}, \tag{6.102c}$$

for $0 \leq l+k \leq n-13$ and if l_0 is the integer part of $l/2+3$ then

$$\begin{aligned} &\wp^D_l(f(t)) + (1+t)\wp^D_l(g(t)) \\ &\quad \leq (1+t)^{-3/2+\rho} C_l \big(\|u\|_{E^\rho_{N_0+l_0+7}} \|K^{(1)} - K^{(2)}\|_{\mathcal{F}^M_l} + \|u\|_{E^\rho_{N_0+l+2}} \|K^{(1)} - K^{(2)}\|_{\mathcal{F}^M_{l_0}}\big), \end{aligned} \tag{6.102d}$$

for $0 \leq l \leq n$, where C_l depends only on ρ, $\|u\|_{E^\rho_{N_0+N}}$ and $\|K^{(2)}\|_{\mathcal{F}^M_n}$. Let n_0 be the integer part of $n/2+5$. Since $n_0 \leq n-14$ for $n \geq 38$ and since $(1+\lambda_0(t))(x) \leq C(1+t)$, with C independent of $t \in \mathbb{R}^+, x \in \mathbb{R}^3$, if follows from inequalities (6.102a)–(6.102d) that the hypotheses of Proposition 6.4 applied to equation (6.100) are satisfied, which proves that $\Delta' \in \mathcal{F}^M_n$. Moreover statements ii) and iii) of Proposition 6.4 and inequalities (6.102a)–(6.102d) show that

$$\|\Delta'\|_{\mathcal{F}^D_n} \leq C_n \|u\|_{E^\rho_{N_0+n+1}} \|K^{(1)} - K^{(2)}\|_{\mathcal{F}^M_n} \tag{6.103a}$$

and that

$$H_l(\Delta', t) \leq (1+t)^{-3/2+\rho} C_l \|u\|_{E^\rho_{N_0+n}} \|K^{(1)} - K^{(2)}\|_{\mathcal{F}^M_n}, \tag{6.103b}$$

$0 \leq l \leq n-22$, where C_l and C_n depend only on ρ, $\|u\|_{E^\rho_{N_0+n}}$ and $\|K^{(i)}\|_{\mathcal{F}^M_n}$, $i = 1, 2$. Since $\Delta \in \mathcal{F}^D_n$ and $\Delta' \in \mathcal{F}^D_n$ satisfy respectively equations (6.98) and (6.100), it follows that the unique solution in $\mathcal{F}_n$ of equation (6.97) is given by $\Phi^{(1)} - \Phi^{(2)} = \Delta + \Delta'$. Inequalities (6.99a), (6.99b), (6.103a) and (6.103b) give that

$$\|\Phi^{(1)} - \Phi^{(2)}\|_{\mathcal{F}^D_n} \leq C_n \|u\|_{E^\rho_{N_0+n+2}} \|K^{(1)} - K^{(2)}\|_{\mathcal{F}^M_n} \tag{6.104a}$$

and that

$$H_l(\Phi^{(1)} - \Phi^{(2)}, t) \leq (1+t)^{-3/2+\rho} C_n \|u\|_{E^\rho_{N_0+n}} \|K^{(1)} - K^{(2)}\|_{\mathcal{F}^M_n}, \tag{6.104b}$$

$0 \leq l \leq n-22$, where C_n depends only on ρ, $\|u\|_{E^\rho_{N_0+n}}$ and $\|K^{(i)}\|_{\mathcal{F}^M_n}$, $i = 1, 2$. Since $n/2 + 3 \leq n - 22$ for $n \geq 50$, the inequality of the proposition follows from inequalities (6.95), (6.104a) and (6.104b). This proves the proposition.

We are now ready to prove the *existence of solutions* (K, Φ) of equations (6.31a), (6.31b) and (6.31c). We recall that ϕ^*, A^* and Δ^{*M} are functions of $u \in E^{\circ\rho}_\infty$.

Corollary 6.8. *Let $n \in \mathbb{N}$, $\varepsilon \in]0, \infty[$, $\rho \in]1/2, 1[$ and let $\mathcal{O}_{n+3}$ (resp. Q_n) be the open ball with center at the origin and radius ε (resp.2ε) in $E^{\circ\rho}_{n+3}$ (resp. $\mathcal{F}_n$). Let $\mathcal{O}_\infty = \mathcal{O}_{n+3} \cap E^{\circ\rho}_\infty$. If $n \geq 50$, then there exists ε such that equations (6.31a), (6.31b) and (6.31c) have a unique solution $(K, \Phi) \in Q_n$ for each $u \in \mathcal{O}_\infty$. This solution satisfies*

$$\|(K, \Phi)\|_{\mathcal{F}_n} \leq C_n \|u\|^2_{E^\rho_{N_0+n+3}}$$

and

$$(1+t)^{3/2-\rho} H_{n-13}(\Phi, t) + (1+t)^{\rho-1/2} u_{n-14}(\Phi, t) \leq C_n \|u\|^2_{E^\rho_{N_0+n-1}}$$

for $t \geq 0$, where C_n depends only on ρ and ε, and where $H_j(\Phi, t)$ and $u_j(\Phi, t)$ are as in Lemma 6.6.

Proof. Let $n \geq 50$ and let Q^M_n be the closed ball with center at the origin and of radius ε in $\mathcal{F}^M_n$. Let C be an upper bound of the two constants C_n and C'_n in Proposition 6.7 and the two constants $C'^{(1)}_n$ and $C'^{(2)}_n$ in inequality (6.94a), for all $u \in \mathcal{O}_\infty$ and $K, K^{(1)}, K^{(2)} \in Q^M_n$. If $u \in \mathcal{O}_\infty$, then it follows from Proposition 6.7 that $K \mapsto \mathcal{N}(u, K)$ is a contraction mapping from Q^M_n into Q^M_n if $C\varepsilon < 1$ and $2\varepsilon^2 C \leq \varepsilon$. Since C is bounded for ε belonging to a fixed bounded interval we can choose ε such that $C\varepsilon \leq 1/2$. It then follows that ε satisfies the above two inequalities. This proves that, for such ε the equation $K = \mathcal{N}(u, K)$ has a unique solution $K \in Q^M_n$ for each $u \in \mathcal{O}_\infty$ and, according to the first of the inequalities in Proposition 6.7, that $\|K\|_{\mathcal{F}^M_n} \leq \frac{1}{2}(\|u\|_{E^\rho_{N_0+n}} + \|K\|_{\mathcal{F}^M_n})$. This shows that $\|K\|_{\mathcal{F}^M_n} \leq \|u\|_{E^\rho_{N_0+n}}$, which introduced into the first inequality of Proposition 6.7 gives that

$$\|K\|_{\mathcal{F}^M_n} \leq 2C \|u\|^2_{E^\rho_{N_0+n+3}} \leq \varepsilon \tag{6.105a}$$

It follows from inequality (6.94a) and from the fact that $\|K\|_{\mathcal{F}_n^M} \leq \|u\|_{E^\rho_{N_0+n}}$ that the unique solution $\Phi \in \mathcal{F}_n^D$ of equation (6.31a) satisfies

$$\begin{aligned}\|\Phi\|_{\mathcal{F}_n^D} &\leq C\|u\|_{E^\rho_{N_0+n+2}}(\|u\|_{E^\rho_{N_0+n}} + \|K\|_{\mathcal{F}_n^M}) \\ &\leq 2C\|u\|^2_{E^\rho_{N_0+n+2}} \leq \varepsilon.\end{aligned} \tag{6.105b}$$

$(K, \Phi) \in Q_n$, since inequalities (6.105a) and (6.105b) shows that

$$\|(K,\Phi)\|_{\mathcal{F}_n} \leq 2\sqrt{2}C\|u\|^2_{E^\rho_{N_0+n+3}} \leq \sqrt{2}\varepsilon < 2\varepsilon,$$

which also gives the first inequality of the corollary. The second inequality of the corollary follows from inequalities (6.94b), with Φ and K instead of $\Phi^{(i)}$ and $K^{(i)}$, and (6.101b), with Φ and K instead of $\Phi^{(2)}$ and $K^{(2)}$. This proves the corollary, since equation (6.31c) is satisfied, by the definition of the space $\mathcal{F}_n$.

According to Corollary 6.8, $v(u) = (K, \Phi)$ defines a mapping $v\colon \mathcal{O}_\infty \to Q_{50} \subset \mathcal{F}_{50}$ into solutions of equations (6.31a), (6.31b) and (6.31c). In the next theorem we give supplementary *differentiability properties* of this map.

Theorem 6.9. *Let $1/2 < \rho < 1$, $L_0 = N_0 + 53$, let $\mathcal{O}_{L_0}$ (resp. Q_{50}) be the open ball with center at the origin and radius ε (resp. 2ε) in $E^{\circ\rho}_{L_0}$(resp.$\mathcal{F}_{50}$), let $\mathcal{O}_\infty = E^{\circ\rho}_\infty \cap \mathcal{O}_{L_0}$ and let $Q_\infty = (\cap_{n\geq 0}\mathcal{F}_n) \cap Q_{50}$. If $\varepsilon > 0$ is sufficiently small, then equations (6.31a), (6.31b) and (6.31c) have a unique solution $(K, \Phi) = v(u) \in Q_{50}$ for each $u \in \mathcal{O}_\infty$, the image of the map $v\colon \mathcal{O}_\infty \to Q_{50}$ is a subset of Q_∞, the map $v\colon \mathcal{O}_\infty \to Q_\infty$ is C^∞ and if moreover $(K^{(l)}, \Phi^{(l)}) = (D^l v)(u; u_1, \ldots, u_l), l \geq 0, n \geq 0, u \in \mathcal{O}_\infty, u_1, \ldots, u_l \in E^{\circ\rho}_\infty$ then:*

i) $\|K^{(l)}\|_{\mathcal{F}_n^M} \leq C_{n,l}\overline{\mathcal{R}}^l_{L_0,N_0+3+n+l}(u_1,\ldots,u_l) + C'_{n,l}\|u\|_{E^\rho_{N_0+3+n+l}}\|u_1\|_{E^\rho_{L_0}} \cdots \|u_l\|_{E^\rho_{L_0}}$,

where $C_{n,l}$ and $C'_{n,l}$ depend only on ρ and $\|u\|_{E^\rho_{L_0}}$, where $C_{n,l} \leq C_n\|u\|^{3-l}_{E^\rho_{L_0}}$ for $0 \leq l \leq 3$ and $C'_{n,l} \leq C_n\|u\|^{2-l}_{E^\rho_{L_0}}$ for $0 \leq l \leq 2$, with C_n depending only on ρ and $\|u\|_{E^\rho_{L_0}}$ and where $\overline{\mathcal{R}}^0_{i,j} = 0$ and

$$\overline{\mathcal{R}}^l_{i,j}(u_1,\ldots,u_l) = \sum_{1\leq q\leq l} \|u_1\|_{E^\rho_i} \cdots \|u_{q-1}\|_{E^\rho_i}\|u_q\|_{E^\rho_j}\|u_{q+1}\|_{E^\rho_i} \cdots \|u_l\|_{E^\rho_i},$$

for $i, j \geq 0$,

ii) $\|\Phi^{(l)}\|_{\mathcal{F}_n^D} \leq C_{n,l}\overline{\mathcal{R}}^l_{L_0,N_0+3+n+l}(u_1,\ldots,u_l) + C'_{n,l}\|u\|_{E^\rho_{N_0+3+n+l}}\|u_1\|_{E^\rho_{L_0}} \cdots \|u_l\|_{E^\rho_{L_0}}$,

where $C_{n,l}$ and $C'_{n,l}$ depend only on ρ and $\|u\|_{E^\rho_{L_0}}$, and where $C_{n,l} \leq C_n\|u\|^{2-l}_{E^\rho_{L_0}}$ for $0 \leq l \leq 2$, and $C'_{n,0} \leq C_n\|u\|_{E^\rho_{L_0}}$ with C_n depending only on ρ and $\|u\|_{E^\rho_{L_0}}$,

iii) $\sup_{t\geq 0}\Big((1+t)^{3/2-\rho} \sum_{\substack{Y\in\Pi' \\ k+|Y|\leq n}} \|(\delta(t))^{3/2}(1+\lambda_1(t)^{k/2}\Phi^{(l)}_Y(t)\|_{L^\infty}$

$$+ (1+t)^{\rho-1/2} \sum_{\substack{Y\in\Pi' \\ k+|Y|\le n}} \|(\delta(t))^{3-\rho}(1+\lambda_1(t))^{k/2}(m+i\gamma^\mu\partial_\mu-\gamma^\mu G_\mu)\Phi_Y^{(l)}(t)\|_{L^\infty}\Big)$$
$$\le C_{n,l}\overline{\mathcal{R}}^l_{L_0,N_0+3+n+l+13}(u_1,\ldots,u_l)+C'_{n,l}\|u\|_{E^\rho_{N_0+16+n+l}}\|u_1\|_{E^\rho_{L_0}}\cdots\|u_l\|_{E^\rho_{L_0}},$$

where $C_{n,l}$ and $C'_{n,l}$ are as in statement ii),

iv) $$\sup_{t\ge0}\Big((1+t)^{3/2-\rho}\sum_{j+k\le n}\wp^D_j\big((1+\lambda_1(t))^{k/2}\Phi^{(l)}(t)\big)\Big)$$
$$\le C_{n,l}\overline{\mathcal{R}}^l_{L_0,N_0+3+n+l}(u_1,\ldots,u_l)+C'_{n,l}\|u\|_{E^\rho_{N_0+3+n+l}}\|u_1\|_{E^\rho_{L_0}}\cdots\|u_l\|_{E^\rho_{L_0}},$$

where $C_{n,l}$ and $C'_{n,l}$ are as in statement ii).

Proof. According to Corollary 6.8, with $n=50$, there exists $\varepsilon>0$ such that equations (6.31a), (6.31b) and (6.31c) have a unique solution $(K,\Phi)\in Q_{50}$ for each $u\in\mathcal{O}_\infty$. This establishes the existence of the map $v\colon\mathcal{O}_\infty\to Q_{50}$ where $v(u)=(K,\Phi)$.

We shall estimate the solution Φ of equation (6.31a) by using Proposition 6.4. To do this we use the results (6.83a)–(6.83d), with $L=K+\Delta^{*M}$ and $j=0$, wich were obtained from Lemma 6.5 for the equation (6.79). This gives with $g=\gamma^\mu(K+\Delta^{*M}-\partial_\mu\vartheta(K+\Delta^{*M}))$:

$$R'_l(t)\le(1+t)^{-3/2+\rho}C_l\|u\|^2_{E^\rho_{L_0}}\|u\|_{E^\rho_{N_0+l+2}},\quad 0\le l\le 50, \tag{6.106a}$$
$$R^\infty_l(t)\le(1+t)^{-1}C_l\|u\|^2_{E^\rho_{L_0}}\|u\|_{E^\rho_{N_0+l}},\quad 0\le l\le 47, \tag{6.106b}$$
$$\wp^D_l\big((1+\lambda_1(t))^{k/2}g(t)\big) \le(1+t)^{-1}C_{l,k}\|u\|^2_{E^\rho_{L_0}}\|u\|_{E^\rho_{N_0+l+k}},\quad 0\le l\le 50, k\ge0, \tag{6.106c}$$

and

$$\wp^D_l(f(t))\le(1+t)^{-3/2+\rho}C_l\|u\|^2_{E^\rho_{L_0}}\|u\|_{E^\rho_{N_0+l+2}},\quad 0\le l\le 50, \tag{6.106d}$$

where C_l and $C_{l,k}$ depend only on ρ and $\|u\|_{E^\rho_{L_0}}$ and where $f_Y, Y\in\Pi'$, is given by (5.111b) in the context of equation (6.31a). It follows from inequalities (6.39a)–(6.39e) and (6.106a)–(6.106b) that in the context of equation (6.31a):

$$\overline{Q}_l(t)\le C_l\|u\|^2_{E^\rho_{L_0}}\|u\|_{E^\rho_{N_0+l+1}},\quad \text{for } 0\le l\le 45, \tag{6.107a}$$
$$R^{(1)}_l(t)\le C_l\|u\|^2_{E^\rho_{L_0}}\|u\|_{E^\rho_{N_0+l+10}},\quad \text{for } 0\le l\le 45, \tag{6.107b}$$
$$h'_l(j_0,t)\le C_l\|u\|^3_{E^\rho_{L_0}}\|u\|_{E^\rho_{N_0+j_0+1}},\quad \text{for } 19\le j\le 51, \tag{6.107c}$$

$j_0+1\le l\le j$, where j_0 is the integer part of $j/2+5$,

$$h''_l(t)\le C_l\|u\|^4_{E^\rho_{L_0}}\|u\|_{E^\rho_{N_0+11}},\qquad \text{for } 0\le l\le 50, \tag{6.107d}$$

and

$$h_l^\infty(j_0,t) \leq C_l \|u\|^3_{E^\rho_{L_0}} \|u\|_{E^\rho_{N_0+l}}, \qquad \text{for } j_0+1 \leq l \leq \min(50,j), \tag{6.107e}$$

$19 \leq j \leq 51$. Here C_l is a constant depending only on ρ and $\|u\|_{E^\rho_{L_0}}$. It follows from statement i) of Proposition 6.4 and inequality (6.107a), for $0 \leq l \leq 45$, from statement ii) of Proposition 6.4 and inequality (6.107e) with $j=50$, for $46 \leq l \leq 50$ that

$$\|\Phi\|_{\mathcal{F}^D_l} \leq C_l \|u\|^2_{E^\rho_{L_0}} \|u\|_{E^\rho_{N_0+l+2}}, \quad 0 \leq l \leq 50, \tag{6.108a}$$

where C_l depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. Statement iii) of Proposition 6.4 and inequality (6.107b) show that

$$\begin{aligned} H_l(\Phi,t) &= \sum_{\substack{Y\in\Pi' \\ |Y|+k\leq l}} \|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2}\Phi_Y(t)\|_{L^\infty} \\ &\leq (1+t)^{-3/2+\rho} C_l \|u\|^2_{E^\rho_{L_0}} \|u\|_{E^\rho_{N_0+l+10}}, \quad 0 \leq l \leq 37, \end{aligned} \tag{6.108b}$$

where C_l depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. Equation (6.31b) and inequality (6.90), (with $K^{(2)} = -\Delta^{*M}, \Phi^{(2)} = 0, K^{(1)} = K,\ \Phi^{(1)} = \Phi, K'^{(2)} = 0, K'^{(1)} = K$), give that

$$\begin{aligned} \|K\|_{\mathcal{F}^M_l} \leq C_l \sup_{t\geq 0} \Big(&\sum_{\substack{n_1+n_2=l \\ n_1\leq n_2}} \big(H_{n_1+3}(\Phi,t) + \wp^D_{n_1}(\Phi(t))\big)\|\Phi\|_{\mathcal{F}^D_{n_2}} \\ &+ \sum_{n_1+n_2=l} \big(H_{n_1+3}(\phi^*,t) + \wp^D_{n_1}(\phi^*(t))\big)\|\Phi\|_{\mathcal{F}^D_{n_2}}\Big), \quad 0 \leq l \leq 50. \end{aligned}$$

This inequality and inequalities (6.108a) and (6.108b) show that

$$\|K\|_{\mathcal{F}^M_l} \leq C_l \|u\|^3_{E^\rho_{L_0}} \|u\|_{E^\rho_{N_0+l+3}}, \quad 0 \leq l \leq 50, \tag{6.109}$$

where C_l depends only on ρ and $\|u\|_{E^\rho_{L_0}}$.

We shall prove, by induction, that the theorem is true for the case of $l=0$. Let $n \geq 50$ and suppose that

$$\|K\|_{\mathcal{F}^M_j} \leq C_j \|u\|^3_{E^\rho_{L_0}} \|u\|_{E^\rho_{N_0+j+3}}, \quad \text{for } 0 \leq j \leq n \tag{6.110a}$$

and that

$$\|\Phi\|_{\mathcal{F}^D_j} \leq C_j \|u\|^2_{E^\rho_{L_0}} \|u\|_{E^\rho_{N_0+j+3}}, \quad \text{for } 0 \leq j \leq n, \tag{6.110b}$$

where C_j depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. The hypothesis is true for $n=50$, according to inequalities (6.108a) and (6.109). The use of Corollary 2.6 and Proposition 6.2 will

not be systematically mentionned in the sequel of this proof. It follows from inequalities (6.83a)–(6.83d), inequalities (6.106a)–(6.106d) and hypotheses (6.110a) and (6.110b) that

$$R'_j(t) \leq (1+t)^{-3/2+\rho} C_j \|u\|^2_{E^\rho_{L_0}} \|u\|_{E^\rho_{N_0+j+3}}, \quad 0 \leq j \leq n, \tag{6.111a}$$

$$R^\infty_j(t) \leq (1+t)^{-1} C_j \|u\|^2_{E^\rho_{L_0}} \|u\|_{E^\rho_{N_0+j+3}}, \quad 0 \leq j \leq n-3, \tag{6.111b}$$

$$\wp^D_j\big((1+\lambda_1(t))^{k/2} g(t)\big) \leq (1+t)^{-1} C_{j,k} \|u\|_{E^\rho_{L_0}} \tag{6.111c}$$

$$\big(\|u\|_{E^\rho_{N_0+3}} \|u\|_{E^\rho_{N_0+j+k}} + \|u\|_{E^\rho_{N_0+j+3}} \|u\|_{E^\rho_{N_0+k}}\big), \quad 0 \leq j \leq n, k \geq 0,$$

$$\wp^D_j(f(t)) \leq (1+t)^{-3/2+\rho} C_j \|u\|^2_{E^\rho_{L_0}} \|u\|_{E^\rho_{N_0+j+3}}, \quad 0 \leq j \leq n, \tag{6.111d}$$

where C_j and $C_{j,k}$ depend only on ρ and $\|u\|_{E^\rho_{L_0}}$. It follows from definition (6.37) of $\chi^{(j)}$ and from hypothesis (6.110a) that

$$\chi_{k,j} \leq C_{k,j} \|u\|_{E^\rho_{N_0+j+k+3}}, \quad 0 \leq j+k \leq n, \tag{6.112}$$

where $C_{k,j}$ depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. Using definitions (6.39a)–(6.39d) of $\overline{Q}_j$, $R^{(1)}_j$, h'_j in the context of equation (6.31a), using hypothesis (6.110a) and using inequalities (6.107a)–(6.107c), (6.111a)–(6.111d) and (6.112) we obtain that

$$\overline{Q}_j(t) \leq C_j \|u\|^2_{E^\rho_{L_0}} \|u\|_{E^\rho_{N_0+j+8}}, \quad 0 \leq j \leq n-5, \tag{6.113a}$$

$$R^{(1)}_j(t) \leq C_j \|u\|^2_{E^\rho_{L_0}} \|u\|_{E^\rho_{N_0+j+16}}, \quad 0 \leq j \leq n-13, \tag{6.113b}$$

$$h'_j(j_0, t) \leq C_j \|u\|^2_{E^\rho_{L_0}} \|u\|_{E^\rho_{N_0+j+2}}, \quad 50 \leq j \leq n+1, \tag{6.113c}$$

where j_0 is the integer part of $j/2+5$ and where C_j depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. Statement iii) of Proposition 6.4 and inequalities (6.112) and (6.113b) give that

$$H_j(\Phi, t) \leq C_j \|u\|^2_{E^\rho_{L_0}} \|u\|_{E^\rho_{N_0+16+j}}, \quad 0 \leq j \leq n-13, \tag{6.114}$$

where C_j depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. For $|Y| = n+1, Y \in \Pi'$, the existence of $\Phi_Y \in C^0(\mathbb{R}^+, D)$ such that $\sup_{t\geq 0}((1+t)^{3/2-\rho}\|\Phi_Y(t)\|_D) < \infty$ follows from the linear inhomogeneous equation for Φ_Y obtained by substitution of the solution K of equation (6.31b) into equation (6.31a), from hypotheses (6.110a) and (6.110b) and from the energy estimates in Theorem 5.1. The existence of $\Phi \in \mathcal{F}^D_{n+1}$ then follows from hypothesis (6.110b). To prove that inequality (6.110b) is true also for $n+1$, we note that it follows from inequality (6.90), with $(K^{(2)} = -\Delta^{*M}, \Phi^{(2)} = 0, K^{(1)} = K, \Phi^{(1)} = \Phi, K'^{(2)} = 0, K'^{(1)} = K)$, hypothesis (6.10b) and inequality (6.114) that

$$\|K\|_{\mathcal{F}^M_{n+1}} \leq C \|u\|_{E^\rho_{N_0+19}} \|\Phi\|_{\mathcal{F}^D_{n+1}} + C_n \|u\|^3_{E^\rho_{L_0}} \|u\|_{E^\rho_{N_0+n+4}}, \tag{6.115}$$

where C and C_n depend only on ρ and $\|u\|_{E^\rho_{L_0}}$. It then follows from inequalities (6.81) (with $j=0$) and (6.115) and from statement iv) of Proposition 6.5 that

$$\wp^D_{n+1}(f(t)) \leq (1+t)^{-3/2+\rho}\big(C\|u\|^2_{E^\rho_{L_0}}\|\Phi\|_{\mathcal{F}_{n+1}} + C_n\|u\|^4_{E^\rho_{L_0}}\|u\|_{E^\rho_{N_0+n+4}}\big) \tag{6.116a}$$

and that

$$\wp^D_{n+1,i}(f(t)) \leq (1+t)^{-3/2+\rho}C_n\|u\|^4_{E^\rho_{L_0}}\|u\|_{E^\rho_{N_0+n+4}}, \quad 1 \leq i \leq n+1, \tag{6.116b}$$

where C and C_n depend only on ρ and $\|u\|_{E^\rho_{L_0}}$. We also obtain, using (6.39e), (6.113b) and (6.115) that

$$h''_{n+1}(t) \leq C\|u\|^4_{E^\rho_{L_0}}\|\Phi\|_{\mathcal{F}^D_{n+1}} + C_n\|u\|^6_{E^\rho_{L_0}}\|u\|_{E^\rho_{N_0+n+4}}, \tag{6.117}$$

where C and C_n depend only on ρ and $\|u\|_{E^\rho_{L_0}}$. According to the definition of $R^0_{n,1}$ in Corollary 5.9 if follows that

$$\begin{aligned}
&(1+t)^{1/2}(R^0_{n,1}(t))^{2(1-\rho)}\big(\wp^D_n(\Phi(t))\big)^{2\rho-1} \\
&\quad \leq (1+t)^{3/2-\rho}\big(\wp^D_n\big((1+\lambda_0(t))^{1/2}g(t)\big)\big)^{2(1-\rho)}\big(\wp^D_n(\Phi(t))\big)^{2\rho-1} \\
&\quad \leq C(1+t)^{3/2-\rho}\big(\wp^D_n\big((1+\lambda_0(t))^{1/2}g(t)\big) + \wp^D_n(\Phi(t))\big),
\end{aligned}$$

where C depends only on ρ. Hypothesis (6.110b) and inequality (6.111c) then give that

$$(1+t)^{1/2}R^0_{n,1}(t)^{2(1-\rho)}\wp^D_n(\Phi(t))^{2\rho-1} \leq C_n\|u\|^2_{E^\rho_{L_0}}\|u\|_{E^\rho_{N_0+n+4}}, \tag{6.118}$$

where C_n depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. We also have

$$\sum_{\substack{n_1+n_2=n+1\\ 1\leq n_1\leq j_0\\ n_2\geq j_0+1}} \chi_{5,n_1}\|\Phi\|_{\mathcal{F}^D_{n_2}} \leq C_n\|u\|^3_{E^\rho_{L_0}}\|u\|_{E^\rho_{N_0+n+4}}, \tag{6.119}$$

where j_0 is the integer part of $(n+1)/2+5$ and where C_n depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. Statement iv) of Proposition 6.4, inequality (6.113c) with $j=n+1$, inequalities (6.116a), (6.116b), (6.117), (6.118), (6.119) and the notation

$$\xi_{n+1,i} = \sup_{t\geq 0}\big((1+t)^{3/2-\rho}\wp^D_{n+1,i}(\Phi(t))\big), \quad 0 \leq i \leq n+1, \tag{6.120}$$

and $\xi_{n+1,i}=0$ for $i \geq n+2$ give

$$\begin{aligned}
&\xi_{n+1,0} \\
&\quad \leq C\|u\|^2_{E^\rho_{L_0}}\xi_{n+1,0} + C_{n+1}(\xi_{n+1,0})^{\varepsilon'}(\xi_{n+1,1})^{1-\varepsilon'} + C_{n+1}\|u\|^2_{E^\rho_{L_0}}\|u\|_{E^\rho_{N_0+n+4}}
\end{aligned} \tag{6.121a}$$

and

$$\xi_{n+1,i} \leq C_{n+1}(\xi_{n+1,0})^{\varepsilon'}(\xi_{n+1,i+1})^{1-\varepsilon'} + C_{n+1}\|u\|^2_{E^\rho_{L_0}}\|u\|_{E^\rho_{N_0+n+4}}, \quad 1 \leq i \leq n+1, \tag{6.121b}$$

where $\varepsilon' = \max(1/2, 2(1-\rho))$ and C and C_{n+1} depend only on ρ and $\|u\|_{E^\rho_{L_0}}$. We choose $\varepsilon > 0$ sufficiently small, such that $C\|u\|^2_{E^\rho_{L_0}} \leq 1/2$ for $u \in \mathcal{O}_\infty$. With the notation $\alpha_{n+1} = 2C_{n+1}\|u\|^2_{E^\rho_{L_0}}\|u\|_{E^\rho_{N_0+n+4}}$ and $\beta_{n+1} = 2C_{n+1}$, it follows then from inequalities (6.121a) and (6.121b) that

$$\xi_{n+1,i} \leq \alpha_{n+1} + \beta_{n+1}(\xi_{n+1,0})^{\varepsilon'}(\xi_{n+1,i+1})^{1-\varepsilon'}, \quad 0 \leq i \leq n+1. \tag{6.122}$$

Proceeding as in solving the system (5.182c) we obtain that $\xi_{n+1,0} \leq C'_{n+1}\alpha_{n+1}$, where C'_{n+1} is a polynomial in β_{n+1}. This proves that

$$\|\Phi\|_{\mathcal{F}^D_{n+1}} \leq C_{n+1}\|u\|^2_{E^\rho_{L_0}}\|u\|_{E^\rho_{N_0+n+4}}, \tag{6.123a}$$

for some C_{n+1} depending only on ρ and $\|u\|_{E^\rho_{L_0}}$. Inequality (6.115) then gives that

$$\|K\|_{\mathcal{F}^M_{n+1}} \leq C_{n+1}\|u\|^3_{E^\rho_{L_0}}\|u\|_{E^\rho_{N_0+n+4}}, \tag{6.123b}$$

where C_{n+1} depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. It now follows, by induction, from the hypothesies (6.110a) and (6.110b) and from inequalities (6.123a) and (6.123b) that inequalities (6.110a) and (6.110b) are true for every $n \in \mathbb{N}$. This proves statements i) and ii) of the theorem for the case of $l = 0$. It also proves that inequalities (6.111a)–(6.111d), (6.113a), (6.113b) and (6.114) are true for every $j \geq 0$ and that inequality (6.112) is true for every $j \geq 0, k \geq 0$. Statement iii), with $l = 0$, follows by considering equation (6.31c) and by using inequality (6.114). To prove statement iv) of the theorem, for the case of $l = 0$, we note that $\tau_{i,j}(t), i = 0, 1$, defined in Theorem 5.5 satisfy $\tau_{i,j}(t) \leq C_j\chi_{3,j}$, according to inequalities (5.116c) and (6.38c), that $T^2_{N,j}$ defined in (5.89a) satisfies $T^2_{N,j} \leq C_{N,j}\chi_{N+1,j}$ according to inequalities (5.138) and (6.41b) and the fact that Γ_j in Corollary 5.9 satisfies $\Gamma_j(t) \leq C_j\chi^{(j+1)}$, according to inequalities (5.125b) and (6.41b), where C_j and $C_{N,j}$ depends only on $\|u\|_{E^\rho_{N_0}}$. This gives, together with Corollary 5.9, if $J_0 \geq 3$, that

$$\begin{aligned} &\wp^D_j\big((1+\lambda_1(t))^{k/2}\Phi(t)\big) \\ &\quad\leq C_{j+k}\sum_{\substack{n_1+n_2=j+k\\ n_1\leq J_0}}(1+\chi_{3,n_1})(\wp^D_{n_2}(\Phi(t)) + R^1_{n_2-k,k}(t)) \\ &\qquad + C'_{j+k}\sum_{\substack{n_1+n_2+n_3+n_4=j+k\\ n_1\leq J_0, n_2\leq j+k-1\\ n_3+n_4\leq j+k-J_0-1}}(1+\chi_{3,n_1})\chi^{(n_2)}(1+\chi_{10,n_3}) \\ &\qquad \big(R'_{n_4+7}(t) + R^2_{n_4+9}(t) + R^\infty_{n_4}(t) + \wp^D_{n_4+8}(\Phi(t))\big), \quad j \geq 0, k \geq 1, \end{aligned} \tag{6.124}$$

where C_{j+k} depends only on ρ and $\chi^{(3)}$ and C'_{j+k} depends only on ρ and $\chi^{(11)}$. It follows from inequalities (6.110b), (6.111a)–(6.111c), (6.112) and (6.124), and from Corollary 2.6 that

$$\wp_j^D\big((1+\lambda_1(t))^{k/2}\Phi(t)\big) \leq (1+t)^{-3/2+\rho} C_{j+k}\|u\|_{E^\rho_{L_0}} \big(\|u\|_{E^\rho_{N_0+j+3}}\|u\|_{E^\rho_{N_0+k}} + \|u\|_{E^\rho_{N_0+3}}\|u\|_{E^\rho_{N_0+j+k}}\big), \quad j \geq 0, k \geq 1, \tag{6.125}$$

where C_{j+k} depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. This proves statement iv) for $l = 0$.

Let $\bar{l} \in \mathbb{N}$. We suppose the induction hypothesis that statements i)–iv) of the theorem are true for $n \geq 0$ and $0 \leq l \leq \bar{l}$, with the exeption of the decrease properties of $C_{n,l}$ and $C'_{n,l}$ which we only suppose for $\bar{l} = 0$. We have proved that the hypothesis is true for $\bar{l} = 0$ and we shall prove that it is true for $0 \leq l \leq \bar{l}+1$ if it is true for $l \leq \bar{l}$. Derivation of equations (6.31a), (6.31b) and (6.31c) give the following equations for $(K^{(l)}, \Phi^{(l)})$ in $\mathcal{F}_n$:

$$(i\gamma^\mu\partial_\mu + m - \gamma^\mu G_\mu)\Phi^{(l)} = \sum_{i=0}^{4} g_i^{(l)}, \tag{6.126a}$$

$$\Box K_\mu^{(l)} = J_\mu^{(l)} \tag{6.126b}$$

and

$$\partial^\mu J_\mu^{(l)} = 0, \tag{6.126c}$$

for $0 \leq l \leq \bar{l}+1$, where $G_\mu = A^*_\mu + K_\mu - \partial_\mu\vartheta(A^* + K)$, where

$$g_0^{(0)} = \gamma^\mu(K_\mu - \partial_\mu\vartheta(K))\phi^*, \tag{6.127a}$$

$$g_0^{(l)} = \gamma^\mu(K_\mu^{(l)} - \partial_\mu\vartheta(K^{(l)}))(\phi^* + \Phi), \quad \text{for} l \geq 1,$$

$$g_1^{(l)} = \sum_{\substack{i_1+i_2=l \\ i_2 \leq l-1}} C_{i_1,i_2}\gamma^\mu(A_\mu^{*i_1} - \partial_\mu\vartheta(A^{*i_1}))\Phi^{i_2}\sigma_l, \tag{6.127b}$$

$$g_2^{(l)} = \sum_{\substack{i_1+i_2=l \\ 1\leq i_2 \leq l-1}} C_{i_1,i_2}\gamma^\mu(K_\mu^{i_1} - \partial_\mu\vartheta(K^{i_1}))\Phi^{i_2}\sigma_l, \tag{6.127c}$$

$$g_3^{(l)} = \sum_{\substack{i_1+i_2=l \\ i_1 \leq l-1}} C_{i_1,i_2}\gamma^\mu(K_\mu^{i_1} - \partial_\mu\vartheta(K^{i_1}))\phi^{*i_2}\sigma_l, \tag{6.127d}$$

$$g_4^{(l)} = \sum_{i_1+i_2=l} C_{i_1,i_2}\gamma^\mu(\Delta_\mu^{*Mi_1} - \partial_\mu\vartheta(\Delta^{*Mi_1}))\phi^{*i_2}\sigma_l, \tag{6.127$_rme$}$$

with K^i (resp. $\Phi^i, A^{*i}, \phi^{*i}, \Delta^{*Mi}$) being equal to the l-linear symmetric map $D^iK(u)$ (resp. $D^i\Phi(u)$ $D^iA^*(u)$, $D^i\phi^*(u)$, $D^i\Delta^{*M}(u)$), σ_l being the normalized symmetrization of $(u_1, \ldots, u_l)$ and C_{i_1,i_2} being the binomial coefficients and where

$$J_\mu^{(l)} = \sum_{i_1+i_2=l} C_{i_1,i_2}\big(\overline{\Phi}^{i_1}\gamma_\mu\phi^{*i_2} + \overline{\phi}^{*i_1}\gamma_\mu\Phi^{i_2} + \overline{\Phi}^{i_1}\gamma_\mu\Phi^{i_2}\big)\sigma_l. \tag{6.127f}$$

To prove the existence of a unique solution $(K^{(l)},\Phi^{(l)})\in\mathcal{F}_n, n\geq 0$, for $l=\overline{l}+1$, of the linear inhomogeneous system (6.126a)–(6.126c), we shall use Proposition 6.4. In the context of this proposition let $R'_j(g_i^{(l)},t)$, (resp. $R_j^{\infty}(g_i^{(l)},t)$, $f_j^{(l)},\cdots$) denotes $R'_j(t)$, (resp. $R_j^{\infty}(t)$, $f(t)$) in the case of the equation

$$(i\gamma^\mu\partial_\mu+m-\gamma^\mu G_\mu)\Phi_i^{(l)}=g_i^{(l)},\quad 0\leq i\leq 4, l=\overline{l}+1. \tag{6.128}$$

We first study this equation for $1\leq i\leq 4$ and for $0\leq l\leq\overline{l}+1$. To simplify the notation we shall write A^{*i_1} (resp. K^{i_1},Δ^{*Mi_1}) instead of $A^{*i_1}(u_1,\ldots,u_{i_1})$ (resp. $K^{i_1}(u_1,\ldots,u_{i_1})$, $\Delta^{*Mi_1}(u_1,\ldots,u_{i_1})$) and ϕ^{*i_2} (resp. Φ^{i_2}) instead of $\phi^{*i_2}(u_{i_1+1},\ldots,u_l)$ (resp. $\Phi^{i_2}(u_{i_1+1},\ldots,u_l)$) where $i_1+i_2=l$. Let

$$\overline{\mathcal{R}}_{i,j}^{(0)}(u)=\|u\|_{E_j^\rho}\quad \text{for} i,j\geq 0 \tag{6.129a}$$

and

$$\overline{\mathcal{R}}_{i,j}^{(l)}(u;u_1,\ldots,u_l)=\overline{\mathcal{R}}_{i,j}^{l}(u_1,\ldots,u_l)+\|u\|_{E_j^\rho}\|u_1\|_{E_i^\rho}\cdots\|u_l\|_{E_i^\rho}, \tag{6.129b}$$

$i,j\geq 0, l\geq 1$.If $g=\gamma^\mu(A_\mu^{*i_1}-\partial_\mu\vartheta(A^{*i_1}))\Phi^{i_2}, i_1+i_2=l\leq\overline{l}+1, i_2\leq l-1$ then it follows from inequality (5.116c), Proposition 6.2 and the induction hypothesis (statement iv) with $l\leq\overline{l}$) that

$$\begin{aligned}&\wp_j^D\big((\delta(t))^{3/2-\rho}(1+\lambda_1(t))^{k/2}g(t)\big)\\&\quad\leq(1+t)^{-3/2+\rho}C_{j+k,l}\\&\qquad\sum_{n_1+n_2=j}\overline{\mathcal{R}}_{N_0,N_0+n_1+1}^{(i_1)}(u;u_1,\ldots,u_{i_1})\overline{\mathcal{R}}_{L_0,N_0+3+n_2+k+i_2}^{(i_2)}(u;u_{i_1+1},\ldots,u_l),\end{aligned}$$

where $C_{j+k,l}$ depending only on ρ,l and $\|u\|_{E_{L_0}^\rho}$. Using Corollary 2.6 it follows that

$$\begin{aligned}&\wp_j^D\big((\delta(t))^{3/2-\rho}(1+\lambda_1(t))^{k/2}g(t)\big)\\&\quad\leq C_{j+k,l}(1+t)^{-3/2+\rho}\overline{\mathcal{R}}_{L_0,N_0+3+j+k+l}^{(l)}(u;u_1,\ldots,u_l),\quad i_1+i_2=l\leq\overline{l}+1, i_2\leq l-1,\end{aligned} \tag{6.130}$$

where $C_{j+k,l}$ is as in the last inequality. Applying inequality (6.130) to each term in the sum defining $g_1^{(l)}$ we obtain that

$$\begin{aligned}&\wp_j^D\big((\delta(t))^{3/2-\rho}(1+\lambda_1(t))^{k/2}g_1^{(l)}(t)\big)\\&\quad\leq(1+t)^{-3/2+\rho}C_{j+k,l}\overline{\mathcal{R}}_{L_0,N_0+3+j+k+l}^{(l)}(u;u_1,\ldots,u_l),\quad j\geq 0,k\geq 0,0\leq l\leq\overline{l}+1,\end{aligned} \tag{6.131}$$

where $C_{j+k,l}$ depends only on ρ and $\|u\|_{E_{L_0}^\rho}$. If $g=\gamma^\mu(K_\mu^{i_1}-\partial_\mu\vartheta(K^{i_1}))\Phi^{i_2}$, $i_1+i_2=l\leq\overline{l}+1$, $1\leq i_1\leq l-1$ then it follows from statement iii) of Lemma 6.6 and from the

induction hypothesis that

$$\begin{aligned}&\wp^D_j\big(\delta(t)(1+\lambda_1(t))^{k/2}g(t)\big)\\&\quad\leq(1+t)^{-3/2+\rho}C_{j+k}\\&\qquad\sum_{n_1+n_2=j}\min\Big(\overline{\mathcal{R}}^{(i_1)}_{L_0,N_0+3+n_1+i_1}(u;u_1,\ldots,u_{i_1})\overline{\mathcal{R}}^{(i_2)}_{L_0,N_0+3+n_2+k+1+i_2+13}(u;u_{i_1+1},\ldots,u_l),\\&\qquad\qquad\overline{\mathcal{R}}^{(i_1)}_{L_0,N_0+3+n_1+3+i_1}(u;u_1,\ldots,u_{i_1})\overline{\mathcal{R}}^{(i_2)}_{L_0,N_0+3+n_2+k+i_2}(u;u_1,\ldots,u_l)\Big).\end{aligned}$$

For terms with $n_1+3+i_1\leq j+l+k$ we choose the second argument in the minimum and use Corollary 2.6 together with $N_0+6\leq\min(N_0+6+n_1+i_1,N_0+3+n_2+i_2+k)\leq\max(N_0+6+n_1+i_1,N_0+3+n_2+i_2+k)\leq N_0+3+j+k+l$. For terms with $n_1+3+i_1>j+l+k$ we choose the first argument in the minimum and use that $N_0+n_2+k+i_2+17\leq N_0+19\leq L_0$. This gives that

$$\begin{aligned}&\wp^D_j\big(\delta(t)(1+\lambda_1(t))^{k/2}g(t)\big)\\&\quad\leq(1+t)^{-3/2+\rho}C_{j+k}\overline{\mathcal{R}}^{(l)}_{L_0,N_0+3+j+k+l}(u;u_1,\ldots,u_l),\quad j,k\geq0,0\leq l\leq\bar{l}+1.\end{aligned}\tag{6.132}$$

Applying inequality (6.132) to each term in the sum defining $g_2^{(l)}$ we obtain that

$$\begin{aligned}&\wp^D_j\big(\delta(t)(1+\lambda_1(t))^{k/2}g_2^{(l)}(t)\big)\\&\quad\leq(1+t)^{-3/2+\rho}C_{j+k,l}\overline{\mathcal{R}}^{(l)}_{L_0,N_0+3+j+k+l}(u;u_1,\ldots,u_l),\quad j,k\geq0,0\leq l\leq\bar{l}+1,\end{aligned}\tag{6.133}$$

where $C_{j+k,l}$ depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. If $g=\gamma^\mu(K^{i_1}_\mu-\partial_\mu\vartheta(K^{i_1}))\phi^{*i_2}$, $i_1+i_2=l$, $i_1\leq l-1$, then using Lemma 6.5 and $\delta(t)\leq C(1+\lambda_1(t)+t)$ we obtain that

$$\wp^D_j\big(\delta(t)(1+\lambda_1(t))^{k/2}g(t)\big)\leq C_j\sum_{n_1+n_2=j}\|K^{i_1}\|_{\mathcal{F}^M_{n_1}}H_{n_2+k+1}(\phi^{*i_2},t),\tag{6.134}$$

where C_j depends only on ρ. The induction hypothesis and Proposition 6.2 give that

$$\begin{aligned}&\wp^D_j\big(\delta(t)(1+\lambda_1(t))^{k/2}g(t)\big)\\&\quad\leq C_{j+k,l}\sum_{n_1+n_2=j}\overline{\mathcal{R}}^{(i_1)}_{L_0,N_0+3+n_1+i_1}(u;u_1,\ldots,u_{i_1})\overline{\mathcal{R}}^{(i_2)}_{N_0,N_0+n_2+k+1}(u;u_{i_{1+1}},\ldots,u_l)\\&\quad\leq(1+t)^{-3/2+\rho}C'_{j+k,l}\overline{\mathcal{R}}^{(l)}_{L_0,N_0+3+j+k+l}(u;u_1,\ldots,u_l),\quad j,k\geq0,0\leq l\leq\bar{l}+1.\end{aligned}\tag{6.135}$$

Applying inequality (6.135) to each term in the sum defining $g_3^{(l)}$ we obtain that

$$\begin{aligned}&\wp^D_j\big(\delta(t)(1+\lambda_1(t))^{k/2}g_3^{(l)}(t)\big)\\&\quad\leq C_{j+k,l}\overline{\mathcal{R}}^{(l)}_{L_0,N_0+3+j+k+l}(u;u_1,\ldots,u_l),\quad j,k\geq0,0\leq l\leq\bar{l}+1,\end{aligned}\tag{6.136}$$

where $C_{j+k,l}$ depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. Using also inequality (6.134) in the case of $g_4^{(l)}$ we obtain from inequalities (6.131), (6.133) and (6.136) that

$$\sum_{i=1}^{4} \wp_j^D\big((\delta(t))^{3/2-\rho}(1+\lambda_1(t))^{k/2} g_i^{(l)}(t)\big) \tag{6.137}$$
$$\le (1+t)^{-\rho+1/2} C_{j+k,l} \overline{\mathcal{R}}^{(l)}_{L_0,N_0+3+j+k+l}(u;u_1,\ldots,u_l), \quad j,k\ge 0, 0\le l\le \bar{l}+1,$$

where $C_{j+k,l}$ depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. Similarly, it follows from statement iii) of Lemma 6.5 and statement iii) of Lemma 6.6 that

$$\wp_j^D\big(\delta(t)(1+\lambda_1(t))^{k/2} g_0^{(l)}(t)\big) \tag{6.138}$$
$$\le C_{j+k,l} \overline{\mathcal{R}}^{(l)}_{L_0,N_0+3+j+k+l}(u;u_1,\ldots,u_l), \quad j,k\ge 0, 0\le l\le \bar{l},$$

where $C_{j+k,l}$ depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. To estimate $R'_j(g_1^{(l)},t)$, where R'_j is defined in Theorem 5.8, let $g=\gamma^\mu(a_\mu-\partial_\mu\vartheta(a))r$ and $\partial_\mu a^\mu=0$. Changing the notation in inequality (6.68) we obtain with $g'=(2m)^{-1}(m-i\gamma^\mu\partial_\mu+\gamma^\mu G_\mu)g, F_\mu=a_\mu-\partial_\mu\vartheta(a)$:

$$\begin{aligned} g'_Y = (2m)^{-1} \sum_{Y_1,Y_2}^{Y} \Big(&\gamma^\nu F_{Y_1\nu}\xi^M_{Y_2}((m+i\gamma^\mu\partial_\mu-\gamma^\mu G_\mu)r) \\ &- 2iF^\mu_{Y_1}\partial_\mu r_{Y_2} + i(\partial_\mu F^\mu_{Y_1})r_{Y_2} \\ &- \frac{i}{4}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)((\partial_\mu a_{Y_1\nu})-(\partial_\nu a_{Y_1\mu}))r_{Y_2}\Big) \\ &+ m^{-1}\sum_{Y_1,Y_2,Y_3}^{Y} G_{Y_1\mu}F^\mu_{Y_2}r_{Y_3}, \quad Y\in\Pi', k\ge 0. \end{aligned} \tag{6.139}$$

Taking weighted supremum norms of F_Z and using inequalities (5.7d) and (5.116c), the facts that $A=A^*+K$ and $y^\mu F_\mu(y)=0$, Lemma 6.3 and inequalities (6.49b) and (6.66a) we obtain that

$$\begin{aligned} &\|\delta(t)(1+\lambda_1(t))^{k/2} g'_Y(t)\|_D \\ &\le C_{|Y|}\sum_{Y_1,Y_2}^{Y}[a]^{|Y_1|+2}(t)\Big(\|(\delta(t))^{\rho-1/2}(1+\lambda_1(t))^{k/2}(\xi^M_{Y_2}(i\gamma^\mu\partial_\mu+m-\gamma^\mu G_\mu)r)(t)\|_D \\ &\quad + (1+t)^{-3/2+\rho}\wp^D_{|Y_2|+1}((1+\lambda_1(t))^{k/2}r(t)) + (1+t)^{-1/2}\wp^D_{|Y_2|}((1+\lambda_1(t))^{(k+1)/2}r(t))\Big) \\ &\quad + C_{|Y|}\sum_{Y_1,Y_2,Y_3}^{Y}\Big((1+t)^{-2+2\rho}[A^*]^{|Y_1|+1}(t)[a]^{|Y_2|+1}(t)\wp^D_{|Y_3|}((1+\lambda_1(t))^{k/2}r(t)) \\ &\qquad + (1+t)^{-3/2+\rho}[a]^{|Y_2|+1}(t) \\ &\qquad \min\big(\|K\|_{\mathcal{F}^M_{|Y_1|+3}}\wp^D_{|Y_3|}((1+\lambda_1(t))^{k/2}r(t)), \|K\|_{\mathcal{F}^M_{|Y_1|}}H_{|Y_3|+k}(r,t)\big)\Big), \end{aligned} \tag{6.140}$$

$Y \in \Pi'$, $k \geq 0$, where $C_{|Y|}$ depends only on ρ. The use of inequality (6.140), with $a = A^{*i_1}$, $r = \Phi^{i_2}$, $i_1 + i_2 = l$, $i_2 \leq l-1$, $0 \leq l \leq \bar{l}+1$, the use of equation (6.126a), with $l \leq \bar{l}$, and inequalities (6.137) and (6.138) to estimate the weighted D-norms of $\xi^M_{Y_2}$ $(i\gamma^\mu\partial_\mu + m - \gamma^\mu G_\mu)\Phi^{i_2}$ and the use of Proposition 6.2 and the induction hypothesis give

$$\begin{aligned}&\wp^D_j\big(\delta(t)(1+\lambda_1(t))^{k/2} g'^{(l)}_1(t)\big) \\ &\quad \leq (1+t)^{-3/2+\rho} C_{j+k,l} \overline{\mathcal{R}}^{(l)}_{L_0,N_0+3+j+k+l+1}(u;u_1,\ldots,u_l), \quad j,k \geq 0, 0 \leq l \leq \bar{l}+1,\end{aligned} \tag{6.141}$$

where $g'^{(l)}_1 = (2m)^{-1}(m - i\gamma^\mu\partial_\mu + \gamma^\mu G_\mu) g^{(l)}_1$, where $C_{j+k,l}$ depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. Statement i) of Lemma 6.5, statement i) of Lemma 6.6 and inequality (6.141) give that

$$\begin{aligned}&R'_j(g^{(l)}_i, t) \\ &\quad \leq (1+t)^{-3/2+\rho} C_{j+k,l} \overline{\mathcal{R}}^{(l)}_{L_0,N_0+3+j+l+1}(u;u_1,\ldots,u_l), \quad j \geq 0, 0 \leq l \leq \bar{l}+1, 1 \leq i \leq 4,\end{aligned} \tag{6.142a}$$

where $C_{j+k,l}$ depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. It follows from inequality (6.137) that

$$\begin{aligned}&R^2_j(g^{(l)}_i, t) \\ &\quad \leq (1+t)^{-1} C_{j,l} \overline{\mathcal{R}}^{(l)}_{L_0,N_0+3+j+l}(u;u_1,\ldots,u_l), \quad j \geq 0, 0 \leq l \leq \bar{l}+1, 1 \leq i \leq 4,\end{aligned} \tag{6.142b}$$

where $C_{j,l}$ depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. Moreover using statement ii) of Lemma 6.5, statement ii) of Lemma 6.6 and Proposition 6.2 we obtain that

$$\begin{aligned}&R^\infty_j(g^{(l)}_i, t) \\ &\quad \leq (1+t)^{-1} C_{j,l} \overline{\mathcal{R}}^{(l)}_{L_0,N_0+3+j+l+13}(u;u_1,\ldots,u_l), \quad j \geq 0, 0 \leq l \leq \bar{l}+1, 1 \leq i \leq 4,\end{aligned} \tag{6.142c}$$

where $C_{j,l}$ is as in (6.142b). Proposition 5.16, in the case where $i = 1$ and Lemma 6.5 and Lemma 6.6 in the case where $2 \leq i \leq 4$ give that

$$\begin{aligned}&\wp^D_j(f^{(l)}_i(t)) \\ &\quad \leq C_{j,l}(1+t)^{-3/2+\rho} \overline{\mathcal{R}}^{(l)}_{L_0,N_0+3+j+l}(u;u_1,\ldots,u_l), \quad j \geq 0, 0 \leq l \leq \bar{l}+1, 1 \leq i \leq 4,\end{aligned} \tag{6.142d}$$

where $C_{j,l}$ is as in (6.142b). Definition (6.39a) of $\overline{Q}_n$ and inequalities (6.142b) and (6.142d) give, for $1 \leq i \leq 4, 0 \leq l \leq \bar{l}+1$, that

$$\overline{Q}_j(g^{(l)}_i, t) \leq C_{j,l} \overline{\mathcal{R}}^{(l)}_{L_0,N_0+3+j+l}(u;u_1,\ldots,u_l), \tag{6.143a}$$

if $0 \leq j \leq 45$ and

$$\overline{Q}_j(g^{(l)}_i, t) \leq C_{j,l} \overline{\mathcal{R}}^{(l)}_{L_0,N_0+3+j+l+5}(u;u_1,\ldots,u_l), \tag{6.143b}$$

if $0 \leq j$, where $C_{j,l}$ is as in (6.142b). It follows from inequalities (6.142a), (6.142b), (6.142c) and (6.143b) and from definition (6.39c) of $R_j^{(1)}$ that

$$R_j^{(1)}(g_i^{(l)}, t) \leq C_{j,l}\overline{\mathcal{R}}^{(l)}_{L_0, N_0+3+j+l+13}(u; u_1, \ldots, u_l), \quad j \geq 0, 0 \leq l \leq \overline{l}+1, 1 \leq i \leq 4, \tag{6.143c}$$

where $C_{j,l}$ is as in (6.142b). Definitions (6.39d) of h'_j and (6.39e) of h''_j and the definition of h_j^∞ in statement ii) of Proposition 6.4 give, j_0 being the integer part of $j/2+5$, that

$$h_j^\infty(g_i^{(l)}, j_0, t) \leq C_{j,l}\overline{\mathcal{R}}^{(l)}_{L_0, N_0+3+j+l}(u; u_1, \ldots, u_l), \quad j \geq 46, 0 \leq l \leq \overline{l}+1, 1 \leq i \leq 4, \tag{6.143d}$$

where $C_{j,l}$ depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. Using first statement i) of Proposition 6.4 with $n=50$ and inequality (6.143a) and using secondly statement ii) of Proposition 6.4 with $n \geq 82$ and inequality (6.143d) we obtain that the solution $\Phi_i^{(l)}$ of equation (6.128), with $1 \leq i \leq 4$ and $0 \leq l \leq \overline{l}+1$ exists and satisfies

$$\|\Phi_i^{(l)}\|_{\mathcal{F}^D_j} \leq C_{j,l}\overline{\mathcal{R}}^{(l)}_{L_0, N_0+3+j+l}(u; u_1, \ldots, u_l), \quad j \geq 0, 0 \leq l \leq \overline{l}+1, 1 \leq i \leq 4, \tag{6.144a}$$

where $C_{j,l}$ depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. Inequality (6.143c) and statement iii) of Proposition 6.4 give that

$$H_j(\Phi_i^{(l)}, t) \leq (1+t)^{-3/2}C_{j,l}\overline{\mathcal{R}}^{(l)}_{L_0, N_0+3+j+l+13}(u; u_1, \ldots, u_l), \quad j \geq 0, 0 \leq l \leq \overline{l}+1, 1 \leq i \leq 4, \tag{6.144b}$$

where $C_{j,l}$ depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. In (6.144a) and (6.144b) $C_{j,0} \leq C'_j\|u\|^2_{E^\rho_{L_0}}$, $C_{j,l} \leq C'_j\|u\|_{E^\rho_{L_0}}$.

To study equation (6.128), with $i=0$ and $l=\overline{l}+1$, we introduce the equation

$$(i\gamma^\mu\partial_\mu + m - \gamma^\mu G_\mu)\Psi(L) = \overline{g} \tag{6.145}$$

for $\Psi(L) \in \mathcal{F}^D_n$, where $\overline{g} = \gamma^\mu(L_\mu - \partial_\mu\vartheta(L))(\Phi+\phi^*)$ and $L \in \mathcal{F}^M_n$. If $n=37$, then it follows from ii) and iii), with $l=0$, of the present theorem and from Lemma 6.5 and Lemma 6.6 that

$$R'_j(\overline{g}, t) \leq (1+t)^{-3/2+\rho}C_j\|u\|_{E^\rho_{L_0}}\|L\|_{\mathcal{F}^M_{j+1}}, \quad 0 \leq j \leq 36, \tag{6.146a}$$

$$R^\infty_j(\overline{g}, t) \leq (1+t)^{-1}C_j\|u\|_{E^\rho_{L_0}}\|L\|_{\mathcal{F}^M_{j+3}}, \quad 0 \leq j \leq 34, \tag{6.146b}$$

$$R^2_j(\overline{g}, t) \leq (1+t)^{-1}C_j\|u\|_{E^\rho_{L_0}}\|L\|_{\mathcal{F}^M_j}, \quad 0 \leq j \leq 37, \tag{6.146c}$$

$$\wp^D_j(\overline{f}(t)) \leq (1+t)^{-3/2+\rho}C_j\|u\|_{E^\rho_{L_0}}\|u\|_{\mathcal{F}^M_j}, \quad 0 \leq j \leq 37, \tag{6.146d}$$

where C_j depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. These inequalities give that

$$\overline{Q}_j(\overline{g},t) \leq C_j\|u\|_{E^\rho_{L_0}}\|L\|_{\mathcal{F}^M_j}, \quad 0 \leq j \leq 37, \tag{6.147a}$$

$$R^{(1)}_j(\overline{g},t) \leq C_j\|u\|_{E^\rho_{L_0}}\|L\|_{\mathcal{F}^M_{j+9}}, \quad 0 \leq j \leq 28, \tag{6.147b}$$

$$h^\infty_j(\overline{g},n_0,t) \leq C_j\|u\|^2_{E^\rho_{L_0}}\|L\|_{\mathcal{F}^M_j}, \quad n_0+1 \leq j \leq n = 37, \tag{6.147c}$$

where n_0 is the integer part of $n/2+5$ and where C_j depend only on ρ and $\|u\|_{E^\rho_{L_0}}$. Statements i), ii) and iii) of Proposition 6.4 and inequalities (6.147a)–(6.147c) then give that

$$\|\Psi(L)\|_{\mathcal{F}^D_j} \leq C_j\|u\|_{E^\rho_{L_0}}\|L\|_{\mathcal{F}^M_j}, \quad 0 \leq j \leq 37, \tag{6.148a}$$

and that

$$H_j(\Psi(L),t) \leq (1+t)^{-3/2+\rho}C_j\|u\|_{E^\rho_{L_0}}\|L\|_{\mathcal{F}^M_{j+9}}, \quad 0 \leq j \leq 27, \tag{6.148b}$$

where C_j depends only on ρ and $\|u\|_{E^\rho_{L_0}}$.

For given solutions $\Phi^{(l)}_i, 1 \leq i \leq 4, 0 \leq l \leq \overline{l}+1$, of equation (6.128) and $\Psi(L)$ of equation (6.145) we introduce the current $\overline{J}^{(\overline{l}+1)}$ by

$$\overline{J}^{(\overline{l}+1)}_\mu(L) = \overline{J}^{(\overline{l}+1)}_{0\mu}(L) + \overline{J}^{(\overline{l}+1)}_{1\mu}, \tag{6.149a}$$

where

$$\overline{J}^{(\overline{l}+1)}_{0\mu}(L) = \overline{\Psi}(L)\gamma_\mu(\phi^*+\Phi) + (\overline{\phi^*+\Phi})\gamma_\mu\Psi(L) \tag{6.149b}$$

and

$$\begin{aligned}\overline{J}^{(\overline{l}+1)}_{1\mu} = &\sum_{\substack{i_1+i_2=\overline{l}+1\\ 1\leq i_1\leq\overline{l}}} C_{i_1,i_2}\big(\overline{\Phi}^{i_1}\gamma_\mu\phi^{*i_2} + \overline{\phi}^{*i_1}\gamma_\mu\Phi^{i_2} + \overline{\Phi}^{i_1}\gamma_\mu\Phi^{i_2}\big)\sigma_{\overline{l}+1}\\ &+\sum_{i=1}^{4}(\overline{\Phi}^{(\overline{l}+1)}_i\gamma_\mu(\phi^*+\Phi) + (\overline{\phi^*+\Phi})\gamma_\mu\Phi^{(\overline{l}+1)}_i),\end{aligned} \tag{6.149c}$$

where C_{i_1,i_2} are the binomial coefficients and where we have used the notation of (6.127b)–(6.127e). Since equation (6.126a) is satisfied for $0 \leq l \leq \overline{l}$, according to the induction hypothesis, since we have proved that there is a (unique) solution $\Phi_i \in \mathcal{F}^D_j$, $j \geq 0$, of equation (6.128) for $1 \leq i \leq 4, 0 \leq l \leq \overline{l}+1$ and since we have proved that equation (6.145) has a (unique) solution $\Psi(L) \in \mathcal{F}^D_0$, it follows that

$$\partial^\mu\overline{J}^{(\overline{l}+1)}_\mu(L) = 0. \tag{6.149d}$$

We shall estimate the $\mathcal{F}_j^M$ norm of the unique solution $N(L) \in \mathcal{F}_j^M$ of the equation

$$\square N_\mu(L) = \overline{J}_\mu^{(\overline{l}+1)}(L). \tag{6.150}$$

It follows from the definition of $\|\cdot\|_{\mathcal{F}_j^M}$ that

$$\begin{aligned}\|N(L)\|_{\mathcal{F}_j^M} \le C\Big(\sum_{|Y|\le j}\sup_{t\ge 0}\big((1+t)^{2-\rho}\|\delta(t)(\xi_Y^M\overline{J}^{(\overline{l}+1)}(L))(t)\|_{L^2} \\ + (1+t)^{1-\rho}\|\delta(t)(\xi_Y^M\overline{J}^{(\overline{l}+1)}(L))(t)\|_{L^{6/5}}\big)^2\Big)^{1/2}.\end{aligned} \tag{6.151}$$

Using inequalities (6.88), (6.144a), (6.144b) and the induction hypothesis we obtain that

$$\begin{aligned}&\Big(\sum_{|Y|\le j}\sup_{t\ge 0}\big((1+t)^{2-\rho}\|\delta(t)(\xi_Y^M\overline{J}_1^{(\overline{l}+1)})(t)\|_{L^2} + (1+t)^{1-\rho}\|\delta(t)(\xi_Y^M\overline{J}_1^{(\overline{l}+1)})(t)\|_{L^{6/5}}\big)^2\Big)^{1/2} \\ &\quad\le C_{j,\overline{l}}\overline{\mathcal{R}}^{(\overline{l}+1)}_{L_0,N_0+3+j+\overline{l}+1}(u;u_1,\ldots,u_{\overline{l}+1}), \quad j\ge 0,\end{aligned} \tag{6.152}$$

where $C_{j,\overline{l}}$ depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. Using once more inequality (6.88) we obtain that

$$\begin{aligned}&\Big(\sum_{|Y|\le j}\sup_{t\ge 0}\big((1+t)^{2-\rho}\|\delta(t)(\xi_Y^M\overline{J}_0^{(\overline{l}+1)}(L))(t)\|_{L^2} \\ &\quad + (1+t)^{1-\rho}\|\delta(t)(\xi_Y^M\overline{J}_0^{(\overline{l}+1)}(L))(t)\|_{L^{6/5}}\big)^2\Big)^{1/2} \\ &\le \sum_{n_1+n_2=j} C_{n_1,n_2}\Big(\big(H_{3+n_1}(\phi^*,t) + \wp^D_{n_1}(\phi^*(t))\big)\|\Psi(L)\|_{\mathcal{F}^D_{n_2}} \\ &\quad + \min\Big(\big(H_{3+n_1}(\Phi,t) + \wp^D_{n_1}(\Phi(t))\big)\|\Psi(L)\|_{\mathcal{F}^D_{n_2}}, \\ &\qquad \|\Phi\|_{\mathcal{F}^M_{n_1}}\big(H_{3+n_2}(\Psi(L),t) + \wp^D_{n_2}((\Psi(L))(t))\big)\Big)\Big),\end{aligned} \tag{6.153}$$

where $C_{0,j} = C_{j,0} = C$ and C_{n_1,n_2} depend only on ρ. The induction hypothesis (for $\overline{l}=0$), Proposition 6.2 and inequalities (6.148a), (6.151), (6.152) and (6.153) give that

$$\begin{aligned}&\|N(L)\|_{\mathcal{F}_j^M} \\ &\quad\le C\|u\|_{E^\rho_{L_0}}\|L\|_{\mathcal{F}_j^M} + C_{\overline{l}}\overline{\mathcal{R}}^{(\overline{l}+1)}_{L_0,N_0+3+j+\overline{l}+1}(u;u_1,\ldots,u_{\overline{l}+1}), \quad j\le 37,\end{aligned} \tag{6.154}$$

where C and $C_{\overline{l}}$ depend only on ρ and $\|u\|_{E^\rho_{L_0}}$. If $\varepsilon > 0$ is sufficiently small, then $C\|u\|_{E^\rho_{L_0}} \le 1/2$ for $u \in \mathcal{O}_\infty$, which together with inequality (6.154) shows that the linear inhomogeneous equation in $\mathcal{F}^M_{37}$

$$K^{(\overline{l}+1)} = N(K^{(\overline{l}+1)}) \tag{6.155}$$

has a unique solution $K^{(\bar{l}+1)} \in \mathcal{F}^M_{37}$ and that this solution satisfies

$$\|K^{(\bar{l}+1)}\|_{\mathcal{F}^M_j} \leq C_{\bar{l}} \overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0, N_0+3+j+\bar{l}+1}(u; u_1, \ldots, u_{\bar{l}+1}), \quad j \leq 37 = n, \tag{6.156a}$$

where $C_{\bar{l}}$ depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. Inequalities (6.148a) and (6.148b) then show that

$$\|\Phi_0^{(\bar{l}+1)}\|_{\mathcal{F}^M_j} \leq C_{\bar{l}} \overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0, N_0+3+j+\bar{l}+1}(u; u_1, \ldots, u_{\bar{l}+1}), \quad j \leq 37 = n, \tag{6.156b}$$

and that

$$\begin{aligned} &H_j(\Phi_0^{(\bar{l}+1)}, t) \\ &\quad \leq (1+t)^{-3/2+\rho} C_{\bar{l}} \overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0, N_0+3+j+\bar{l}+10}(u; u_1, \ldots, u_{\bar{l}+1}), \quad j \leq 27 = n-10, \end{aligned} \tag{6.156c}$$

where $C_{\bar{l}}$ depends only on ρ and $\|u\|_{E^\rho_{L_0}}$.

We now suppose that (6.156a) and (6.156b) are true for $j \leq n$ when $n \in \mathbb{N}$, $n \geq 37$. We study equation (6.128) for $i = 0$ and $l = \bar{l}+1$. It follows from statements i) and ii), with $l = 0$, of the present theorem and from Lemma 6.5 and Lemma 6.6 that

$$R'_j(g_0^{(\bar{l}+1)}, t) \leq (1+t)^{-3/2+\rho} C_{j,\bar{l}} \overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0, N_0+3+j+1+\bar{l}+1}(u; u_1, \ldots, u_{\bar{l}+1}), \; j \leq n-14, \tag{6.157a}$$

$$R^\infty_j(g_0^{(\bar{l}+1)}, t) \leq (1+t)^{-1} C_{j,\bar{l}} \overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0, N_0+3+j+4+\bar{l}+1}(u; u_1, \ldots, u_{\bar{l}+1}), \quad j \leq n-13, \tag{6.157b}$$

$$R^2_j(g_0^{(\bar{l}+1)}, t) \leq (1+t)^{-1} C_{j,\bar{l}} \overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0, N_0+3+j+\bar{l}+1}(u; u_1, \ldots, u_{\bar{l}+1}), \quad j \leq n, \tag{6.157c}$$

$$\wp^D_j(f_0^{(\bar{l}+1)}(t)) \leq (1+t)^{-3/2+\rho} C_{j,\bar{l}} \overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0, N_0+3+j+\bar{l}+1}(u; u_1, \ldots, u_{\bar{l}+1}), \quad j \leq n, \tag{6.157d}$$

where $C_{j,\bar{l}}$ depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. These inequalities give that

$$R^{(1)}_j(g_0^{(\bar{l}+1)}, t) \leq C_{j,\bar{l}} \overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0, N_0+3+j+9+\bar{l}+1}(u; u_1, \ldots, u_{\bar{l}+1}), \quad j \leq n-21, \tag{6.158a}$$

$$h'_j(g_0^{(\bar{l}+1)}, j_0, t) \leq C_{j,\bar{l}} \overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0, N_0+3+j+\bar{l}+1}(u; u_1, \ldots, u_{\bar{l}+1}), \quad j = n+1, \tag{6.158b}$$

where j_0 is the integer part of $j/2+5$,

$$h''_j(g_0^{(\bar{l}+1)}, t) \leq C_{j,\bar{l}} \overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0, N_0+3+j+\bar{l}+1}(u; u_1, \ldots, u_{\bar{l}+1}), \quad j = n+1, \tag{6.158c}$$

where $C_{j,\bar{l}}$ depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. It follows from Lemma 6.5 and Lemma 6.6 that

$$\begin{aligned} &\wp^D_{n+1}(f_0^{(\bar{l}+1)}(t)) + (1+t)\wp^D_n((1+\lambda_1(t))^{1/2} g_0^{(\bar{l}+1)}(t)) \\ &\quad \leq (1+t)^{-3/2+\rho}\big(C\|u\|_{E^\rho_{L_0}} \|K^{(\bar{l}+1)}\|_{\mathcal{F}^M_{n+1}} + C_{n,\bar{l}} \overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0, N_0+3+n+1+\bar{l}+1}(u; u_1, \ldots, u_{\bar{l}+1})\big) \end{aligned} \tag{6.158d}$$

and

$$\begin{aligned}&\wp^{D}_{n+1,i}(f_0^{(\bar{l}+1)}(t)) &&(6.158\text{e})\\&\qquad\le C_{n,\bar{l}}(1+t)^{-3/2+\rho}\overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0,N_0+3+n+1+\bar{l}+1}(u;u_1,\ldots,u_{\bar{l}+1}),\quad i\ge 1,\end{aligned}$$

where C and $C_{n,\bar{l}}$ depend only on ρ and $\|u\|_{E^{\rho}_{L_0}}$. It follows from the induction hypothesis in n and from inequalities (6.151), (6.152) and (6.153) that

$$\|K^{(\bar{l}+1)}\|_{\mathcal{F}^M_{n+1}}\le C\|u\|_{E^{\rho}_{L_0}}\|\Phi_0^{(\bar{l}+1)}\|_{\mathcal{F}^D_{n+1}}+C_{j,\bar{l}}\overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0,N_0+3+n+1+\bar{l}+1}(u;u_1,\ldots,u_{\bar{l}+1}),\quad(6.159)$$

where C and $C_{n,\bar{l}}$ depend only on ρ and $\|u\|_{E^{\rho}_{L_0}}$. Applying statement iv) of Proposition 6.4 to equation (6.128) with $i=0$ and $l=\bar{l}+1$, using inequalities (6.158b)–(6.159) and using the convention $\xi_{n+1,i}=0$ for $i\ge n+2$ and

$$\xi_{n+1,i}=\sup_{t\ge 0}\big((1+t)^{3/2-\rho}\wp^D_{n+1,i}(\Phi_0^{(\bar{l}+1)}(t))\big),\quad 0\le i\le n+1,\qquad(6.160)$$

we obtain that

$$\begin{aligned}&\xi_{n+1,0} &&(6.161\text{a})\\&\quad\le C\|u\|_{E^{\rho}_{L_0}}\xi_{n+1,0}+C_{n,\bar{l}}(\xi_{n+1,0})^{\varepsilon'}(\xi_{n+1,1})^{1-\varepsilon'}+C_{n,\bar{l}}\overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0,N_0+3+n+1+\bar{l}+1}(u;u_1,\ldots,u_{\bar{l}+1})\end{aligned}$$

and

$$\begin{aligned}&\xi_{n+1,i} &&(6.161\text{b})\\&\quad\le C_{n,\bar{l}}(\xi_{n+1,0})^{\varepsilon'}(\xi_{n+1,i+1})^{1-\varepsilon'}+C_{n,\bar{l}}\overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0,N_0+3+n+1+\bar{l}+1}(u;u_1,\ldots,u_{\bar{l}+1}),\quad 1\le i\le n+1,\end{aligned}$$

where $\varepsilon'=\min(1/2,2(1-\rho))$ and where C and $C_{n,\bar{l}}$ depend only on ρ and $\|u\|_{E^{\rho}_{L_0}}$. Let $\varepsilon>0$ be such that $C\|u\|_{E^{\rho}_{L_0}}\le 1/2$ for $u\in\mathcal{O}_\infty$ and let $\beta_{n+1}=2C_{n,\bar{l}}$ and $\alpha_{n+1}=2C_{n,\bar{l}}\overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0,N_0+3+n+1+\bar{l}+1}(u;u_1,\ldots,u_{\bar{l}+1})$. It then follows from inequalities (6.161a) and (6.161b) that

$$\xi_{n+1,i}\le\alpha_{n+1}+\beta_{n+1}(\xi_{n+1,0})^{\varepsilon'}(\xi_{n+1,i+1})^{1-\varepsilon'},\quad 0\le i\le n+1.\qquad(6.162)$$

Proceeding as in solving system (5.182c) we obtain that $\xi_{n+1,0}\le C'_{n+1}\alpha_{n+1}$, where C'_{n+1} is a polynomial in β_{n+1}. This proves together with inequality (5.159), that

$$\|(K^{(\bar{l}+1)},\Phi^{(\bar{l}+1)})\|_{\mathcal{F}_{n+1}}\le C_{n+1,\bar{l}+1}\overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0,N_0+3+n+1+\bar{l}+1}(u;u_1,\ldots,u_{\bar{l}+1}),\qquad(6.163)$$

which by induction in n proves that inequalities (6.157a) and (6.157b) are true for each $j \in \mathbb{N}$, since they are true for $j \leq 37$. Inequalities (6.157a)–(6.158a) are then also true for each $j \in \mathbb{N}$. Inequalities (6.157a) and (6.157b) for $j \in \mathbb{N}$ and inequality (6.144a), together with $\Phi^{(\bar{l}+1)} = \sum_{0 \leq i \leq 4} \Phi_i^{(\bar{l}+1)}$, give that

$$\|(K^{(\bar{l}+1)}, \Phi^{(\bar{l}+1)})\|_{\mathcal{F}_j} \leq C_{n+1,\bar{l}+1} \overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0,N_0+3+j+\bar{l}+1}(u; u_1, \ldots, u_{\bar{l}+1}), \quad j \geq 0, \tag{6.164a}$$

where $C_{n+1,\bar{l}+1}$ depend only on ρ and $\|u\|_{E^\rho_{L_0}}$. Inequality (6.158a), with $j \in \mathbb{N}$, and statement iii) of proposition 6.4 prove that inequality (6.156c) is true for each $j \in \mathbb{N}$. This shows, together with inequality (6.144b) that

$$H_j(\Phi^{(\bar{l}+1)}, t) \leq (1+t)^{-3/2+\rho} C_{j,\bar{l}+1} \overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0,N_0+3+j+\bar{l}+14}(u; u_1, \ldots, u_{\bar{l}+1}), \quad j \geq 0, \tag{6.164b}$$

where $C_{j,\bar{l}+1}$ depends only on ρ and $\|u\|_{E^\rho_{L_0}}$. Inequality (6.124) is true, when Φ is replaced by $\Phi^{(\bar{l}+1)}$ and $R^1_{n_2-k,k}(t)$ (resp. $R'_{n_4+7}(t)$, $R^2_{n_4+9}(t)$, $R^\infty_{n_4}(t)$) is replaced by $R^1_{n_2-k,k}(\Phi^{(\bar{l}+1)}, t)$ (resp. $R'_{n_4+7}(\Phi^{(\bar{l}+1)}, t)$, $R^2_{n_4+9}(\Phi^{(\bar{l}+1)}, t)$, $R^\infty_{n_4}(\Phi^{(\bar{l}+1)}, t)$). This inequality, inequalities (6.112), (6.157a)–(6.157c), (6.164b) and the definition of $R^1_{n,k}$ in Corollary 5.9 give that

$$\begin{aligned} &\wp^D_j\big((1+\lambda_1(t))^{k/2}\Phi^{(\bar{l}+1)}(t)\big) \\ &\quad \leq (1+t)^{-3/2+\rho} C_{j,\bar{l}+1} \overline{\mathcal{R}}^{(\bar{l}+1)}_{L_0,N_0+3+j+k+\bar{l}+1}(u; u_1, \ldots, u_{\bar{l}+1}), \quad j,k \geq 0, \end{aligned} \tag{6.164c}$$

where $C_{j,\bar{l}+1}$ depend only on ρ and $\|u\|_{E^\rho_{L_0}}$.

Inequalities (6.164a)–(6.164c) prove that the induction hypothesis in l is true for $l = \bar{l}+1$. Since we already have proved that it is true for $l = 0$ it follows by induction in l that it is true for each $l \in \mathbb{N}$. To complete the proof we only have to prove the announced decrease properties of $C_{n,l}$ and $C'_{n,l}$, when $\|u\|_{E^\rho_{L_0}} \to 0$. We observe that since the solution $(\Phi, K) \in \mathcal{O}_\infty$ of equations (6.126a)–(6.126c), with $l = 0$, is unique, since $(\Phi^{(l)}, K^{(l)}) \in \mathcal{F}_j$, $l \geq 1$, is the unique solution of equations (6.126a)–(6.126c) with $l \geq 1$ and since, when $u = 0$, $(\Phi^{(l)}, K^{(l)}) = 0$ for $0 \leq l \leq 2$ and $(\Phi^{(3)}, 0)$ are solutions it follows that $\Phi^{(l)} = 0$ for $0 \leq l \leq 2$ and $K^{(l)} = 0$ for $0 \leq l \leq 3$, when $u = 0$. The use of Taylor formula now gives the announced properties of $C_{n,l}$ and $C'_{n,l}$ when $\|u\|_{E^\rho_{L_0}} \to 0$. This proves the theorem.

In the situation of Theorem 6.9 let $u \in \mathcal{O}_\infty$ and let $v(u) = (K, \Phi)$. According to the variable substitution (6.30) we introduce

$$\begin{aligned} A_\mu(t) &= K_\mu(t) + A^*_\mu(t), \quad 0 \leq \mu \leq 3 \\ \psi'(t) &= \Phi(t) + \phi^*(t), \end{aligned} \tag{6.165}$$

and we introduce

$$\begin{aligned} (A_Y)_\mu &= (\xi^M_Y A)_\mu, \\ (\dot{A}_Y)_\mu &= (\xi^M_{P_0 Y} A), \\ \psi'_Y &= \xi^D_Y \psi', \end{aligned} \tag{6.166}$$

for $Y \in U(\mathfrak{p})$. When we want to indicate the dependence of $A_\mu(t), (A_Y)_\mu(t)$ etc. on u we shall write $(A(u))_\mu(t), (A_Y(u))_\mu(t)$ etc. The functions A_μ, $0 \leq \mu \leq 3$, and ψ' satisfy, as we shall prove it in next theorem, the following equations on $\mathbb{R}^+ \times \mathbb{R}^3$:

$$\square A_\mu = (\psi')^+ \gamma^0 \gamma_\mu \psi', \tag{6.167a}$$

$$(i\gamma^\mu \partial_\mu + m)\psi' = \big(A_\mu - (\partial_\mu \vartheta(A))\big)\gamma^\mu \psi', \tag{6.167b}$$

and

$$\partial_\mu A^\mu = 0. \tag{6.167c}$$

We recall that $A_{0,Y}, \dot{A}_{0,Y}, \phi'_{0,Y}$ are free solutions given in (4.137a), (4.137b) and (4.137c).

In the sequel of this chapter N_0 will not necessarily denote the same integer as in Proposition 6.2.

Theorem 6.10. *Let $1/2 < \rho < 1$. Then there exists $N_0 \geq 0$ and an open ball $\mathcal{O}_{N_0}$ in $E^{\circ\rho}_{N_0}$ with center at the origin such that A, $\dot{A}$, ψ' defined by (6.166) satisfy*

$$\begin{aligned} &\|\!|(D^l(A - A_0, \psi' - \phi'_0))(u; u_1, \ldots, u_l)\|\!|_{\rho', \varepsilon, L} \\ &\quad \leq C_{l+L}(\mathcal{R}^l_{N_0, L+l}(u_1, \ldots, u_l) + \|u\|_{E^\rho_{N_0+L+l}} \|u_1\|_{E^\rho_{N_0}} \cdots \|u_l\|_{E^\rho_{N_0}}), \end{aligned} \tag{6.168}$$

for all $l \geq 0$, $L \geq 0$, $1/2 < \rho' \leq 1$, $\varepsilon = (\varepsilon(0), \varepsilon(1))$, $\varepsilon(0) > 0$, $\varepsilon(1) \geq \rho$, $u \in \mathcal{O}_\infty = \mathcal{O}_{N_0} \cap E^\rho_\infty$, $u_1, \ldots, u_l \in E^{\circ\rho}_\infty$, where

$$\begin{aligned} \|\!|(a, \Phi)\|\!|_{\rho', \varepsilon, L} = \sum_{i=0,1} \sum_{\substack{Y \in \boldsymbol{\sigma}^i \\ |Y|+k \leq L}} \sup_{t \geq 0} \Big(&(1+t)^{\rho' - 1/2} \|(\xi^M_Y a, \xi^M_{P_0 Y} a)(t)\|_{M^{\rho'}_0} \\ &+ (1+t)^{2(1-\rho)} \|(1 + \lambda_1(t))^{k/2} (\xi^D_Y \Phi)(t)\|_{D_0} \\ &+ \|(\delta(t))^{1+i-\varepsilon(i)} (1+t)^{\varepsilon(i)(1-i)} (\xi^M_Y a)(t)\|_{L^\infty} \\ &+ \|(\delta(t))^{3/2 + 2(1-\rho)} (1 + \lambda_1(t))^{k/2} (\xi^D_Y \Phi)(t)\|_{L^\infty} \Big) \end{aligned}$$

and where C_{l+L} depends only on ρ', ε, ρ and $\|u\|_{E^\rho_{N_0}}$. Moreover the function $u \mapsto (A - A_0, \psi' - \phi'_0)(u)$ from $\mathcal{O}_\infty$ to the Fréchet space with seminorms $\|\!| \cdot \|\!|_{\rho', \varepsilon, L}$, indexed by $L \geq 0$ has a zero of order two at $u = 0$,

$$(D^l A_Y, D^l \dot{A}_Y, D^l \psi'_Y)(u; u_1, \ldots, u_l) \in C^0(\mathbb{R}^+, E^\rho), \quad Y \in \Pi'$$

and equations (6.167a), (6.167b) and (6.167c) are satisfied.

Proof. Since

$$\begin{aligned} &(A_Y - A_{0,Y}, \dot{A}_Y - \dot{A}_{0,Y}, \psi'_Y - \phi'_{0,Y}) \\ &\quad = (K_Y, K_{P_0 Y}, \Phi_Y) + (A^*_Y - A_{0,Y}, A^*_{P_0 Y} - A_{0, P_0 Y}, \phi^*_Y - \phi'_{0,Y}) \end{aligned}$$

according to (6.165) and (6.166), it follows from Theorem 6.9 that it is sufficient to prove inequality (6.168) with A_Y (resp. $\dot{A}_Y, \psi'_Y$) replaced by A^*_Y (resp. $A^*_{P_0 Y}, \phi^*_Y$). Since

$$\begin{aligned}
&(A^*_Y - A_{0,Y}, A^*_{P_0 Y} - A_{0,P_0 Y}, \phi^*_Y - \phi'_{0,Y}) \\
&\quad = (\Delta^{*M}_Y, \Delta^{*M}_{P_0 Y}, \Delta^{*D}_Y) + (A^*_{0,Y} - A_{J+1,Y}, A^*_{0,P_0 Y} - A_{J+1,P_0 Y}, \phi^*_{0,Y} - \phi'_{J+1,Y}) \\
&\qquad + \sum_{0 \le n \le J} (A_{n+1,Y} - A_{n,Y}, A_{n+1,P_0 Y} - A_{n,P_0 Y}, \phi'_{n+1,Y} - \phi'_{n,Y}),
\end{aligned}$$

inequality (6.168) now follows from Theorem 4.10, Lemma 6.1 and Proposition 6.2 as does also the statement concerning the second order zero.

It follows from equations (6.1a)–(6.2b), from equations (4.137b) and (4.137c) for $A_{n+1,\mu}$ and ϕ'_{n+1} and from definitions (4.138a) and (4.138b) of Δ^M_n and Δ^D_n that equations (6.167a) and (6.167b) are satisfied. Equations (6.167a) and (6.167b) give $\Box \partial_\mu A\mu = 0$, i.e. with the notation (6.166), $\Box \partial_\mu (A_{\mathbb{I}})^\mu = 0$. It then follows from inequality (6.168) with $\rho' = 1$, $l = 1$, $Y = \mathbb{I}$ that

$$\partial_\mu (A_{\mathbb{I}})^\mu = \partial_\mu (A_{0,\mathbb{I}})^\mu = 0,$$

where the last equality follows from the definition of $E^{\circ\rho}$. The continuity from $\mathbb{R}^+$ to E^ρ follows from the fact that $(K, \Phi) \in \mathcal{F}_n$ for $n \ge 0$. This proves the theorem.

Next we introduce the *manifold on which the gauge condition* defined by (1.3a) and (1.3b) is *satisfied*. Let $V^\rho_N, N \ge 1$, be the subset of all elements $(f, \dot{f}, \alpha) \in E^\rho_N, 1/2 < \rho < 1$, such that

$$\Delta f_0 - \sum_{1 \le i \le 3} \partial_i \dot{f}_i + |\alpha|^2 = 0, \tag{6.169}$$

$$\dot{f}_0 - \sum_{1 \le i \le 3} \partial_i f_i = 0$$

and let $V^\rho_\infty = \cap_{N \ge 1} V^\rho_N$. We also introduce the map

$$F^{-1}: (g, \dot{g}, \beta) \mapsto (f, \dot{f}, \alpha), f_0 = g_0 - \Delta^{-1} |\beta|^2, f_i = g_i, 1 \le i \le 3, \dot{f} = \dot{g}, \alpha = \beta, \tag{6.170}$$

which maps E^ρ_N onto E^ρ_N, $N \ge 1$ as will be proved.

Theorem 6.11. *Let I be the identity map on E_N and let F^{-1} be defined by (6.170). Then $F = I + F^{(2)}$ and $F^{-1} = I - F^{(2)}$ where $F^{(2)}$ is a (real) bilinear continuous map from E_N to $E_N, N \ge 1$. $F^{(2)}$ satisfies*

$$\begin{aligned}
&\|F^{(2)}(u_1, u_2)\|_{E^\rho_N} \\
&\quad \le C_N \big(\|\alpha_1\|_{D_N} \|\alpha_2\|_{D_1} + \|\alpha_1\|_{D_1} \|\alpha_2\|_{D_N} \big), \quad N \ge 1, u_i = (f_i, \dot{f}_i, \alpha_i), \quad i = 1, 2,
\end{aligned}$$

where C_N are independent of u_1, u_2. Moreover $V^\rho_N, N \ge 1$, (with its topology inherited from E^ρ_N) is a differentiable Hilbert manifold globally diffeomorphic to $E^{\circ\rho}_N$ by the map $F: V^\rho_N \to E^{\circ\rho}_N$ and $V^\rho_N = V^\rho_1 \cap E^\rho_N$.

Proof. Let $u = (f, \dot{f}, \alpha), u_i = (f_i, \dot{f}_i, \alpha_i), i = 1, 2, u, u_1, u_2 \in E_N$. By definition (6.170) of F it follows that

$$\begin{aligned} F(u) &= u + F^{(2)}(u,u), \\ F^{-1}(u) &= u - F^{(2)}(u,u), \\ F^{(2)}(u_1,u_2) &= \big((\frac{1}{2}\Delta^{-1}(\alpha_1^+\alpha_2 + \alpha_2^+\alpha_1), 0,0,0), 0, 0\big). \end{aligned} \tag{6.171}$$

According to Theorem 2.9

$$\begin{aligned} \|F^{(2)}(u_1,u_2)\|_{E_N^\rho} &= \|\big((\frac{1}{2}\Delta^{-1}(\alpha_1^+\alpha_2 + \alpha_2^+\alpha_1), 0,0,0), 0\big)\|_{M_N^\rho} \\ &\le C_N \sum_{0 \le |\mu| \le |\nu| \le n} \||\nabla|^\rho x^\mu \partial^\nu \Delta^{-1}(\alpha_1^+\alpha_2 + \alpha_2^+\alpha_1)\|_{L^2} \end{aligned}$$

where $\mu, \nu \in \mathbb{N}^3, x^\mu = x_1^{\mu_1}x_2^{\mu_2}x_3^{\mu_3}, \partial^\nu = \partial_1^{\nu_1}\partial_2^{\nu_2}\partial_3^{\nu_3}$. Since $|\mu| \le |\nu|$ it follows that

$$\|F^{(2)}(u_1,u_2)\|_{E_N^\rho} \le C'_N \sum_{0 \le |\mu| \le |\nu| \le n} \||\nabla|^{\rho-2} x^\mu \partial^\nu (\alpha_1^+\alpha_2 + \alpha_2^+\alpha_1)\|_{L^2}$$

for some constants C'_N and it follows by continuity, using the fact that $\||\nabla|^{\rho-2}h\|_{L^2} \le C_p\|h\|_{L^p}, p = 6(7-2\rho)^{-1}$, for $1/2 < \rho \le 2$, where h has compact support and the constant C_p is independent of the support:

$$\|F^{(2)}(u_1,u_2)\|_{E_N^\rho} \le C''_N \sum_{|\mu| \le |\nu| \le n} \|x^\mu \partial^\nu (\alpha_1^+\alpha_2 + \alpha_2^+\alpha_1)\|_{L^p},$$

$1/2 < \rho < 1, p = 6(7-2\rho)^{-1}$. Using the inequalities

$$\begin{aligned} \|h_1h_2\|_{L^p} &\le \|h_1\|_{L^2}\|h_2\|_{L^{3/(2-\rho)}}, \\ \|h\|_{L^{3/(2-\rho)}} &\le C_\rho\|h_1\|_{W^{1,2}}, \end{aligned}$$

for $1/2 < \rho < 1$, we obtain

$$\begin{aligned} &\|F^{(2)}(u_1,u_2)\|_{E_N^\rho} \\ &\quad \le C_N \sum_{0 \le i \le N/2} \big(\|\alpha_1\|_{D_{N-i}}\|\alpha_2\|_{D_{i+1}} + \|\alpha_1\|_{D_{1+i}}\|\alpha_2\|_{D_{N-i}}\big), \quad N \ge 1. \end{aligned}$$

Corollary 2.6 now gives (with new constants C_N independent of u_1, u_2)

$$\|F^{(2)}(u_1,u_2)\|_{E_N^\rho} \le C_N\big(\|\alpha_1\|_{D_N}\|\alpha_2\|_{D_1} + \|\alpha_1\|_{D_1}\|\alpha_2\|_{D_N}\big), \quad N \ge 1.$$

This proves the inequality of the theorem.

Let $u=(f,\dot{f},\alpha)\in V_N^\rho, N\geq 1$. Then $v=(g,\dot{g},\beta)=F(u)\in E_N^\rho$ and according to (6.171): $g_0=f_0+\Delta^{-1}|\alpha|^2$, $g_i=f_i$ for $1\leq i\leq 3$ and $\dot{g}=\dot{f}$, $\beta=\alpha$. It follows from (6.169) that

$$\Delta g_0-\sum_{1\leq i\leq 3}\partial_i\dot{g}_i=0,$$
$$\dot{g}_0-\sum_{1\leq i\leq 3}\partial_i g_i=0,$$

which, according to the definition of $E_N^{\circ\rho}$ proves that $v\in E_N^{\circ\rho}$. Similarly it follows that $F^{-1}(u)\in V_N^\rho$ if $u\in E_N^{\circ\rho}$. This proves the theorem since $E_N^{\circ\rho}=E_N^\rho\cap E_0^\rho$.

We next define a *modified wave operator* Ω_1 for the M-D equations (1.1a), (1.1b) and (1.1c), when $t\to\infty$. Let $(A_Y(u))(t),(\dot{A}_Y(u))(t),(\psi'_Y(u))(t),N_0$ and $\mathcal{O}_{N_0}$ be as in Theorem 6.10 and let

$$\Omega_1(u)=\big((A(u))(0),(\dot{A}(u))(0),e^{-i\vartheta(A,0)}(\psi'(u))(0)\big),\quad u\in\mathcal{O}_\infty=\mathcal{O}_{N_0}\cap E_\infty^\rho. \tag{6.172}$$

Theorem 6.12. *Let $1/2<\rho<1$. There exists $N_0\geq 0$ and a neighbourhood $\mathcal{O}_{N_0}$ of zero in $E_{N_0}^{\circ\rho}$ such that $\Omega_1\colon\mathcal{O}_\infty\to E_\infty^\rho$ is a one to one C^∞ mapping satisfying*

$$\begin{aligned}&\|(D^l\Omega_1)(u;u_1,\ldots,u_l)\|_{E_L}\\&\quad\leq F_{L,l}(\|u\|_{E_{N_0}})\mathcal{R}^l_{N_0,l+L}(u_1,\ldots,u_l)+F'_{L,l}(\|u\|_{E_{N_0}})\|u\|_{E_{N_0+l+L}}\|u_1\|_{E_{N_0}}\cdots\|u_l\|_{E_{N_0}}\end{aligned}$$

for each $L\geq 0$, $l\geq 0$, $u_1,\ldots,u_l\in E_\infty^{\circ\rho}$, where $F_{L,l}$ and $F'_{L,l}$ are increasing continuous functions. Moreover $\Omega_1(0)=0$ and $\Omega_1(\mathcal{O}_\infty)\subset V_\infty^\rho$.

Proof. We first prove that Ω_1 is one to one. Let $u,u'\in\mathcal{O}_\infty$, where $\mathcal{O}_\infty$ is as in Theorem 6.10. If $\Omega_1(u)=\Omega_1(u')$, then it follows because of the uniqueness of the local (in time) solution of the Maxwell-Dirac equations (1.1a), (1.1b) and (1.1c), according to Theorem 1 of [10] that the solution $(A(u),\psi'(u))$ and $(A(u'),\psi'(u'))$ of equations (6.167a), (6.167b) and (6.167c) are equal. Theorem 6.10 then give that the free solutions $(A_{0,\mathbb{I}}(u),\psi'_{0,\mathbb{I}}(u))$ and $(A_{0,\mathbb{I}}(u'),\psi'_{0,\mathbb{I}}(u'))$ are equal, which according to their definition after formula (4.137c) proves that $u=u'$.

Let
$$a_N=\Big(\sum_{\substack{Y\in\Pi'\\|Y|\leq N}}\|(A_Y(u),\dot{A}_Y(u),\psi'_Y(u))(0)\|^2_{E_0^\rho}\Big)^{1/2},\quad N\geq 0. \tag{6.173}$$

Introduce also
$$\psi_Y(u)=\xi_Y^D(e^{-i\vartheta(A)}\psi'(u))$$

and
$$\vartheta'_Y(u)=\xi_Y\vartheta(A),\quad Y\in U(\mathfrak{p}).$$

By Leibniz rule we obtain that

$$\begin{aligned}\|(\psi'_Y(u))(0)\|_D &= \|(\xi^D_Y e^{-i\vartheta(A,0)}\psi(u))(0)\|_D \\ &\geq \|(\xi^D_Y \psi(u))(0)\|_D - C_{|Y|}\sum \|(\vartheta'_{Y_1}(u)\cdots\vartheta'_{Y_l}(u)\psi_Z(u))(0)\|_D,\end{aligned} \tag{6.174}$$

$Y \in \Pi'$, $|Y| \geq 1$, where the sum is taken over $1 \leq l \leq |Y|$, $Y_i \in \Pi'$, $|Y_1|+\cdots+|Y_l|+|Z| \leq |Y|$, $|Z| \leq |Y|-1$, $|Y_i| \geq 1$. It follows from Lemma 4.4 and Theorem 6.10 that

$$\sup_{x\in\mathbb{R}^3}(1+|x|)^{1/2-\rho}|(\vartheta'_{Y_i}(u))(0,x)| \leq F'_{|Y_i|,0}(\|u\|_{E^\rho_{N_0}})\|u\|_{E^\rho_{N_0+|Y_i|}},$$

which gives using that $T^D_Z(\Omega_1(u)) = (\psi_Z(u))(0)$:

$$\begin{aligned}&\|(\vartheta'_{Y_1}(u)\cdots\vartheta'_{Y_l}(u)\psi_Z(u))(0)\|_D \\ &\quad\leq G_{|Y|}(\|u\|_{E^\rho_{N_0}})\|u\|_{E^\rho_{N_0+|Y_1|}}\cdots\|u\|_{E^\rho_{N_0+|Y_l|}}\|(1+|\cdot|)^{l(\rho-1/2)}T^D_Z(\Omega_1(u))\|_D,\end{aligned}$$

where $G_{|Y|}$ is a polynomial. Since $|Y_i| \geq 1$ it follows from Corollary 2.6 that

$$\|u\|_{E^\rho_{N_0+|Y_1|}}\cdots\|u\|_{E^\rho_{N_0+|Y_l|}} \leq C_{|Y|}\|u\|^{l-1}_{E^\rho_{N_0+1}}\|u\|_{E^\rho_{N_0+1+|Y_1|+\cdots+|Y_l|-l}},$$

which after redefinition of the polynomial $G_{|Y|}$ and by the fact that $0 \leq |Y_1|+\cdots+|Y_l|-l \leq |Y|-|Z|-l$ gives:

$$\begin{aligned}&\|(\vartheta'_{Y_1}(u)\cdots\vartheta'_{Y_l}(u)\psi_Z(u))(0)\|_D \\ &\quad\leq G_{|Y|}(\|u\|_{E^\rho_{N_0+1}})\|u\|_{E_{N_0+1+|Y|-|Z|-l}}\|(1+|\cdot|)^{l(\rho-1/2)}T^D_Z(\Omega_1(u))\|_D.\end{aligned} \tag{6.175}$$

The inequality $\|(1+|\cdot|)^b f\|_{L^2} \leq \|(1+|\cdot|)f\|^b_{L^2}\|f\|^{1-b}_{L^2}$, $0 \leq b \leq 1$, gives:

$$\begin{aligned}&\|(1+|\cdot|)^{l(\rho-1/2)}T^D_Z(\Omega_1(u))\|_D \\ &\quad\leq \|(1+|\cdot|)^{1+(l-1)(\rho-1/2)}T^D_Z(\Omega_1(u))\|^{\rho-1/2}_D\|(1+|\cdot|)^{(l-1)(\rho-1/2)}T^D_Z(\Omega_1(u))\|^{3/2-\rho}_D,\end{aligned}$$

which together with Theorem 2.9, statement i) of Corollary 2.21 and inequality (6.175) shows that

$$\begin{aligned}&\|(\vartheta'_{Y_1}(u)\cdots\vartheta'_{Y_l}(u)\psi_Z(u))(0)\|_D \\ &\quad\leq G_{|Y|}(\|u\|_{E^\rho_{N_0+1}})(C_{l,|Z|}(\|\Omega_1(u)\|_{E^\rho_0}))^{\rho-1/2}(C_{l-1,Z}(\|\Omega_1(u)\|_{E^\rho_0}))^{3/2-\rho} \\ &\qquad\|u\|_{E_{N_0+1+|Y|-|Z|-l}}\|\Omega_1(u)\|^{\rho-1/2}_{E_{l+|Z|}}\|\Omega_1(u)\|^{3/2-\rho}_{E_{l-1+|Z|}}.\end{aligned}$$

Since

$$\|\Omega_1(u)\|_{E^\rho_0} = \|(A(u),\dot{A}(u),\psi'(u))(0)\|_{E^\rho_0},$$

it follows from the preceeding inequality and from Theorem 6.10 that

$$\begin{aligned}&\|(\vartheta'_{Y_1}(u)\cdots\vartheta'_{Y_l}(u)\psi_Z(u))(0)\|_D \\ &\qquad \leq G'_{|Y|}(\|u\|_{E_{N_0+1}})\|u\|_{E_{N_0+1+|Y|-|Z|-l}}\|\Omega_1(u)\|^{\rho-1/2}_{E_{l+|Z|}}\|\Omega_1(u)\|^{3/2-\rho}_{E_{l-1+|Z|}},\end{aligned} \tag{6.176}$$

where $G'_{|Y|}$ is a continuous function. Since $(x+y)^a \leq x^a + y^a$ for $x, y \geq 0$, $0 \leq a \leq 1$, it follows from Corollary 2.6 that

$$\begin{aligned}&\|u\|_{E_{N_0+1+|Y|-|Z|-l}}\|\Omega_1(u)\|^{\rho-1/2}_{E_{l+|Z|}}\|\Omega_1(u)\|^{3/2-\rho}_{E_{l-1+|Z|}} \\ &\quad \leq C_{|Y|}(\|u\|_{E_{N_0+1}}\|\Omega_1(u)\|^{\rho-1/2}_{E_{|Y|}}\|\Omega_1(u)\|^{3/2-\rho}_{E_{|Y|-1}} + \|u\|_{E_{N_0+1+|Y|}}\|\Omega_1(u)\|_{E_0}),\end{aligned}$$

which together with Theorem 6.10 and inequality (6.176) give

$$\begin{aligned}&\|(\vartheta'_{Y_1}(u)\cdots\vartheta'_{Y_l}(u)\psi_Z(u))(0)\|_D \\ &\quad \leq G_{|Y|}(\|u\|_{E_{N_0+1}})\|u\|_{E_{N_0+1+|Y|}} + G'_{|Y|}(\|u\|_{E_{N_0+1}})\|\Omega_1(u)\|^{\rho-1/2}_{E_{|Y|}}\|\Omega_1(u)\|^{3/2-\rho}_{E_{|Y|-1}},\end{aligned} \tag{6.177}$$

where $G_{|Y|}$ and $G'_{|Y|}$ are continuous functions.

Definition (6.173) and inequalities (6.174) and (6.177) give after redefining N_0+1 by N_0 and noting that

$$\wp_N(T(\Omega_1(u))) = \Big(\sum_{\substack{Y\in\Pi' \\ |Y|\leq N}} \|(A_Y(u), \dot{A}_Y(u), \psi_Y(u))(0)\|^2_D\Big)^{1/2}:$$

$$\begin{aligned}&\wp_N(T(\Omega_1(u))) \\ &\quad \leq a_N + G_N(\|u\|_{E_{N_0}})\|u\|_{E_{N_0+N}} + G'_N(\|u\|_{E_{N_0}})\|\Omega_1(u)\|^{\rho-1/2}_{E_N}\|\Omega_1(u)\|^{3/2-\rho}_{E_{N-1}}, \quad N \geq 1,\end{aligned} \tag{6.178}$$

where G_N and $G_{N'}$ are some continuous functions.

Let $\mathcal{O}_{N_0}$ be sufficiently small so that $\Omega_1(u) \in \mathcal{O}$, where $\mathcal{O}$ is given by Theorem 2.22. This is possible according to theorem 6.10. It follows from inequality (6.178) statement ii) of Theorem 2.22 and Theorem 6.10 that

$$\begin{aligned}&\wp_1(T(\Omega_1(u))) \\ &\quad \leq a_1 + G_1(\|u\|_{E_{N_0}})\|u\|_{E_{N_0+1}} + H(\|u\|_{E_{N_0}})\wp_1(T(\Omega_1(u)))^{\rho-1/2}\|\Omega_1(u)\|^{3/2-\rho}_{E_0},\end{aligned} \tag{6.179}$$

where H is a continuous function. This inequality gives

$$\begin{aligned}&\wp_1(T(\Omega_1(u))) \\ &\quad \leq 2a_1 + 2G_1(\|u\|_{E_{N_0}})\|u\|_{E_{N_0+1}} + (H(\|u\|_{E_{N_0}}))^2\|\Omega_1(u)\|_{E_0}.\end{aligned} \tag{6.180}$$

As a matter of fact if $\|\Omega_1(u)\|_{E_0} = 0$ then this is obvious. Let $\|\Omega_1(u)\|_{E_0} > 0$ and let $x = \wp_1(T(\Omega_1(u)))/\|\Omega_1(u)\|_{E_0}$. Since $x \geq 1$ and $0 < \rho - 1/2 < 1/2$ we obtain

$$x \leq (a_1 + G_1(\|u\|_{E_{N_0}})\|u\|_{E_{N_0+1}})/\|\Omega_1(u)\|_{E_0} + H(\|u\|_{E_{N_0}})x^{1/2},$$

which gives the result. Inequality (6.180) and Theorem 6.10 (used for a_1 and $\|\Omega_1(u)\|_{E_0}$) now show that

$$\wp_1(T(\Omega_1(u))) \leq F(\|u\|_{E_{N_0}})\|u\|_{E_{N_0+1}}, \tag{6.181}$$

where F is a continuous function. Since $\Omega_1(u) \in \mathcal{O}$, it follows from (6.181) and statement ii) of Theorem 2.22 that

$$\|\Omega_1(u)\|_{E_1} \leq F'_{1,0}(\|u\|_{E_{N_0}})\|u\|_{E_{N_0+1}}, \tag{6.182}$$

where $F'_{1,0}$ is a continuous function.

Inequalities (6.178), (6.181) and (6.182) and statement iii) of Theorem 2.22 give, after replacing $N_0 + 1$ by N_0, that

$$\begin{aligned} &\wp_N(T(\Omega_1(u))) \\ &\quad \leq H_N(\|u\|_{E_{N_0}})\|u\|_{E_{N_0+N}} + H'_N(\|u\|_{E_{N_0}})\big(\wp_N(T(\Omega_1(u)))\big)^{\rho-1/2}\big(\wp_{N-1}(T(\Omega_1(u)))\big)^{3/2-\rho}, \end{aligned} \tag{6.183}$$

$N \geq 1$, where we have estimated a_N by Theorem 6.10 and where H_N and H'_N are continuous functions.

In the same way as we obtained inequality (6.180) from inequality (6.179) we obtain from inequality (6.183) that

$$\begin{aligned} &\wp_N(T(\Omega_1(u))) \\ &\quad \leq 2H_N(\|u\|_{E_{N_0}})\|u\|_{E_{N_0+N}} + (H'_N(\|u\|_{E_{N_0}}))^2\wp_{N-1}(T(\Omega_1(u))), \quad N \geq 1. \end{aligned}$$

Since $\wp_0(T(\Omega_1(u))) = \|\Omega_1(u)\|_{E_0} \leq H(\|u\|_{E_{N_0}})\|u\|_{E_{N_0}}$, according to Theorem 6.10, where H is a continuous function it now follows that

$$\wp_N(T(\Omega_1(u))) \leq G_N(\|u\|_{E_{N_0}})\|u\|_{E_{N_0+N}}, \quad N \geq 0, \tag{6.184}$$

where G_N are continuous functions. Since $\Omega_1(u) \in \mathcal{O}$ it follows from (6.184) and statement iii) of Theorem 2.22 (replacing $N_0 + 1$ by N_0) that

$$\|\Omega_1(u)\|_{E_N} \leq F'_{N,0}(\|u\|_{E_{N_0}})\|u\|_{E_{N_0+N}}, \quad N \geq 0. \tag{6.185}$$

This proves the theorem in the case of $l = 0$. The proof for the case $l > 0$ is so similar that we omit it. That the gauge conditions (1.3a) and (1.3b) are satisfied follows from the fact that equation (6.167c) is satisfied according to Theorem 6.10. This shows that $\Omega_1(\mathcal{O}_\infty) \subset V^\rho_\infty$, which proves the theorem.

In order to prove that Ω_1 has a *local inverse*, we shall use, for the space E^ρ_∞, the inverse mapping theorem in the case of Fréchet spaces. To do that we first extend the map $\Omega_1 \colon \mathcal{O}_\infty \to V^\rho_\infty$, where $\mathcal{O}_\infty$ is an open neighbourhood of zero in $E^{\circ\rho}_\infty$ to a map from $\mathcal{O}'_\infty$ to E^ρ_∞, where $\mathcal{O}'_\infty$ is a neighbourhood of zero in E^ρ_∞ and then prove that the derivative of this extended map has a right inverse.

Theorem 6.13. *Let* $1/2 < \rho < 1$. *There exists* $N_0 \geq 0$ *and* $M_0 \geq 0$ *and there exists neighbourhoods* $\mathcal{O}_{N_0}$ *and* $\mathcal{U}_{M_0}$ *of zero in* $E^{\circ\rho}_{N_0}$ *and* $V^\rho_{M_0}$ *respectively, such that the map*

$\Omega_1: \mathcal{O}_\infty \to \mathcal{U}_\infty$, $\mathcal{O}_\infty = \mathcal{O}_{N_0} \cap E^\rho_\infty$, $\mathcal{U}_\infty = \mathcal{U}_{M_0} \cap E^\rho_\infty$, satisfies the conclusions of Theorem 6.12 and has a C^∞ inverse $\Omega_1^{-1}: \mathcal{U}_\infty \to \mathcal{O}_\infty$ satisfying

$$\begin{aligned}&\|(D^l(\Omega_1^{-1} \circ F^{-1}))(u; u_1, \ldots, u_l)\|_{E_L} \\ &\quad \le C_{L,l} \mathcal{R}^l_{M_0, l+L}(u_1, \ldots, u_l) + C'_{L,l} \|u\|_{E_{M_0+l+L}} \|u_1\|_{E_{M_0}} \cdots \|u_l\|_{E_{M_0}}\end{aligned}$$

for each $L \ge 0, l \ge 0$ and each $u \in F(\mathcal{U}_\infty), u_1, \ldots, u_l \in E^{\circ\rho}_\infty$, where F^{-1} is defined by (6.170) and where $C_{L,l}$ and $C'_{L,l}$ are constants depending only on ρ and $\|u\|_{E^\rho_{M_0}}$.

Proof. Let Q be the orthogonal projection on $E^{\circ\rho}_0$ in E^ρ_0, let $N_0 \ge 1$, $\mathcal{O}_{N_0}$, $\mathcal{O}_\infty$ be as in Theorem 6.12, let the map F be as in Theorem 6.11, let $\mathcal{O}'_{N_0}$ be an open neighbourhood of zero in $E^\rho_{N_0}$ such that $\mathcal{O}'_{N_0} \cap E^{\circ\rho}_{N_0} = \mathcal{O}_{N_0}$ and let $\mathcal{O}'_\infty = \mathcal{O}'_{N_0} \cap E^\rho_\infty$.

$$G(u) = F(\Omega_1(Qu)) + (1-Q)u, u \in \mathcal{O}'_\infty, \tag{6.186}$$

defines a C^∞ function $G: \mathcal{O}'_\infty \to E^\rho_\infty$ according to Theorem 6.11 and Theorem 6.12 and since Q is a linear continuous mapping from E^ρ_N to $E^\rho_N, N \ge 0$. It follows from Theorem 6.11, Theorem 6.12 and Corollary 2.6 that

$$\begin{aligned}&\|(D^l G)(u; u_1, \ldots, u_l)\|_{E_L} \\ &\quad \le C_{l,L} \mathcal{R}^l_{N_0, l+L}(u_1, \ldots, u_l) + C'_{L,l} \|u\|_{E^\rho_{N_0+L+l}} \|u_1\|_{E^\rho_{N_0}} \cdots \|u_l\|_{E_{N_0}},\end{aligned} \tag{6.187}$$

for all $u \in \mathcal{O}'_\infty, u_1, \ldots, u_l \in E^\rho_\infty$, $L \ge 0$, $l \ge 0$, for some constants $C_{l,L}$ and $C'_{l,L}$ depending only on $\|u\|_{E^\rho_{N_0}}$. Since $\Omega_1(Qu) \in V^\rho_\infty$ for $u \in \mathcal{O}'_\infty$ according to Theorem 6.12 it follows from Theorem 6.11 that

$$\begin{aligned}QG(u) &= F(\Omega_1(Qu)), \\ (1-Q)G(u) &= (1-Q)u, \quad u \in \mathcal{O}'_\infty,\end{aligned} \tag{6.188a}$$

which shows that

$$\begin{aligned}QDG(u).v &= DF(\Omega_1(Qu)).(D\Omega_1(Qu).Qv), \\ (1-Q)DG(u).v &= (1-Q)v,\end{aligned} \tag{6.188b}$$

$u \in \mathcal{O}'_\infty, v \in E^\rho_\infty$.

We shall prove that $DG(u) \in L(E^\rho_\infty, E^\rho_\infty)$, $u \in \mathcal{O}'_\infty$ has a right inverse $w'(u) \in L(E^\rho_\infty, E^\rho_\infty)$, i.e. $DG(u).w'(u)v = v$ for $v \in E^\rho_\infty$ and $u \in \mathcal{O}'_\infty$. For $w(u) \in L(E^{\circ\rho}_\infty, E^{\circ\rho}_\infty)$, $u \in \mathcal{O}_\infty$ we define $w'(u) = w(Qu)Q + 1 - Q$ for $u \in \mathcal{O}'_\infty$. According to (6.188b) $w'(u)$ is then a right inverse of $DG(u)$, $u \in \mathcal{O}'_\infty$ if and only if

$$DF(\Omega_1(Qu)).(D\Omega_1(Qu).w(Qu)Qv) = Qv, \quad u \in \mathcal{O}'_\infty, v \in E^\rho_\infty,$$

i.e. if

$$DF(\Omega_1(u)).(D\Omega_1(u).w(u)v) = v, \quad u \in \mathcal{O}_\infty, v \in E^{\circ\rho}_\infty.$$

This equation and Theorem 6.11 give

$$D\Omega_1(u).w(u)v = DF^{-1}(F(\Omega_1(u))).v, \quad u \in \mathcal{O}_\infty, v \in E_\infty^{\circ\rho}, \tag{6.189}$$

where $D\Omega_1(u).w(u)v$ belongs to the tangent space of V_∞^ρ at $\Omega_1(u)$. Let $H(u;v) = DF^{-1}(F(\Omega_1(u))).v$. It follows from Theorem 6.11 that $H(u;v) \in E_\infty^\rho$ is an element of the tangent space of V_∞^ρ at the point $\Omega_1(u)$. Theorem 6.11, Theorem 6.12 and Corollary 2.6 give the following result, where D is differentiation only with respect to u:

$$\begin{aligned} &\|(D^l H)(u;v;u_1,\ldots,u_l)\|_{E_L} \\ &\quad\le C_{l,L}\big(\mathcal{R}^{l+1}_{N_0,L+l}(v,u_1,\ldots,u_l) + \|u\|_{E_{N_0+L+l}}\|v\|_{E_{N_0}}\|u_1\|_{E_{N_0}}\cdots\|u_l\|_{E_{N_0}},\big) \end{aligned} \tag{6.190}$$

for all $l \ge 0$, $L \ge 0$, where $C_{l,L}$ depends only on $\|u\|_{E_{N_0}}$ and where N_0, which we have chosen sufficiently large, is independent of $l, L, u, v, u_1, \ldots, u_l$.

Since $H(u;v)$ belongs to the tangent space of V_∞^ρ at the point $\Omega_1(u)$ we can take $H(u;v) = (a(0), \dot a(0), \Psi(0))$ as initial conditions for the derivative of the Maxwell-Dirac equations in the form (1.12):

$$\frac{d}{dt}(a(t),\dot a(t),\Psi(t)) = DT_{P_0}(A(t),\dot A(t),\psi(t)).(a(t),\dot a(t),\Psi(t)), \quad t \ge 0, \tag{6.191a}$$

where

$$\frac{d}{dt}(A(t),\dot A(t),\psi(t)) = T_{P_0}(A(t),\dot A(t),\psi(t)), \quad t \ge 0, \tag{6.191b}$$

and $(A(t),\dot A(t),\psi(t))$ is given by Theorem 6.10 with $Y = \mathbb{I}$ and $\psi(t) = e^{-i\vartheta(A,t)}\psi'(t)$ and where

$$(A(0),\dot A(0),\psi(0)) = \Omega_1(u), \quad u \in \mathcal{O}_\infty. \tag{6.191c}$$

The variable substitution $\psi'(t) = e^{i\vartheta(A,t)}(\psi(t)$ gives that the derivative $\Psi'(t)$ of $\psi'(t)$ satisfies

$$\Psi'(t) = e^{i\vartheta(A,t)}\Psi(t) + i\vartheta(a,t)\psi'(t), \quad t \ge 0. \tag{6.192}$$

It follows from (6.191a), (6.191b) and (6.192) that

$$\Box a_\mu = (\psi')^+\gamma^0\gamma_\mu\Psi' + (\Psi')^+\gamma^0\gamma_\mu\psi' \tag{6.193a}$$

and

$$(i\gamma^\mu\partial_\mu + m)\Psi' = (A_\mu + B_\mu)\gamma^\mu\Psi' + (a_\mu + b_\mu)\gamma^\mu\psi', \tag{6.193b}$$

where $B_\mu = -\partial_\mu\vartheta(A)$ and $b_\mu = -\vartheta(a)$. Moreover

$$\partial_\mu A^\mu = 0 \quad \text{and} \quad \partial_\mu a^\mu = 0. \tag{6.193c}$$

We shall solve the system (6.193a), (6.193b) and (6.193c) for initial conditions $(a(0),\dot a(0),\Psi'(0))$ at $t=0$, defined by $H(u;v) = (a(0),\dot a(0),\Psi(0))$ and formula (6.192).

Let

$$a_Y = \xi^M_Y a, \quad \dot a_Y = \xi^M_{P_0Y} a, \quad \Psi'_Y = \xi^D_Y\Psi', \quad \Psi_Y = \xi^D_Y\Psi \tag{6.194a}$$

and

$$\vartheta_Y(a) = \xi_Y \vartheta(a), \quad b_Y = \xi_Y^M b, \quad Y \in U(\mathfrak{p}). \tag{6.194b}$$

Since the initial condition $H(u;v)$, where $u \in \mathcal{O}_\infty$, $v \in E_\infty^{\circ\rho}$ for equation (6.193a) and (6.193b) is a function of u and v, this is also the case for $a_Y, \dot{a}_Y$, Ψ_Y and Ψ'_Y. We introduce the notation

$$a_Y^{(l)} \quad (\text{resp.}\dot{a}_Y^{(l)}, \Psi_Y^{(l)}, \Psi_Y^{\prime(l)}), \quad l \geq 0, \tag{6.195}$$

for the l^{th} derivative of a_Y (resp. $\dot{a}_Y, \Psi_Y, \Psi'_Y$) with respect to u, in the directions $u_1, \ldots, u_l \in E_\infty^{\circ\rho}$.

It follows from (6.192) and from Leibniz rule that

$$\begin{aligned} &\Psi_Y^{\prime(l)}(0) \\ &\quad = \sum_{l_1+l_2=l} C_{l_1,l_2} \sum_{Y_1,Y_2}^{Y} ((\xi_{Y_1} e^{i\vartheta(A)})^{(l_1)}(0)\Psi_{Y_2}^{(l_2)} + i\vartheta_{Y_1}(a,0)^{(l)} \psi_{Y_2}^{\prime(l)}(0)), \quad l \geq 0, Y = \Pi', \end{aligned} \tag{6.196}$$

where $f^{(l)}$ is defined as in (6.195). Since $\Psi_Y(0)$ is the derivative of $T_Y^D(A(0), \dot{A}(0), \psi(0))$ in the direction $(a(0), \dot{a}(0), \psi(0))$, where $Y \in U(\mathfrak{p})$ and T_Y^D is the Dirac component of T_Y, it follows from i) of Lemma 2.19, Corollary 2.6 and from inequality (6.190) that

$$\begin{aligned} &\|\Psi_Y^{(l)}(0)\|_{D_L} \\ &\quad \leq C_{l,|Y|,L} \mathcal{R}_{N_0,L+l+|Y|}^{l+1}(v, u_1, \ldots, u_l) + C'_{l,|Y|,L} \|u\|_{E^\rho_{N_0+L+l+|Y|}} \|v\|_{E^\rho_{N_0}} \|u_1\|_{E^\rho_{N_0}} \cdots \|u_l\|_{E^\rho_{N_0}}, \end{aligned} \tag{6.197}$$

for $Y \in \Pi'$, $L \geq 0$, $l \geq 0$, $u \in \mathcal{O}_\infty$, $v, u_1, \ldots, u_l \in E_\infty^{\circ\rho}$. Here $C_{l,|Y|,L}$ and $C'_{l,|Y|,L}$ are constants depending only on $\|u\|_{E^\rho_{N_0}}$ and N_0 is an integer independent of L, Y, l, u, $v, u_1, \ldots, u_l$. It now follows by Corollary 2.6 and as in the proof of (6.175) that

$$\begin{aligned} &\|(\xi_{Y_1} e^{i\vartheta(A)})^{(l_1)}(0)\Psi_{Y_2}^{(l_2)}(0)\|_{D_L} \\ &\quad \leq C_{l,|Y|,L} \mathcal{R}_{N_0,L+l+|Y|}^{l+1}(v, u_1, \ldots, u_l) + C'_{l,|Y|,L} \|u\|_{E^\rho_{N_0+L+l+|Y|}} \|v\|_{E^\rho_{N_0}} \|u_1\|_{E^\rho_{N_0}} \cdots \|u_l\|_{E^\rho_{N_0}}, \end{aligned} \tag{6.198}$$

for $Y_1, Y_2 \in \Pi', |Y_1| + |Y_2| \leq |Y|, l \geq 0, l_1 + l_2 = l, L \geq 0$, and some constants $C_{l,|Y|,L}$ and $C'_{l,|Y|,L}$ and integer N_0 having the same properties as those in (6.197). Similarly it is proved that the second term in the sum in (6.196) also satisfy estimate (6.198). This gives together with (6.190) that

$$\begin{aligned} &\|(a_Y^{(l)}(0), \dot{a}_Y^{(l)}(0), \Psi_Y^{\prime(l)}(0))\|_{E^\rho_L} \\ &\quad \leq C_{l,|Y|,L} \mathcal{R}_{N_0,L+l+|Y|}^{l+1}(v, u_1, \ldots, u_l) + C'_{l,|Y|,L} \|u\|_{E^\rho_{N_0+L+l+|Y|}} \|v\|_{E^\rho_{N_0}} \|u_1\|_{E^\rho_{N_0}} \cdots \|u_l\|_{E^\rho_{N_0}}, \end{aligned} \tag{6.199}$$

for $Y \in \Pi', L \geq 0, l \geq 0, u \in \mathcal{O}_\infty, v, u_1, \ldots, u_l \in E_\infty^{\circ\rho}$. Here $C_{l,|Y|,L}$ and $C'_{l,|Y|,L}$ are constants depending only on $\|u\|_{E^\rho_{N_0}}$ and N_0 is an integer independent of $L, Y, l, u, v, u_1, \ldots, u_l$.

Next we shall derive estimates for equations (6.193a), (6.193b) and (6.193c). Application of $\xi_Y^M, Y \in \Pi'$, to equation (6.193a) gives for $1/2 < \rho' \leq 1$:

$$\|(a_Y(t), \dot a_Y(t))\|_{M_0^{\rho'}} \tag{6.200}$$
$$\leq \|(a_Y(0), \dot a_Y(0))\|_{M_0^{\rho'}} + \sum_{0\leq\mu\leq 3} \int_0^t \||\nabla|^{\rho-1}\big(\xi_Y^M((\psi')^+\gamma^0\gamma_\mu \Psi' + (\Psi')^+\gamma^0\gamma_\mu\psi')\big)(s)\|_{L^2} ds.$$

Since $\||\nabla|^{\rho'-1} f\|_{L^2} \leq C_p \|f\|_{L^p}$, $p = 6(5-2\rho')^{-1}$, and since

$$\|fg\|_{L^p} \leq \|f\|_{L^\infty}^{(1+2\rho')/3} \|f\|_{L^2}^{(2-2\rho')/3} \|g\|_{L^2},$$

we obtain:

$$\||\nabla|^{\rho'-1}\big(\xi_Y^M\big((\psi')^+\gamma^0\gamma_\mu\Psi' + \Psi'^+\gamma^0\gamma_\mu\psi'\big)\big)(s)\|_{L^2}$$
$$\leq C_{|Y|} \sum_{\substack{|Y_1|+|Y_2|\leq|Y| \\ |Y_2|<|Y|}} \|\psi'_{Y_1}(s)\|_{L^\infty}^{(1+2\rho')/3} \|\psi'_{Y_1}(s)\|_{L^2}^{(2-2\rho')/3} \|\Psi'_{Y_2}(s)\|_{L^2}$$
$$+ C_0 \|\psi'_{\mathbb{I}}(s)\|_{L^\infty}^{(1+2\rho')/3} \|\psi'_{\mathbb{I}}(s)\|_{L^2}^{(2-2\rho')/3} \|\Psi'_Y(s)\|_{L^2}.$$

This inequality, inequality (6.200) and Theorem 6.10, give

$$\|(a_Y(t), \dot a_Y(t))\|_{M_0^{\rho'}} \tag{6.201}$$
$$\leq \|(a_Y(0), \dot a_Y(0))\|_{M_0^{\rho'}}$$
$$+ \sum_{\substack{L+|Y_2|\leq|Y| \\ |Y_2|<|Y|}} F_L(\|u\|_{E^\rho_{N_0}}) \|u\|_{E^\rho_{N_0+L}} \int_0^t (1+s)^{-(1+2\rho')/2} \|\Psi'_{Y_2}(s)\|_{L^2} ds$$
$$+ F_0(\|u\|_{E^\rho_{N_0}}) \|u\|_{E^\rho_{N_0}} \int_0^t (1+s)^{-(1+2\rho')/2} \|\Psi'_Y(s)\|_{L^2} ds,$$

where $1/2 < \rho' \leq 1$ and $Y \in \Pi'$, $1/2 < \rho < 1$. It follows from (6.201) that

$$\wp_n^{M^{\rho'}}(a(t), \dot a(t)) \tag{6.202}$$
$$\leq \wp_n^{M^{\rho'}}(a(0), \dot a(0)) + C_n \sum_{0\leq i\leq n-1} \|u\|_{E^\rho_{N_0+n-i}} \int_0^t (1+s)^{-(1+2\rho')/2} \wp_i^D(\Psi'(s)) ds$$
$$+ C_0 \|u\|_{E^\rho_{N_0}} \int_0^t (1+s)^{-(1+2\rho')/2} \wp_n^D(\Psi'(s)) ds, \quad n \geq 0, t \geq 0,$$

where C_n, $n \geq 0$, are constants depending only on $\|u\|_{E^\rho_{N_0}}$. Statement i) of Corollary 5.18, with $G_\mu = A_\mu + B_\mu$, $t_0 = 0$, $\varepsilon = 0$, $\eta = 0$, $\rho' = \rho$, and equation (6.193b) give for $n \geq 0$, $1/2 < \rho < 1$, $t \geq 0$:

$$\wp_n^D(\Psi'(t)) \tag{6.203a}$$
$$\leq C_n\Big(\wp_n^D(\Psi'(0)) + \sum_{0\leq i\leq n-1} \|u\|_{E^\rho_{N_0+n-i}} \wp_i^D(\Psi'(0)) + \sum_{0\leq i\leq n} \|u\|_{E_{N_0+n-i}} \tau_i(t)\Big),$$

where

$$\begin{aligned}\tau_i(t) = {} & \wp_i^{M^\rho}(a(0),\dot a(0)) + \wp_i^{M^1}(a(0),\dot a(0)) \\ & + \int_0^t (1+s)^{\rho-2}\Big(\sup_{0\le s'\le s} \big(\wp_i^{M^\rho}(a(s'),\dot a(s')) + \wp_i^{M^1}(a(s'),\dot a(s'))\big) \\ & + \sum_{\substack{Y\in\Pi' \\ |Y|\le i}} \sup_{0\le s'\le s} \big((1+s')^{3/2-\rho}\|\Box a_Y(s')\|_{L^2(\mathbb{R}^3,\mathbb{R}^4)}\big)\Big) ds\end{aligned} \tag{6.203b}$$

and where C_n, $n\ge 0$, depends only on $\|u\|_{E^\rho_{N_0}}$. Estimating $\|(1+t+|\cdot|)\Box a_Y(t)\|_{L^2(\mathbb{R}^3,\mathbb{R}^4)}$, by using equation (6.193a) and Theorem 6.10, it follows from (6.202), (6.203a) and (6.203b), using Corollary 2.6 and denoting

$$e_n(t) = \sup_{0\le s\le t}\big(\wp_n^{E^\rho}(a(s),\dot a(s),\Psi'(s))\big) + \sup_{0\le s\le t}\big(\wp_n^{E^1}(a(s),\dot a(s),\Psi'(s))\big),$$

that:

$$\begin{aligned} e_n(t) \le {} & C_n\Big(e_n(0) + \sum_{0\le i\le n-1}\|u\|_{E^\rho_{N_0+n-i}} e_i(0)\Big) \\ & + C_n \sum_{0\le i\le n-1}\|u\|_{E^\rho_{N_0+n-i}} \int_0^t (1+s)^{-\varepsilon} e_i(s)ds \\ & + C_n\|u\|_{E^\rho_{N_0}}\int_0^t (1+s)^{-\varepsilon} e_n(s)ds, \quad n\ge 0, t\ge 0,\end{aligned} \tag{6.204}$$

where $\varepsilon = \min(2-\rho, Y_2+\rho) > 1$ and where C_n depends only on $\|u\|_{E^\rho_{N_0}}$. It follows by Grönvall inequality that

$$\begin{aligned} e_n(t) \le {} & C'_n\Big(e_n(0) + \sum_{0\le i\le n-1}\|u\|_{E^\rho_{N_0+n-i}} e_i(0) \\ & + \sum_{0\le i\le n-1}\|u\|_{E^\rho_{N_0+n-i}} \int_0^t (1+s)^{-\varepsilon} e_i(s)ds\Big) \quad, t\ge 0, n\ge 0,\end{aligned} \tag{6.205}$$

where C'_n, $n\ge 0$, are constants depending only on $\|u\|_{E^\rho_{N_0}}$. We obtain from (6.205) by induction and using Corollary 2.6 that

$$e_n(t) \le C_n\Big(e_n(0) + \sum_{0\le i\le n-1}\|u\|_{E^\rho_{N_0+n-i}} e_i(0)\Big), \quad t\ge 0, n\ge 0, \tag{6.206}$$

for some constants depending only on $\|u\|_{E^\rho_{N_0}}$ and for some integer N_0 independent of n, t, u. Combining (6.149) and (6.206) we obtain using Corollary 2.6 that the solution of (6.193a), (6.193b) and (6.193c) with initial conditions $(a(0),\dot a(0),\Psi'(0))$ sastify

$$\wp_n^{E^\rho}(a(t),\dot a(t),\Psi'(t)) \le C_n\|v\|_{E^\rho_{N_0+n}} + C'_n\|u\|_{E^\rho_{N_0+n}}\|u\|_{E^\rho_{N_0}}, \quad t\ge 0, n\ge 0, \tag{6.207}$$

where C_n, C'_n, are constants depending only on $\|u\|_{E^\rho_{N_0}}$ and where N_0 is an integer independent of n, t, u.

We can now use directly Corollary 5.2, with $w(t,s) = e^{\mathcal{D}(t-s)}$, on the different terms on the right-hand side of the equation

$$(i\gamma^\mu \partial_\mu + m)\Psi'_Y = \xi^D_Y\big((A_\mu + B_\mu)\gamma^\mu \Psi' + (a_\mu + b_\mu)\gamma^\mu \psi'\big), \quad Y \in \Pi', \tag{6.208}$$

which proves that there exists $\beta' \in D_\infty$ such that

$$\lim_{t\to\infty} \|\Psi'_Y(t) - e^{\mathcal{D}t} T^{D1}_Y(\beta')\|_D = 0. \tag{6.209}$$

Since the integral in (6.200) converges as $t \to \infty$, it follows using (6.209) that there exists $(g, \dot g) \in M^\rho_\infty$ such that

$$\lim_{t\to\infty} \|(a(t), \dot a(t), \Psi'(t)) - U^1_{\exp(tP_0)} T^1_Y(v')\|_{M^\rho_0} = 0, \tag{6.210}$$

where $v' = (g, \dot g, \beta) \in E^\rho_\infty$. It follows from (6.207) and (6.210) that

$$\|v'\|_{E^\rho_{N_0}} \le C_n \|v\|_{E^\rho_{N_0+n}} + C'_n \|u\|_{E^\rho_{N_0+n}} \|v\|_{E^\rho_{N_0}}, \quad n \ge 0, \tag{6.211}$$

where C_n and C'_n are constants depending only on $\|u\|_{E^\rho_{N_0}}$ and where N_0 is an integer independent of n and u.

We now define $w(u)v = v'$, $u \in \mathcal{O}_\infty$, which proves that equation (6.189) has a solution $w(u)$ with the property (6.209). Differentiation of equations (6.193a), (6.193b) and (6.193c) with respect to u in the direction $u_1, \ldots, u_l \in E^{\circ\rho}_\infty$ and induction give in the same way as (6.211) was obtained that

$$\begin{aligned}
&\|(D^l(w(u)v))(u_1, \ldots, u_l)\|_{E^\rho_n} \\
&\quad \le C_{l,n} \mathcal{R}^{l+1}_{N_0, l+n}(v, u_1, \ldots, u_l) + C'_{l,n} \|u\|_{E^\rho_{N_0+l+n}} \|v\|_{E^\rho_{N_0}} \|u_1\|_{E^\rho_{N_0}} \cdots \|u_l\|_{E^\rho_{N_0}},
\end{aligned} \tag{6.212}$$

for $l \ge 0$, $n \ge 0$, $u \in \mathcal{O}_\infty$, v, $u_1, \ldots, u_l \in E^{\circ\rho}_\infty$. Here $C_{l,n}$ and $C'_{l,n}$ are constants depending only on $\|u\|_{E^\rho_{N_0}}$ and N_0 is an integer independent of l, n, u, v, $u_1, \ldots, u_l$. By construction of $w'(u)$, $u \in \mathcal{O}'_\infty$, it now follows that $FG(u).w'(u)v = v$ for $v \in E^\rho_\infty$ and it follows from (6.212) that

$$\begin{aligned}
&\|(D^l(w'(u)v))(u_1, \ldots, u_l)\|_{E^\rho_n} \\
&\quad \le C_{l,n} \mathcal{R}^{l+1}_{N_0, l+n}(v, u_1, \ldots, u_l) + C'_{l,n} \|u\|_{E^\rho_{N_0+l+n}} \|v\|_{E^\rho_{N_0}} \|u_1\|_{E^\rho_{N_0}} \cdots \|u_l\|_{E^\rho_{N_0}},
\end{aligned} \tag{6.213}$$

for $l \ge 0$, $n \ge 0$, $u \in \mathcal{O}'_\infty$, v, $u_1, \ldots, u_l \in E^\rho_\infty$. Here $C_{l,n}$, $C'_{l,n}$ and N_0 have the same properties as in (6.212).

Theorem 2.5, property (6.187) of the map $G\colon \mathcal{O}'_\infty \to E^\rho_\infty$ and the existence of a right inverse $w'(u) \in L(E^\rho_\infty, E^\rho_\infty)$ of $DG(u) \in L(E^\rho_\infty, E^\rho_\infty)$ with property (6.123) prove, according to the implicit function theorem for Fréchet spaces (Theorem 4.1.1 of [17]), that

there exists a positive integer M_0, an open neighbourhood $\mathcal{U}'_{M_0}$ of zero in $E^\rho_{M_0}$ and a C^∞ map $H: \mathcal{U}'_\infty = \mathcal{U}'_{M_0} \cap E^\rho_\infty \to \mathcal{O}'_\infty$ such that $G(H(u)) = u$ for $u \in \mathcal{U}'_\infty$. Let $\mathcal{Q}'_\infty = G^{-1}[\mathcal{U}'_\infty]$ be the inverse image of $\mathcal{U}'_\infty$ by G. Since G is continuous and $\mathcal{U}'_\infty$ is an open neignbourhood of zero in E^ρ_∞ and since $G(0) = 0$, it follows that $\mathcal{Q}'_\infty$ is an open neignbourhood of zero in E^ρ_∞. The map $G: \mathcal{Q}'_\infty \to \mathcal{U}'_\infty$ is onto because $H(\mathcal{U}'_\infty) \subset \mathcal{Q}'_\infty$ and $G(H(u)) = u$ for $u \in \mathcal{U}'_\infty$, and according to Theorem 6.11 and Theorem 6.12 G is also one-to-one. Since $G: \mathcal{Q}'_\infty \to \mathcal{U}'_\infty$ is a bijection and $G(H(u)) = u$ for $u \in \mathcal{U}'_\infty$, it follows that $H: \mathcal{U}'_\infty \to \mathcal{O}'_\infty$ is a bijection. This proves that the C^∞ function $G: \mathcal{Q}'_\infty \to \mathcal{U}'_\infty$ has a C^∞ inverse $G^{-1}: \mathcal{U}'_\infty \to \mathcal{Q}'_\infty$.

Similarly, as we obtained (6.213), we now obtain that

$$\begin{aligned}&\|(D^l G^{-1})(u; u_1, \ldots, u_l)\|_{E^\rho_N} \\ &\quad \leq C_{l,n} \mathcal{R}^l_{M_0, l+n}(u_1, \ldots, u_l) + C'_{l,n} \|u\|_{E^\rho_{M_0+l+n}} \|u_1\|_{E^\rho_{M_0}} \cdots \|u_l\|_{E^\rho_{M_0}},\end{aligned} \tag{6.214}$$

for $l \geq 0$, $n \geq 0$, $u \in \mathcal{U}'_\infty$, $u_1, \ldots, u_l \in E^\rho_\infty$. Here $C_{l,n}$ and $C'_{l,n}$ are constants depending only on $\|u\|_{E^\rho_{M_0}}$ and M_0 is an integer independent of l, n, u, $u_1, \ldots, u_l$.

Let us define $\mathcal{U}_\infty = F^{-1}(\mathcal{U}'_\infty \cap E^{\circ\rho}_\infty)$ and let us redefine $\mathcal{O}_\infty$ by $\mathcal{O}_\infty = \mathcal{Q}'_\infty \cap E^{\circ\rho}_\infty$. According to definition (6.186) of G, Theorem 6.11 and the fact that $G: \mathcal{Q}'_\infty \to \mathcal{U}'_\infty$ is a diffeomorphism, it follows that $\Omega_1: \mathcal{O}_\infty \to \mathcal{U}_\infty$ is a diffeomorphism, which satisfies the inequality of the theorem since G^{-1} satisfies inequality (6.214). This proves the theorem.

The construction of a modified wave operator $\Omega_1: \mathcal{O}_\infty \to \mathcal{U}_\infty$, for $t \to \infty$, being a diffeomorphism according to Theorem 6.13, could of course as well has been done for $t \to -\infty$. To distinguish between the *two modified wave operators* so constructed, we use the notation $\Omega_1^{(\varepsilon)}: \mathcal{O}_{1,\infty(\varepsilon)} \to \mathcal{U}_{\infty(\varepsilon)}$ for the wave operator and $A_{(\varepsilon)}(u), \psi'_{(\varepsilon)}(u)$ for the functions given by definition (6.165), with $u \in \mathcal{O}_{1,\infty(\varepsilon)}$, for the case $t \to \varepsilon\infty$, $\varepsilon = \pm$. $\mathcal{O}_{1,\infty(\varepsilon)}$ and $\mathcal{U}_{\infty(\varepsilon)}$ are given by Theorem 6.13, with N_ε and M_ε instead of N_0 and M_0 respectively. By definition we then have

$$\Omega_1^{(\varepsilon)}(u) = \big((A_{(\varepsilon)}(u))(0), (\dot{A}_{(\varepsilon)}(u))(0), e^{-i\vartheta(A_{(\varepsilon)}(u),0)}(\psi'_{(\varepsilon)}(u))(0)\big), \quad \varepsilon = \pm, \tag{6.215}$$

which should be compared with (6.172). Theorem 6.10 is then true, after the obvious modification that $(A_{0,Y}(t), \dot{A}_{0,Y}(t), \phi'_{0,Y}(t))$ is replaced by $U^1_{\exp(tP_0)} T^1_Y(u)$, $(A_Y, \dot{A}_Y, \psi'_Y)$ is replaced by $(A_{(\varepsilon)Y}, \dot{A}_{(\varepsilon)Y}, \psi'_{(\varepsilon)Y})$. N_0 is replaced by N_ε and $t \geq 0$ is replaced by $\varepsilon t \geq 0$. This gives the following corollary:

Corollary 6.14. *The function $u \longmapsto (A_{(\varepsilon)}(u), \dot{A}_\varepsilon(u), \psi'_{(\varepsilon)}(u))$ satisfies the conclusions of Theorem 6.10 for $t \to \varepsilon\infty$ and $\Omega_1^{(\varepsilon)}: \mathcal{O}_{1,\infty(\varepsilon)} \to \mathcal{U}_{\infty(\varepsilon)}$ satisfies the conclusions of Theorem 6.13, where $\varepsilon = \pm$.*

Corollary 6.14 permits to *solve the Cauchy problem for the Maxwell-Dirac equations* (1.1.a)–(1.1.c) for times $t \in \mathbb{R}$ and intial conditions v belonging to the open neighbourhood $\mathcal{U}_{\infty(0)} = \mathcal{U}_{\infty(+)} \cap \mathcal{U}_{\infty(-)}$ of zero in V^ρ_∞. We note that it follows using Corollary 6.14 and the notation proceeding it, that $\mathcal{U}_{\infty(0)} = \mathcal{U}_{M_0} \cap V^\rho_\infty$, where $M_0 = \max(M_+, M_-)$ and $\mathcal{U}_{M_0} = \mathcal{U}_{M_+} \cap \mathcal{U}_{M_-}$ is a neighbourhood of zero in $V^\rho_{M_0}$. Let $\Omega_1^{(\varepsilon)}: \mathcal{O}_{1,\infty(\varepsilon)} \to \mathcal{U}_{\infty(0)}$, $\varepsilon = \pm$, be the

diffeomorphism, which is the restriction of the former $\Omega_1^{(\varepsilon)}$ to $\mathcal{O}_{1,\infty(\varepsilon)} = \big(\Omega_1^{(\varepsilon)}\big)^{-1}[\mathcal{U}_{\infty(0)}]$. We recall that for differentiation of functions defined on V_∞^ρ, that V_∞^ρ is diffeomorphic to $E_\infty^{\circ\rho}$, according to Theorem 6.11.

Theorem 6.15. *Let $1/2 < \rho < 1$. If $v = (f, \dot f, \alpha) \in \mathcal{U}_{\infty(0)}$, then there exists a unique solution $h(v) = (A, \dot A, \psi) \in C^0(\mathbb{R}, (1-\Delta)^{-1/4}E_0^1) \cap C^1(\mathbb{R}, (1-\Delta)^{1/4}E_0^1)$ of the M-D equations (1.1a)–(1.1c), with initial data v at $t = 0$. Moreover $h \in C^\infty(\mathcal{U}_{\infty(0)}, C_b^\infty(\mathbb{R}, V_\infty^\rho))$, where b stands for the topology of convergence on bounded subsets of $\mathbb{R}$, the conclusion of Theorem 6.10, with $(A_{0,Y}(t), \dot A_{0,Y}(t), \phi'_{0,Y}(t))$ replaced by $U^1_{\exp(tP_0)}T^1_Y(u_\varepsilon)$, u replaced by $u_\varepsilon = \big(\Omega_1^{(\varepsilon)}\big)^{-1}(v)$, N_0 replaced by N_ε and $t \geq 0$ replaced by $\varepsilon t \geq 0$, is true for $\varepsilon = \pm 1$ and if $h^{(l)}(t)$ is the l^{th} derivative of the function $v \mapsto (h(v))(t)$ at $v \in \mathcal{U}_{\infty(0)}$ in the directions of the elements $v_1, \cdots, v_l$ of the tangent space of V_∞^ρ at v, then there exists $N \geq 0$ such that*

$$\|h^{(l)}(t)\|_{E_n^\rho} \leq C_{n+l,t}\big(\mathcal{R}^l_{N,n+l}(v_1, \cdots, v_l) + \|v\|_{E^\rho_{N+n+l}}\|v_1\|_{E_N^\rho}\cdots\|v_l\|_{E_N^\rho}\big),$$

for $t \in \mathbb{R}$, n, $l \in \mathbb{N}$, where $C_{n+l,t}$ depends only on ρ and $\|v\|_{E_N^\rho}$.

Proof. Let $g_Y^{(l)}$ be the l^{th} derivative of the function $v \mapsto g_Y(v) = (A_Y, A_{P_0Y}, \xi_Y^D(e^{i\vartheta(A)}\psi))$, $Y \in \Pi'$. It then follows by definition (6.215) of $\Omega_{1,(\varepsilon)}$ and by applying Theorem 6.10 and Corollary 6.14 with $u_\varepsilon = \Omega^{-1}_{1(\varepsilon)}(v)$ that $g_Y^{(l)} \in C^0(\mathbb{R}, E_0^\rho)$, for $Y \in \Pi'$ and $l \geq 0$ and that $g_{\mathbb{I}}(v)$ is a solution of equations (6.167a)–(6.176c).

We prove that $h_Y^{(0)} = T_Y(h(v)) \in C^0(\mathbb{R}, E_0^\rho)$ for $Y \in \Pi'$. Let $V_Y = (A_Y, A_{P_0Y}, \psi_Y)$, where $\psi_Y = \xi_Y^D\psi$, $g_Y(v) = (A_Y, A_{P_0Y}, \psi'_Y)$, $\psi'_Y = \xi_Y^D\psi'$, $\psi = e^{-i\vartheta(A)}\psi'$. It then follows that

$$\begin{aligned}&\wp_n^{E^\rho}(V(t)) &(6.216)\\ &\quad\leq \wp_n^{E^\rho}\big((g(v))(t)\big) + C_n \sum \|\vartheta_{Y_1}(A,t)\cdots\vartheta_{Y_l}(A,t)\psi'_Z(t)\|_D, \quad n \geq 0, t \in \mathbb{R},\end{aligned}$$

where $\vartheta_{Y_i}(A,t) = (\xi_{Y_i}\vartheta(A))(t)$, C_n is a numerical constant and the sum is taken over $1 \leq l \leq n$, $Y_i \in \Pi'$, $Z \in \Pi'$, $|Y_1| + \cdots + |Y_l| + |Z| \leq n$, $|Y_i| \geq 1$. It follows from Lemma 4.4 and Theorem 6.10 that

$$\|(\delta(t))^{1/2-\rho}\vartheta_{Y_i}(A,t)\|_{L^\infty} \leq C_{|Y_i|}\|u_\varepsilon\|_{E^\rho_{N_0+|Y_i|}}, \tag{6.217}$$

where $u_\varepsilon = \Omega^{-1}_{1(\varepsilon)}(v)$ and where $C_{|Y_i|}$ depends only on ρ and $\|u_\varepsilon\|_{E^\rho_{N_0}}$. Inequalities (6.216) and (6.217) give, similarly as in the proof of Theorem 6.12 (see (6.174)–(6.175)) that

$$\wp_n^{E^\rho}(V(t)) \leq \wp_n^{E^\rho}((g(v))(t)) + C_n \sum_{\substack{1\leq l\leq n\\ j\leq n-l}} \|u_\varepsilon\|_{E^\rho_{N_0+1+n-|Z|-l}}\wp_j^D\big((\delta(t))^{l(\rho-1/2)}\psi'(t)\big), \tag{6.218}$$

where C_n depends only on ρ and $\|u_\varepsilon\|_{E^\rho_{N_0+1}}$. Since $(\delta(t))(x) \leq C(1 + (\lambda_1(t))(x) + |t|)$, where C is independent of t and x, and since $0 < \rho - 1/2 < 1/2$, it follows from Theorem 6.10 that

$$\wp_j^D\big((\delta(t))^{l(\rho-1/2)}\psi'(t)\big) \leq (1+t)^{l(\rho-1/2)}C_{j+l}\|u_\varepsilon\|_{E^\rho_{N_0+j+l}}, \quad j, l \geq 0,$$

where C_{j+l} depends only on ρ and $\|u_\varepsilon\|_{E^\rho_{N_0}}$. This inequality and inequality (6.218) give that

$$\wp_n^{E^\rho}(V(t)) \leq \wp_n^{E^\rho}((g(v))(t)) + (1+t)^{n(\rho-1/2)} C_n \|u_\varepsilon\|_{E^\rho_{N_0+1}} \|u_\varepsilon\|_{E^\rho_{N_0+1+n}}, \quad n \geq 0,$$

where C_n depends only on ρ and $\|u_\varepsilon\|_{E^\rho_{N_0+1}}$. Estimating the first term on the right-hand side of this inequality by Theorem 6.10 then gives that

$$\wp_n^{E^\rho}(V(t)) \leq C_n (1+t)^{n(\rho-1/2)} \|u_\varepsilon\|_{E^\rho_{N_0+1+n}}, \quad n \geq 0, \tag{6.219}$$

where C_n depends only on ρ and $\|u\|_{E^\rho_{N_0+1}}$. Choosing N sufficiently large, it follows from Theorem 6.13 that

$$\wp_n^{E^\rho}(V(t)) \leq C_n (1+t)^{n(\rho-1/2)} \|v\|_{E^\rho_{N+n}}, \quad n \geq 0, t \in \mathbb{R}, \tag{6.220}$$

where C_n depends only on $\|v\|_{E^\rho_N}$ and ρ. Let $Y(t) = \exp(t\mathrm{ad}_{P_0})Y$, as in definition (1.11). Then $V_Y(t) = T_{Y(t)}((h(v))(t))$, which together with inequalities (6.220) proves that

$$\begin{aligned} \|T_Y((h(v))(t))\|_D &\leq C_{|Y|}(1+t)^{|Y|} \|T_{Y(t)}((h(v))(t))\|_D \\ &\leq C'_{|Y|}(1+t)^{|Y|(\rho+1/2)} \|v\|_{E^\rho_{N+|Y|}}, \quad Y \in \Pi', t \in \mathbb{R}, \end{aligned} \tag{6.221}$$

where $C'_{|Y|}$ depends only on ρ and $\|u\|_{E^\rho_N}$. This proves that

$$\wp_n^{E^\rho}(h^{(0)}(t)) \leq C_n (1+t)^{n(\rho+1/2)} \|v\|_{E^\rho_{N+n}}, \quad n \geq 0, t \in \mathbb{R}, \tag{6.222}$$

where C_n depends only on ρ and $\|v\|_{E^\rho_N}$.

According to inequality (6.222), $\|h^{(0)}_{\mathbb{I}}(t)\|_{E^\rho_0} \leq C_0 \|v\|_{E^\rho_N}$, which shows that we can choose $\mathcal{U}_{\infty(0)}$ such that the hypothesis of Theorem 2.22 is verified. Statements ii) and iii) of Theorem 2.22 give that

$$\|h^{(0)}_{\mathbb{I}}(t)\|_{E^\rho_1} \leq C_1 (1+t)^{\rho+1/2} \|v\|_{E^\rho_{N+1}}, \quad t \in \mathbb{R} \tag{6.223a}$$

and then that

$$\|h^{(0)}_{\mathbb{I}}(t)\|_{E^\rho_n} \leq C_{n,t} \|v\|_{E^\rho_{N+n}}, \quad n \geq 1, t \in \mathbb{R}, \tag{6.223b}$$

where C_1 and $C_{n,t}$ depend only on $\|v\|_{E^\rho_{N+1}}$. It follows from inequality (6.223a) and (6.223b) and from statement i) of Theorem 2.22 that

$$\|h^{(0)}_Y(t)\|_{E^\rho_n} \leq C_{n,|Y|,t} \|v\|_{E^\rho_{N+n+|Y|}}, \quad n \in \mathbb{N}, Y \in \Pi', t \in \mathbb{R}, \tag{6.224}$$

where $C_{n,|Y|,t}$ depends only on $\|v\|_{E^\rho_{N+1}}$. This proves that $h^{(0)}_Y \in C^0(\mathbb{R}, E^\rho_\infty)$ for $Y \in \Pi'$.

Since $g_{\mathbb{I}}(v)$ is a solution of equations (6.167a)–(6.167c) it follows that $h^{(0)}_{\mathbb{I}}$ is a solution of the M-D equations, i.e. $(d/dt)h^{(0)}_{\mathbb{I}}(t) = T_{P_0}(h^{(0)}_{\mathbb{I}}(t))$. Equations (1.10) then gives that $(d/dt)^k h^{(0)}_{\mathbb{I}}(t) = T_{P_0^{k+1}}(h^{(0)}_{\mathbb{I}}(t)) = h^{(0)}_{P_0^{k+1}}(t)$, which proves that $h^{(0)}_{\mathbb{I}} = h(v) \in C^\infty(\mathbb{R}, V^\rho_\infty)$. This proves the theorem for the case of $l = 0$. Since the case $l \geq 1$ is so similar, we omit it.

According to Theorem 6.15 and Theorem 6.10 the solution $h(v)$, $v \in \mathcal{U}_{\infty(0)}$, satisfies the asymptotic conditions

$$\|(A(t), \dot{A}(t)) - U^{M1}_{\exp(tP_0)}(f_{1(\varepsilon)}, \dot{f}_{1(\varepsilon)})\|_{M^\rho_0} + \|\psi(t) - e^{-i\vartheta(A,t)} U^{D1}_{\exp(tP_0)} \alpha_{1(\varepsilon)}\|_{D_0} \to 0, \quad (6.225)$$

when $\varepsilon t \to \infty$, $\varepsilon = \pm$, where $u_{1(\varepsilon)} = (f_{1(\varepsilon)}, \dot{f}_{1(\varepsilon)}, \alpha_{1(\varepsilon)}) = (\Omega^{(\varepsilon)}_1)^{-1}(v)$.

To obtain *asymptotic representations of the Poincaré group,* which do not require the integration of the M-D equations for their construction, we shall replace (6.225) by

$$\|(A(t), \dot{A}(t)) - U^{M1}_{\exp(tP_0)}(f_\varepsilon, \dot{f}_\varepsilon)\|_{M^\rho_0} + \|\psi(t) - e^{-i\vartheta(A^{(\varepsilon)},t)} U^{D1}_{\exp(tP_0)} \alpha_\varepsilon\|_{D_0} \to 0, \quad (6.226)$$

when $\varepsilon t \to \infty$, $\varepsilon = \pm$, where $A^{(+)}$ is given by (1.22a), with $(f_+, \dot{f}_+, \alpha_+)$ instead of $(f, \dot{f}, \alpha)$, where $f_+ = f_{1(+)}$, $\dot{f}_+ = \dot{f}_{1(+)}$ and where we shall determine α_+. Similar relations are valid for the case $\varepsilon = -$, and we shall only state the following results for $\varepsilon = +$. We recall that the function χ_0 in formula (1.22a) is given by $\chi_0 = 1$ in this chapter. To state next proposition let

$$E_+(f_{1(+)}, \dot{f}_{1(+)}, \alpha_{1(+)}) = (f_{1(+)}, \dot{f}_{1(+)}, \alpha_+), \quad (6.227a)$$

$$\hat{\alpha}_+(k) = \sum_{\varepsilon=\pm} e^{i\vartheta^\infty(A^{(+)} - A, (\omega(k), -\varepsilon k))} P_\varepsilon(k) \hat{\alpha}_{1(+)}(k), \quad (6.227b)$$

for $u_{1(+)} = (f_{1(+)}, \dot{f}_{1(+)}, \alpha_{1(+)}) \in \mathcal{O}_{1,\infty(+)}$, where $h(v) = (A, \dot{A}, \psi)$ is the solution of the M-D equations given by Theorem 6.15 with $v = (\Omega^{(+)}_1)^{-1}(u_{1(+)})$, where ϑ^∞ is given by (1.23b) and where $A^{(+)} = A^{(+)}(u_{1(+)})$ is given by (1.22a) with $u_{1(+)}$ instead of $(f_+, \dot{f}_+, \alpha_+)$.

Proposition 6.16. *Let $1/2 < \rho < 1$. One can choose the open neighbourhood $\mathcal{O}_{1,\infty(+)}$ of zero in $E^{\circ\rho}_\infty$ such that E_+ is a diffeomorphism of $\mathcal{O}_{1,\infty(+)}$ onto an open neighbourhood $\mathcal{O}_{\infty(+)}$ of zero in $E^{\circ\rho}_\infty$, such that there exists $N_+ \in \mathbb{N}$ and such that:*

i) $$\|(D^l E_+)(u; u_1, \cdots, u_l)\|_{E^\rho_n} \leq C_{n+l}\big(\mathcal{R}^l_{N_+,n+l}(u_1, \cdots, u_l) + \|u\|_{E^\rho_{N_+ + n+l}} \|u_1\|_{E^\rho_{N_+}} \cdots \|u_l\|_{E^\rho_{N_+}}\big),$$

for n, $l \in \mathbb{N}$, $u \in \mathcal{O}_{1,\infty(+)}, u_1, \cdots, u_l \in E^{\circ\rho}_\infty$, where C_{n+l} depends only on ρ and $\|u\|_{E^\rho_{N_+}}$,

ii) $$\|(D^l E_+^{-1})(u; u_1, \cdots, u_l)\|_{E^\rho_n} \leq C_{n+l}\big(\mathcal{R}^l_{N_+,n+l}(u_1, \cdots, u_l) + \|u\|_{E^\rho_{N_+ + n+l}} \|u\|_{E^\rho_{N_+}} \cdots \|u_l\|_{E^\rho_{N_+}}\big),$$

for $n,\ l \in \mathbb{N}$, $u \in \mathcal{O}_{\infty(+)}$, $u_1, \ldots, u_l \in E_\infty^{\circ\rho}$, where C_{n+l} depends only on ρ and $\|u\|_{E^\rho_{N_+}}$,

iii) *Let $\chi_0 \in C^\infty(\mathbb{R})$, $0 \le \chi_0 \le 1$, $\chi_0(s) = 0$ for $s \le 4$, $\chi_0(s) = 1$ for $s \ge 14$, let $1/2 < \kappa < 1$, $\chi_1(t,x) = \chi_0((t^2 - |x|^2)/t^{2\kappa})$ for $t \ge 1$ and $|x| < t$, $\chi_1(t,x) = 0$ elsewhere, let $(A_0^{(+)}, \dot{A}_0^{(+)}, \psi_0^{(+)})(t) = U^1_{\exp(tP_0)} u_+$, $u_+ = E_+(u)$, let $(A, \dot{A}, \psi) = h(\Omega_1^{(+)} \circ E_+^{-1}(u_+))$ be the solution given by Theorem 6.15 and let $\varphi = \vartheta(A_0^{(+)}) + \chi_1 \vartheta(A^{(+)2})$, where $A^{(+)2}$ is given by (1.22a). Then*

$$
\begin{aligned}
&\|\!|(D^l(A - A_0^{(+)} e^{i\varphi}\psi - \psi_0^{(+)})(u_+; u_{+1}, \ldots, u_{+l})\|\!|_{\rho', r, L} \\
&\quad \le C_{L+l}\big(\mathcal{R}^l_{N_+, L+l}(u_{+1}, \ldots, u_{+l}) + \|u_+\|_{E^\rho_{N_+ + L + l}} \|u_{+1}\|_{E^\rho_{N_+}} \cdots \|u_{+l}\|_{E^\rho_{N_+}}\big),
\end{aligned}
$$

for $L,\ l \in \mathbb{N}$, $1/2 < \rho' \le 1$, $r = (r(0), r(1))$, $r(0) > 0$, $r(1) \ge \rho$, $u_+ \in \mathcal{O}_{\infty(+)}$, $u_{+1}, \cdots, u_{+l} \in E_\infty^{\circ\rho}$, where $\|\!| \cdot \|\!|_{\rho', r, L}$ is given in Theorem 6.10 and where C_{L+l} depends only on ρ', r, ρ and $\|u_+\|_{E^\rho_{N_+}}$,

iv) *Let $\dot{A} = \xi^M_{P_0} A$ and $\dot{A}_0^{(+)} = \xi^M_{P_0} A_0^{(+)}$. Then*

$$
\|(A(t), \dot{A}(t)) - (A_0^{(+)}(t), \dot{A}_0^{(+)}(t))\|_{M^\rho} + \|\psi(t) - (e^{-i\varphi}\psi_0^{(+)})(t)\|_D \to 0,
$$

when $t \to \infty$, for $u \in \mathcal{O}_{\infty(+)}$.

Proof. We shall first estimate norms of $(D^l(A - A^{(+)}))(u; u_1, \ldots, u_l)$, for $l \in \mathbb{N}$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_l \in E^\rho_\infty$, where $A^{(+)}(u)$ is given by (1.22a) with $\chi_0 = 1$ and with $u = (f, \dot{f}, \alpha)$ instead of $(f_+, \dot{f}_+, \alpha_+)$ and where $(A(u), \dot{A}(u), \psi(u)) = h(\Omega_1^{(+)}(u))$ is, according to Theorem 6.15 the solution of the M-D equations in V^ρ_∞ with initial conditions $\Omega_1^{(+)}(u) \in \mathcal{U}_{\infty(0)}$. Recall that A_n, $n \ge 0$, and A^* are the functions of $u \in \mathcal{O}_{1,\infty(+)}$ given by (4.137b) and (6.30) respectively and let $A^{(l)}$ (resp. $A^{(+)(l)}, A^{*(l)}$) be the l^{th} derivative of A (resp. $A^{(+)}, A^*$) at u in the directions $u_1, \ldots, u_l$. It follows from Lemma 6.3, statement i) of Theorem 6.9 and by a suitable definition of N_+ that

$$
\begin{aligned}
&\|\delta(t)(1 + \big|t - |\cdot|\big|)^{1/2}(\xi^M_Y(A^{(l)} - A^{*(l)}))(t)\|_{L^\infty} + \|(\xi^M_Y(A^{(l)} - A^{*(l)}))(t)\|_{L^2} \qquad (6.228)\\
&\quad \le (1+t)^{-1+\rho} C_{l+|Y|}\big(\mathcal{R}^l_{N_+, |Y|+l}(u_1, \ldots, u_l) + \|u\|_{E^\rho_{N_+ + |Y| + l}} \|u_1\|_{E^\rho_{N_+}} \cdots \|u_l\|_{E^\rho_{N_+}}\big),
\end{aligned}
$$

for $t \ge 0$, $l \in \mathbb{N}$, $Y \in \Pi'$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_1, \ldots, u_l \in E^\rho_\infty$, where $C_{l+|Y|}$ depends only on ρ and $\|u\|_{E^\rho_{N_+}}$. Inequality (6.29) and the inequality following (6.29), with $\rho' = 1$ and the analog inequalities for $l \ge 1$ give that

$$
\begin{aligned}
&\|\delta(t)(1 + \big|t - |\cdot|\big|)^{1/2}(\xi^M_Y(A^{*(l)} - A_J^{(l)}))(t)\|_{L^\infty} + \|(\xi^M_Y(A^{*(l)} - A_J^{(l)}))(t)\|_{L^2} \qquad (6.229)\\
&\quad \le (1+t)^{-1+\rho} C_{l+|Y|}\big(\mathcal{R}^l_{N_+, |Y|+l}(u_1, \ldots, u_l) + \|u\|_{E^\rho_{N_+ + |Y| + l}} \|u_1\|_{E^\rho_{N_+}} \cdots \|u_l\|_{E^\rho_{N_+}}\big),
\end{aligned}
$$

for $t \geq 0$, $l \in \mathbb{N}$, $Y \in \Pi'$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_1, \ldots, u_l \in E^\rho_\infty$, where $C_{l+|Y|}$ depends only on ρ and $\|u\|_{E^\rho_{N_+}}$. Theorem 4.10 gives that

$$\begin{aligned}
&\|(\delta(t))^{1+\overline{\chi}-\eta+i}(\xi^M_Y(A^{(l)}_J - A^{(l)}_1))(t)\|_{L^\infty}(1+t)^n \\
&\quad + \|(\xi^M_Y(A^{(l)}_J - A^{(l)}_1), \xi^M_{P_0 Y}(A^{(l)}_J - A^{(l)}_1))(t)\|_{M^{\rho'}_0}(1+t)^{\rho'-1/2+\overline{\chi}} \\
&\quad \leq C_{l+|Y|}\big(\mathcal{R}^l_{N_+,|Y|+l}(u_1, \ldots, u_l) + \|u\|_{E^\rho_{N_++|Y|+l}}\|u_1\|_{E^\rho_{N_+}} \cdots \|u_l\|_{E^\rho_{N_+}}\big)
\end{aligned} \tag{6.230}$$

for $t \geq 0$, $l \in \mathbb{N}$, $Y \in \boldsymbol{\sigma}^i$, $i = 0, 1$, $0 \leq \rho' \leq 1$, $\rho' - 1/2 + \overline{\chi} > 0$, $\eta > 0$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_1, \ldots, u_l \in E^\rho_\infty$, where $\overline{\chi} = 2(1-\rho)$ and where $C_{l+|Y|}$ depends only on η, ρ, ρ' and $\|u\|_{E^\rho_{N_+}}$. We note that by (4.137b), with $n = 0$, and by definitions (1.22a) of $A^{(+)}$ it follows that $A_1 - A^{(+)} = A^2_1 - A^{(+)2}$, where A^2_1 (resp.$A^{(+)2}$) is the bilinear term of A_1 (resp.$A^{(+)}$) in u. Let, for $u_i = (f_i, \dot{f}_i, \alpha_i) \in E^\rho_\infty$,

$$\begin{aligned}
&(J_\mu(u_1 \otimes u_2))(t) \\
&\quad = \frac{1}{2}\big((U^{D1}_{\exp(tP_0)}\alpha_1)^+\gamma_0\gamma_\mu(U^{D1}_{\exp(tP_0)}\alpha_2) + (U^{D1}_{\exp(tP_0)}\alpha_2)^+\gamma_0\gamma_\mu(U^{D1}_{\exp(tP_0)}\alpha_1)\big), \quad t \in \mathbb{R},
\end{aligned} \tag{6.231a}$$

and

$$\begin{aligned}
&\big((J^{(+)}_\mu(u_1 \otimes u_2))(t)\big)(x) \\
&\quad = \frac{1}{2}\big(\frac{m}{t}\big)^3\big(\frac{t}{\sqrt{t^2-|x|^2}}\big)^5 \sum_{\varepsilon=\pm}\Big(\big(P_\varepsilon(p(t,x))\hat{\alpha}_1(p(t,x))\big)^+\gamma_0\gamma_\mu\big(P_\varepsilon(p(t,x))\hat{\alpha}_2(p(t,x))\big) \\
&\qquad + \big(P_\varepsilon(p(t,x))\hat{\alpha}_2(p(t,x))\big)^+\gamma_0\gamma_\mu\big(P_\varepsilon(p(t,x))\hat{\alpha}_1(p(t,x))\big)\Big),
\end{aligned} \tag{6.231b}$$

for $t > 0$, $t^2 - |x|^2 > 0$ and $\big((J^{(+)}_\mu(u_1 \otimes u_2))(t)\big)(x) = 0$ for $t > 0$, $t^2 - |x|^2 \leq 9$. Then $\big(J^{(+)}_\mu(u_1 \otimes u_2)\big)(t) \in D_\infty$ for $t > 0$. We have $\square\xi^\mu_{Y_1}A^2_1(u_1 \otimes u_2) = \xi^M_{Y_1}J(u_1 \otimes u_2)$ in $\mathbb{R}^4$ for $Y_1 \in U(\mathfrak{p})$ and $\square\xi^M_{Y_1}A^{(+)2} = \xi^M_{Y_1}J^{(+)}(u_1 \otimes u_2)$ in $\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3 \big| t^2 - |x|^2 \geq 4\}$ for $Y_1 \in U(\mathfrak{p})$. This covariance property for $Y_1 \in \Pi' \cap +U(\mathfrak{sl}(2,\mathbb{C}))$, $Y = ZY_1$, with $Z \in \Pi' \cap U(\mathbb{R}^4)$ and Corollary 4.2 and this covariance property for $Y_1 \in \Pi'$, $Y = Y_1$ and statement iii) of Lemma 4.3 give:

$$\begin{aligned}
&\|\big(\xi^M_Y(A^{(2)}_1(u_1 \otimes u_2) - A^{(+)2}(u_1 \otimes u_2))\big)(t)\|_{L^\infty(|x|^2 \leq t^2-4)}(1+t)^2 \\
&\quad + \|\big(\xi^M_Y(A^{(2)}_1(u_1 \otimes u_2) - A^{(+)2}(u_1 \otimes u_2))\big)(t)\|_{L^2(|x|^2 \leq t^2-4)}(1+t)^{1/2} \\
&\quad \leq C_{|Y|}\big(\|\alpha_1\|_{D_{N_+}}\|\alpha_2\|_{D_{N_++|Y|}} + \|\alpha_1\|_{D_{N_++|Y|}}\|\alpha_2\|_{D_{N_+}}\big),
\end{aligned} \tag{6.232}$$

for u_1, $u_2 \in E^\rho_\infty$, $Y \in \Pi'$, $t \geq 2$, where $C_{|Y|}$ is independent of t, u_1, u_2. Since the support of $s \mapsto (d/ds)^n\chi(s)$, $s \in \mathbb{R}$, is a subset of the interval $[1, 2]$, it follows that

$$\big|(\partial^{\alpha_0}_0\partial^{\alpha_1}_1\partial^{\partial_2}_2\partial^{\alpha_3}_3\chi(\rho))(t,x)\big| \leq C_{|\alpha|}(1+t)^{|\alpha|}, \quad t \geq 0, x \in \mathbb{R}^3, \tag{6.233}$$

where we have defined $(\chi(\rho))(t,x)=0$ for $t^2-|x|^2<0$. Since $\xi_Y\rho=0$ for $Y\in U(sl(2,\mathbb{C}))$ and since the volume of $\{x\in\mathbb{R}^3 \big| t^2-4\leq |x|^2\leq t^2\}$ is bounded by $C(1+t)$, $t\geq 0$, where C is independent of t, it follows from (4.47) of Corollary 4.2 that

$$\begin{aligned}&\|(\xi^M_{ZY}A^{(+)2}(u_1\otimes u_2))(t)\|_{L^p(|x|^2\geq t^2-4)} \\ &\quad\leq C_{|Z|+|Y|}\sum_{0\leq n\leq |Z|}(1+t)^{-1+1/p+|Z|-n} \\ &\qquad\big(\|\alpha_1\|_{D_{N_+}}\|\alpha_2\|_{D_{N_++n+|Y|}}+\|\alpha_1\|_{D_{N_++n+|Y|}}\|\alpha_2\|_{D_{N_+}}\big),\end{aligned} \tag{6.234}$$

for $1\leq p\leq\infty$, $t\geq 0$, $u_1,u_2\in E^\rho_\infty$, $Y\in\Pi'\cap U(sl(2,\mathbb{C}))$, $Z\in\Pi'\cap U(\mathbb{R}^4)$, where $C_{|Z|+|Y|}$ depends only on p. It follows similarly from Theorem 4.10, reminding that there is no cut-off function in A^2_1, that

$$\begin{aligned}&\|(\xi^M_Y A^2_1(u_1\otimes u_2))(t)\|_{L^p(t^2-4\leq|x|^2\leq t^2)} \\ &\quad\leq C_{|Y|}(1+t)^{-1+1/p}\big(\|\alpha_1\|_{D_{N_+}}\|\alpha_2\|_{D_{N_++|Y|}}+\|\alpha_1\|_{D_{N_++|Y|}}\|\alpha_2\|_{D_{N_+}}\big),\end{aligned} \tag{6.235}$$

for $1\leq p\leq\infty$, $t\geq 0$, $u_1,u_2\in E^\rho_\infty$, $Y\in\Pi'$, where $C_{|Y|}$ depends only on p.

According to (6.227a) and (6.227b) let $E_+(u)=(f,\dot f,\alpha_+(u))$ for $u=(f,\dot f,\alpha)\in\mathcal{O}_{1,\infty(+)}$, where

$$(\alpha_+(u))^\wedge(k)=\sum_{\varepsilon=\pm}e^{i\vartheta^\infty(A^{(+)}-A,(\omega(k),-\varepsilon k))}P_\varepsilon(k)\hat\alpha(k). \tag{6.236}$$

To prove that E_+ is a C^∞ function from $\mathcal{O}_{1,\infty(+)}$ to E^ρ_∞ satisfying the inequality of statement i), it is sufficient to prove this for the function α_+ from $\mathcal{O}_{1,\infty(+)}$ to D_∞. If $\hat\beta(k)=\sum_\varepsilon F_\varepsilon(k)P_\varepsilon(k)\hat\alpha(k)$, then if follows using definition (1.5d) of $T^{D1}_{M_{0i}}$ that

$$(T^{D1}_{M_{0i}}\beta)^\wedge(k)=\sum_\varepsilon\Big((-\varepsilon\omega(k)\frac{\partial}{\partial k_i}F_\varepsilon(k))P_\varepsilon(k)\hat\alpha(k)+F_\varepsilon(k)P_\varepsilon(k)(T^{D1}_{M_{0i}}\alpha)^\wedge(k)\Big), \tag{6.237a}$$

and if $F_\varepsilon(k)=f(\omega(k),-\varepsilon k)$ then

$$-\varepsilon\omega(k)\frac{\partial}{\partial k_i}F_\varepsilon(k)=(\xi_{M_{0i}}f)(\omega(k),-\varepsilon k). \tag{6.237b}$$

A similar relation is valid for the rotations which gives that

$$\begin{aligned}&(T^{D1}_X\beta)^\wedge(k) \\ &\quad=\sum_\varepsilon((\xi_X f)(\omega(k),-\varepsilon k)P_\varepsilon(k)\hat\alpha(k)+f(\omega(k),-\varepsilon k)P_\varepsilon(k)(T^{D1}_X\alpha)^\wedge(k),\quad X\in sl(2,\mathbb{C}).\end{aligned} \tag{6.237c}$$

Using the SL$(2,\mathbb{C})$ covariance of the function ϑ^∞ it follows from (6.237c), with $f(y)=\vartheta^\infty(H,y)$ that

$$(T^{D1}_{ZY}\beta)(k)=\sum^{Y}_{Y_1,Y_2}\sum_\varepsilon\vartheta^\infty(\xi^M_{Y_1}H,(\omega(k),-\varepsilon k))P_\varepsilon(k)(T^{D1}_{ZY_2}\alpha)^\wedge(k), \tag{6.238}$$

for $Y \in \Pi' \cap U(sl(2,\mathbb{C}))$ and $Z \in \Pi' \cap U(\mathbb{R}^4)$, where $\hat{\beta}(k) = \sum_\varepsilon \vartheta^\infty(H,(\omega(k),-\varepsilon k))$ and $H\colon \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^4$ is sufficiently smooth and decreasing. It follows from equalities (6.236) and (6.238) that $\big(T^{D1}_{ZY}((D^l\alpha_+)u; u_1,\ldots,u_l))\big)^\wedge(k)$ is a sum of terms

$$C_{Y_1,\ldots,Y_{j+1};i_1,\ldots,i_{j+1}}\vartheta^\infty(\xi^M_{Y_1}H^{(i_1)},l_\varepsilon(k))\cdots\vartheta^\infty(\xi^M_{Y_j}H^{(i_j)},l_\varepsilon(k)) \\ e^{i\vartheta^\infty(H,l_\varepsilon(k))}P_\varepsilon(k)(T^{D1}_{ZY_{j+1}}\alpha^{(i_{j+1})})^\wedge(k), \tag{6.238}$$

$Z \in \Pi' \cap U(\mathbb{R}^4)$, $Y \in \Pi' \cap U(sl(2,\mathbb{C}))$, where $H = A^{(+)} - A$, $l_\varepsilon(k) = (\omega(k),-\varepsilon k)$, where $H^{(i_1)},\ldots,H^{(i_j)}$, $\alpha^{(i_{j+1})}$ are the derivatives of order $i_1,\ldots,i_j$ respectively i_{j+1} at u, each depending respectively on $i_1,\cdots,i_j$, i_{j+1} distinct arguments among $u_1,\ldots,u_l$ and where $i_1+\cdots+i_{j+1} = l$, $i_1,\ldots,i_{j+1} \geq 0$, $Y_1,\ldots,Y_{j+1} \in \Pi' \cap U(sl(2,\mathbb{C}))$, $|Y_1|+\cdots+|Y_{j+1}| = |Y|$, $|Y_q| + i_q \geq 1$ for $q \leq j$. Since the function $(t,x) \mapsto \vartheta^\infty(H,(t,x))$ is homogeneous of degree zero, we can use statement ii) of Lemma 3.1 to the function $(\omega(k),-\varepsilon k) \mapsto \vartheta^\infty(H, m^{-1}(\omega(k),-\varepsilon k)) = \vartheta^\infty(H,(\omega(k),-\varepsilon k))$. This gives

$$|\vartheta^\infty(\xi^M_Y H^{(l)},(\omega(k),-\varepsilon k))| \\ \leq 2m\big(\ln\big(1+\frac{\omega(k)}{m}\big)+\frac{1}{2\tau}\big)\sup_{\substack{t\geq 1\\ |x|<t}}\big((1+t)(1+t-|x|)^\tau|(\xi^M_Y H^{(l)})(t,x)|\big), \quad \tau > 0, \tag{6.240}$$

for $Y \in \Pi' \cap U(sl(2,\mathbb{C}))$, $k \in \mathbb{R}^3$, $\varepsilon = \pm$. Since $t-|x| \leq 4(1+t)^{-1}$, for $t \geq 1$, $t^2-|x|^2 \leq 2$ and $|x| \leq t$ it follows from inequalities (6.228), (6.229), (6.230), (6.232), (6.234) and (6.235), choosing $\tau = 1-\rho$ in (6.240) that

$$\big|\vartheta^\infty(\xi^M_Y(A^{(+)(l)}-A^{(l)}),(\omega(k),-\varepsilon k))\big| \\ \leq C_{|Y|+l}\ln\big(1+\frac{\omega(k)}{m}\big)\big(\mathcal{R}^l_{N_+,|Y|+l}(u_1,\ldots,u_l)+\|u\|_{E^\rho_{N_++|Y|+l}}\|u_1\|_{E^\rho_{N_+}}\cdots\|u_l\|_{E^\rho_{N_+}}\big), \tag{6.241}$$

for $l \geq 0$, $Y \in \Pi' \cap U(sl(2,\mathbb{C}))$, $k \in \mathbb{R}^3$, $\varepsilon = \pm$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_1,\ldots,u_l \in E^\rho_\infty$, where $C_{|Y|+l}$ depends only on ρ and $\|u\|_{E^\rho_{N_+}}$. Reindexing for the moment the sequence $u_1,\ldots,u_l$ we can suppose that $H^{(i_1)};\ldots;H^{(i_j)}$ depend on the arguments $u_1,\ldots,u_{i_1}$; $\ldots$; $u_{i_1+\cdots+i_{j-1}+1},\ldots,$ $u_{i_1+\cdots+i_j}$ respectively and that $\alpha^{(i_{j+1})}$ depends on $u_{l-i_{j+1}+1},\ldots,u_l$. Using the notation

$$\overline{\mathcal{R}}^{(l)}_{N_+,N_++|Y|+l}(u;u_1,\ldots,u_l) = \mathcal{R}^l_{N_+,|Y|+l}(u_1,\ldots,u_l)+\|u\|_{E^\rho_{N_++|Y|+l}}\|u_1\|_{E^\rho_{N_+}}\cdots\|u_l\|_{E^\rho_{N_+}}$$

using the fact that $\ln(1+\omega(k)/m) \leq \omega(k)/m$ and that

$$\overline{\mathcal{R}}^{(l)}_{N_+,N_++|Y|+l}(u;u_1,\ldots,u_l) \leq \overline{\mathcal{R}}^{(l)}_{N_++1,N_++1+|Y|+l-1}(u;u_1,\ldots,u_l),$$

it follows from Corollary 2.6 and inequality (6.241) that the expression (6.239) is majorized by

$$C_{|Y|+l}\overline{\mathcal{R}}^{(l)}_{N_++1,N_++1+|Z|+|Y|+l}(u;u_1,\ldots,u_l). \tag{6.242}$$

Here we have also used the fact that

$$\|\omega(-i\partial)^j T^{D1}_{ZY_{j+1}}\ \alpha^{(i_{j+1})}\|_D \le C_{|Z|+|Y_{j+1}|+j}\ \overline{\mathcal{R}}^{(i_{j+1})}_{N_+,N_++|Z|+|Y_{j+1}|+i_{j+1}}(u; u_{l-i_{j+1}+1},\ldots,u_l)$$

and that $|Y_q|+i_q \ge 1$ for $q \le j$. Since the expression (6.242) is independent of permutations of $u_1,\ldots,u_l$, this expression majorizes all the expressions (6.239). This proves, after redefinition of N_+, that $E_+\colon \mathcal{O}_{1,\infty(+)} \to E^\rho_\infty$ is C^∞ and that statement i) of the proposition is true.

We shall next study the asymptotic behavior of $e^{i\varphi(t)}\psi(t)$ in statement iii) of the proposition for $u_+ = (f_+, \dot f_+, \alpha_+) = E_+(u)$, $u = (f, \dot f, \alpha) \in \mathcal{O}_{1,\infty(+)}$ and $v = \Omega_1^{(+)}(u)$. It follows from (4.47) of Corollary 4.2 that

$$\begin{aligned}&\|(\xi^M_Y A^{(+)2}(u_1\otimes u_2))(t)\|_{L^p(|x|^2\le t^2-4)} \\ &\quad\le (1+t)^{-1-i+3/p} C_{|Y|}\big(\|\alpha_1\|_{D_{N_+}}\|\alpha_2\|_{D_{N_++|Y|}} + \|\alpha_1\|_{D_{N_++|Y|}}\|\alpha_2\|_{D_{N_+}}\big),\end{aligned} \tag{6.243}$$

for $1 \le p \le \infty$, $t \ge 0$, $u_j = (f_j, \dot f_j, \alpha_j) \in E^{\circ\rho}_\infty$, $j = 0, 1$, $Y \in \boldsymbol{\sigma}^i$, $i = 0, 1$, where N_+ and $C_{|Y|}$ are independent of p, t, u_j, Y. Let $\chi_1(t,x) = \chi_0((t^2-|x|^2)/t^{2\kappa})$ for $t \ge 1$ and $\chi_1(t,x) = 0$ elsewhere for $t \ge 0$. Since $\chi_1(t,x) = 0$ for $t^2 - |x|^2 \le 4t^{2\kappa}$, it follows that $t^2 - |x|^2 \ge 4t^{2\kappa} > 16$ in the support of φ. Here we have used the fact that $1/2 < \kappa < 1$. Inequalities (6.234) and (6.243) then give for $Y \in \Pi'$ and for $(t,x) \in \operatorname{supp}\chi_1$:

$$\begin{aligned}&\big|\xi_Y\vartheta(A^{(+)2}(u_1\otimes u_2),(t,x))\big| \\ &\quad\le (1+t)^{-i+\kappa'} C_{|Y|}\big(\|\alpha_1\|_{D_{N_+}}\|\alpha_2\|_{D_{N_++|Y|}} + \|\alpha_1\|_{D_{N_++|Y|}}\|\alpha_2\|_{D_{N_+}}\big),\end{aligned} \tag{6.244}$$

for $t \ge 0$, $x \in \mathbb{R}^3$, $Y \in \boldsymbol{\sigma}^i$, $i = 0, 1$, where $\kappa' > 0$ can be made arbitrary small by choosing $\kappa \in]1/2, 1[$ sufficiently close to 1 and where $C_{|Y|}$ depends only on κ. According to the definition of χ_1 we obtain that

$$|\xi_Y\chi_1|(t,x) \le C_{|Y|}(1+t)^{-i(2\kappa-1)} \quad \text{for} Y \in \boldsymbol{\sigma}^i, i \in \mathbb{N}. \tag{6.245}$$

Hence, redefining κ' and identifying the second order part of the map φ with a bilinear symmetric map φ^2, it follows from inequality (6.244) that

$$\begin{aligned}&|\xi_Y\varphi^2(u_1\otimes u_2)|(t,x) \\ &\quad\le (1+t)^{-i+\kappa'} C_{|Y|}\big(\|\alpha_1\|_{D_{N_+}}\|\alpha_2\|_{D_{N_++|Y|}} + \|\alpha_1\|_{D_{N_++|Y|}}\|\alpha_2\|_{D_{N_+}}\big),\end{aligned} \tag{6.246}$$

for $t \ge 0$, $x \in \mathbb{R}^3$, $Y \in \boldsymbol{\sigma}^i$, $i \in \{0,1\}$, $u_j = (f_j, \dot f_j, \alpha_j) \in E^\rho_\infty$, where κ' can be made arbitrary small by choosing $\kappa \in]1/2, 1[$ sufficiently close to 1 and where $C_{|Y|}$ depends only on κ. Let $(t,x) \in \operatorname{supp}(1-\chi_1)$ and $|x| \le t$. Then $0 \le t - |x| \le 16t^{2\kappa-1}$, which shows that $(1+\lambda_1(t))(x) \ge C(1+t)^{2(1-\kappa)}$ for such t and x. Hence by the definition of λ_1, it follows that $(1+t+|x|)^{2(1-\kappa)} \le C(1+\lambda_1,(t))(x)$ for $(t,x) \in \operatorname{supp}(1-\chi_1)$ and $t \ge 0$, where C is independent of t and x. This gives together with Theorem 6.10 and Lemma 4.4 that

$$\begin{aligned}&\|(1+\lambda_1(t))^{-\tau/(2(1-\kappa))}\big(\xi_Y(1-\chi_1)\ \vartheta(A^{(l)}-A^{(l)}_0)\big)(t)\|_{L^\infty} \\ &\quad+ \|(\delta(t))^{2-\rho}\big(\xi_Z(1-\chi_1)\vartheta(A^{(l)}-A^{(l)}_0)\big)(t)\|_{L^\infty} \\ &\qquad\le C_L\big(\mathcal{R}^l_{N_+,L}(u_1,\ldots,u_l) + \|u\|_{E^\rho_{N_++L}}\|u_1\|_{E^\rho_{N_+}}\cdots\|u_l\|_{E^\rho_{N_+}}\big),\end{aligned} \tag{6.247}$$

for $t \geq 0$, $\tau > 0$, $Y \in \Pi'$, $Z \in \boldsymbol{\sigma}^1$, $u_1, \ldots, u_l \in E_\infty^{\circ\rho}$, $\max(|Y|,|Z|) + l \leq L$, where $A^{(l)} - A_0^{(l)} = (D^l(A - A_0))(u; u_1, \ldots, u_l)$ and where C_L depends only on τ, ρ, κ and $\|u\|_{E^\rho_{N_+ + L}}$. It follows from inequalities (6.228), (6.229), (6.230) and (6.232) that if $H^{(l)} = (D^l(A^{(+)} - A))(u; u_1, \ldots, u_l)$ then

$$|\xi_Y^M H^{(l)}|(t,x) \leq (1+t)^{-3+2\rho-j(\rho-1/2)}(1+(\lambda_1(t))(x))^{1-\rho+j(\rho-1/2)} \quad (6.248a)$$
$$C_{|Y|,l}\big(\mathcal{R}^l_{N_+,|Y|+l}(u_1, \ldots, u_l) + \|u\|_{E^\rho_{N_+ + |Y|+l}} \|u_1\|_{E^\rho_{N_+}} \cdots \|u_l\|_{E^\rho_{N_+}}\big),$$

for $t^2 - |x|^2 \geq 4$, $t \geq 0$, $l \geq 0$, $Y \in \boldsymbol{\sigma}^j$, $j \in \{0,1\}$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_1, \ldots, u_l \in E_\infty^{\circ\rho}$, where $C_{|Y|,l}$ depends only on ρ and $\|u\|_{E^\rho_{N_+}}$. It follows from Theorem 6.10 and inequality (6.234) that

$$|\xi_{ZY}^M H^{(l)}|(t,x) \leq C_{|Z|+|Y|+l} \sum_{0 \leq n \leq |Z|} (1+t)^{-1+|Z|-n} \quad (6.248b)$$
$$\big(\mathcal{R}^l_{N_+,n+|Y|+l}(u_1, \ldots, u_l) + \|u\|_{E^\rho_{N_+ + n+|Y|+l}} \|u_1\|_{E^\rho_{N_+}} \cdots \|u_l\|_{E^\rho_{N_+}}\big),$$

for $t^2 - |x|^2 \leq 4$, $t \geq 0$, $l \geq 0$, $Z \in \Pi' \cap U(\mathbb{R}^4)$, $Y \in \Pi' \cap U(sl(2,\mathbb{C}))$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_1, \ldots, u_l \in E_\infty^{\circ\rho}$, where $C_{|Z|+|Y|+l}$ depends only on ρ and $\|u\|_{E^\rho_{N_+}}$. If $(t,x), t > 0$, is inside the light cone then it follows from inequality (6.248a) and (6.248b) that the line integral $\vartheta^\infty(\xi_Y^M H^{(l)}, (t,x))$ converges absolutely for $Y \in \Pi' \cap U(sl(2,\mathbb{C}))$. The function $(t,x) \mapsto \vartheta^\infty(\xi_Y^M H^{(l)}, (t,x))$ from the interior of the forward light cone into $\mathbb{R}$ is homogeneous of degree zero and $\xi_Y \vartheta^\infty(H^{(l)}) = \vartheta^\infty(\xi_Y^M H^{(l)})$ for $Y \in \Pi' \cap U(sl(2,\mathbb{C}))$. Expressing the derivatives ∂_μ as linear functions of $\xi_{M_{0i}}$ and the dilatation operator as in equality (5.67), it follows from inequality (6.248a) with $Y \in \Pi' \cap U(sl(2,\mathbb{C}))$, inequality (6.248b) with $Z = \mathbb{I}$ and by using $t^2 - |x|^2 \geq 4t^{2\kappa}$, $t \geq 0$, in the support of χ_1, that

$$\big(1 + \ln(1 + t/(t-|x|))\big)^{-1} t^{(2\kappa-1)|Z|} |\xi_{ZY} \vartheta^\infty(H^{(l)})|(t,x),$$

$Z \in \Pi' \cap U(\mathbb{R}^4)$, $Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$ is uniformly bounded in (t,x) on the support of χ_1. According to inequality (6.245) and using the bounds in inequalities (6.248a) and (6.248b) we obtain that

$$\big|\xi_{ZY} \chi_1 \vartheta^\infty(H^{(l)})\big|(t,x) \quad (6.249)$$
$$\leq \big(1 + \ln(1 + t/(t-|x|))\big)(1+t)^{-|Z|(2\kappa-1)}$$
$$C_{|ZY|+l}\big(\mathcal{R}^l_{N_+,|Z|+|Y|+l}\ (u_1, \ldots, u_l) + \|u\|_{E^\rho_{N_+ + |Z|+|Y|+l}} \|u_1\|_{E^\rho_{N_+}} \cdots \|u_l\|_{E^\rho_{N_+}}\big),$$

for $(t,x) \in \text{supp}\ (\chi_1)$, $l \geq 0$, $Z \in \Pi' \cap U(\mathbb{R}^4)$, $Y \in \Pi' \cap U(sl(2,\mathbb{C}))$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_1, \ldots, u_l \in E_\infty^{\circ\rho}$, where $C_{|ZY|+l}$ depends only on ρ and $\|u\|_{E^\rho_{N_+}}$. Similarly as we obtained (6.246) from inequality (6.244), it follows from inequalities (6.248a) and (6.248b) and from $1 + \lambda_1(sy) \leq C(1 + \lambda_1(y))$ for y inside the light cone and for $0 \leq s \leq 1$, that

$$\big|\xi_Y \chi_1 \vartheta(H^{(l)})\big|(t,x) \quad (6.250)$$
$$\leq (1+t)^{-2(1-\rho)}(1 + \lambda_1(t,x))^{1/2}$$
$$C_{|Y|+l}\big(\mathcal{R}^l_{N_+,|Y|+l}(u_1, \ldots, u_l) + \|u\|_{E^\rho_{N_+ + |Y|+l}} \|u_1\|_{E^\rho_{N_+}} \cdots \|u_l\|_{E^\rho_{N_+}}\big),$$

for $t \geq 0$, $x \in \mathbb{R}^3$, $Y \in \boldsymbol{\sigma}^1$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_1, \ldots, u_l \in E_\infty^{\circ\rho}$, where κ is chosen sufficiently close to one and where $C_{|Y|+l}$ depends only on κ, ρ and $\|u\|_{E^\rho_{N_+}}$. Since $\varphi - \vartheta(A) = \chi_1 \vartheta(A^{(+)} - A) + (1-\chi_1)\vartheta(A_0 - A)$, it follows from inequalities (6.247) and (6.250) that

$$\begin{aligned}\|(\delta(t))^{2(1-\rho)}(1+\lambda_1(t))^{-1/2}(\xi_Y\big(D^l(\varphi-\vartheta(A))(u;u_1,\ldots,u_l)\big)(t)\|_{L^\infty} \\ \leq C_{|Y|+l}\big(\mathcal{R}^l_{N_+,|Y|+l}(u_1,\ldots,u_l) + \|u\|_{E^\rho_{N_++|Y|+l}}\|u_1\|_{E^\rho_{N_+}}\cdots\|u_l\|_{E^\rho_{N_+}}\big),\end{aligned} \tag{6.251}$$

for $t \geq 0$, $Y \in \boldsymbol{\sigma}^1$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_1, \ldots, u_l \in E_\infty^{\circ\rho}$, where $C_{|Y|+l}$ depends only on ρ and $\|u\|_{E^\rho_{N_+}}$. It also follows from inequality (6.247), with $\tau/(2(1-\kappa))$ replaced by τ, and from inequalities (6.248a) and (6.248b), that

$$\begin{aligned}\|(1+\lambda_1(t))^{-\tau}\big(\xi_Y(D^l(\varphi-\vartheta(A)))(u;u_1,\cdots,u_l)\big)(t)\|_{L^\infty} \\ \leq C_{|Y|+l}\big(\mathcal{R}^l_{N_+,|Y|+l}(u_1,\ldots,u_l) + \|u\|_{E^\rho_{N_++|Y|+l}}\|u_1\|_{E^\rho_{N_+}}\cdots\|u_l\|_{E^\rho_{N_+}}\big),\end{aligned} \tag{6.252}$$

for $t \geq 0$, $Y \in \Pi' \cap U(sl(2,\mathbb{C}))$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_1, \ldots, u_l \in E_\infty^{\circ\rho}$, $\tau > 0$, where $C_{|Y|+l}$ depends only on τ, ρ and $\|u\|_{E^\rho_{N_+}}$. Let $\psi_0(t) = U^{D1}_{\exp(tP_0)}\alpha$. Using Theorem 6.10, Corollary 2.6, inequalities (6.251) and (6.252), with $\tau = 1/2$ and redefining N_+ we obtain, with the notation

$$q_Y^{(l)}(t) = \big(\xi_Y^D\big(D^l(e^{i(\varphi-\vartheta(A))}(e^{i\vartheta(A)}\psi - \psi_0))\big)(u;u_1,\ldots,u_l)\big)(t), \tag{6.253}$$

that

$$\begin{aligned}(1+t)^{2(1-\rho)}\|(1+\lambda_1(t))^{k/2}q_Y^{(l)}(t)\|_D + \|(\delta(t))^{3/2+2(1-\rho)}(1+\lambda_1(t))^{k/2}q_Y^{(l)}(t)\|_{L^\infty} \\ \leq C_{|Y|+l+k}\big(\mathcal{R}^l_{N_+,|Y|+l+k}(u_1,\ldots,u_l) + \|u\|_{E^\rho_{N_++|Y|+l+k}}\|u_1\|_{E^\rho_{N_+}}\cdots\|u_l\|_{E^\rho_{N_+}}\big),\end{aligned} \tag{6.254}$$

for $t \geq 0$, $l \in \mathbb{N}$, $k \in \mathbb{N}$, $Y \in \Pi'$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_1, \ldots, u_l \in E_\infty^{\circ\rho}$, where $C_{|Y|+l+k}$ depends only on ρ and $\|u\|_{E^\rho_{N_+}}$. Similarly, if

$$r^{(l)}_{Y,Z}(t) = \big(\xi^D_{YZ}(D^l(e^{i(\varphi-\vartheta(A))}\psi_0)) - \xi^D_Y(D^l(e^{i(\varphi-\vartheta(A))}\xi^D_Z\psi_0))\big)(t), \tag{6.255}$$

then

$$\begin{aligned}(1+t)^{2(1-\rho)}\|(1+\lambda_1(t))^{k/2}r^{(l)}_{Y,Z}(t)\|_D + \|(\delta(t))^{3/2+2(1-\rho)}(1+\lambda_1(t))^{k/2}r^{(l)}_{Y,Z}(t)\|_{L^\infty} \\ \leq |Z|C_{|Y|+|Z|+l+k}\big(\mathcal{R}^l_{N_+,|Y|+|Z|+l+k}(u_1,\ldots,u_l) + \|u\|_{E^\rho_{N_++|Y|+|Z|+l+k}}\|u_1\|_{E^\rho_{N_+}}\cdots\|u_l\|_{E^\rho_{N_+}}\big),\end{aligned} \tag{6.256}$$

for $t \geq 0$, $l \in \mathbb{N}$, $k \in \mathbb{N}$, $Y \in \Pi' \cap U(sl(2,\mathbb{C}))$, $Z \in \Pi' \cap U(\mathbb{R}^4)$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_1, \ldots, u_l \in E_\infty^{\circ\rho}$, where $C_{|Y|+|Z|+l+k}$ depends only on ρ and $\|u\|_{E^\rho_{N_+}}$. Development of the expression

$$\xi^D_Y\big((D^l(e^{i(\varphi-\vartheta(A))}\xi^D_Z\psi_0))(u;u_1,\ldots,u_l)\big), \quad Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C})),$$

shows that it is a sum of expressions

$$C_{Y_1,\dots,Y_{j+1};i_1,\dots,i_{j+1}}\big(\xi_{Y_1}(\varphi^{(i_1)}-\vartheta(A^{(i_1)}))\big)\cdots\big(\xi_{Y_j}(\varphi^{(i_j)}-\vartheta(A^{(i_j)}))\big) \tag{6.257}$$
$$e^{i(\varphi-\vartheta(A))}(\xi^D_{Y_{j+1}Z}\psi_0^{(i_{j+1})}), \quad Y, Y_1\dots,Y_{j+1}\in\Pi'\cap U(\mathfrak{sl}(2,\mathbb{C})),$$

where the coefficients and the dependence of the arguments are as in expression (6.239). According to Theorem A.1 there exists two functions $\beta^{(i_{j+1})}_{(\varepsilon)X}\in C^\infty(\mathbb{R}^+\times\mathbb{R}^3-\{0\})$, $\varepsilon=\pm$, given by formula (A.1), homogeneous of degree $-3/2$ and with support in the forward light cone such that, if

$$(\phi^{(i_{j+1})}_{(\varepsilon)X}(t))(x)=e^{i\varepsilon m(t^2-|x|^2)^{1/2}}\beta^{(i_{j+1})}_{(\varepsilon)X}(t,x),$$

then there exists $N\in\mathbb{N}$ such that

$$\|(1+\lambda_1(t))^{k/2}\big(P_\varepsilon(-i\partial)U^{D1}_{\exp(tP_0)}T^{D1}_X\alpha^{(i_{j+1})}-\phi^{(i_{j+1})}_{(\varepsilon)X}(t)\big)\|_D\le t^{-1}C_k\|\alpha^{(i_{j+1})}\|_{D_{N+k+|X|}}$$

for $t>0$, $k\in\mathbb{N}$, $X\in\Pi'$, where C_k is independent of t and X. This gives together with inequality (6.252), taking $0<\tau\le 1/2$ and after redefining N_+ by $\max(N_++1,N)$ that, if

$$Q_j\big(Y_1,\dots,Y_j;i_1,\dots,i_j;(t,x)\big) \tag{6.258a}$$
$$=\big((\xi_{Y_1}(\varphi^{(i_1)}-\vartheta(A^{(i_1)})))\cdots(\xi_{Y_j}(\varphi^{(i_j)}-\vartheta(A^{(i_j)})))\big)(t,x),$$

then

$$\|(1+\lambda_1(t))^{k/2}Q_j\big(Y_1,\dots,Y_j;i_1,\dots,i_j;(t,\cdot)\big)\big(P_\varepsilon(-i\partial)(\xi^D_{Y_{j+1}Z}\ \psi_0^{(i_{j+1})})(t)-\phi^{(i_{j+1})}_{(\varepsilon)Y_{j+1}Z}(t)\big)\|_D$$
$$\le t^{-1}C_{|Y|+|Z|+k+l}\overline{\mathcal{R}}^{(l)}_{N_+,N_++|Y|+|Z|+k+l}(u;u_1,\dots,u_l),\quad t>0, \tag{6.258b}$$

where $Y_1,\dots,Y_{j+1}$, $i_1,\dots,i_{j+1}$ are as in expression (6.239), where we have used the notation $\overline{\mathcal{R}}^{(l)}$ as in expression (6.242) and where we have used that $|Y_q|+i_q\ge 1$ for $q\le j$. Here $C_{|Y|+|Z|+k+l}$ depends only on ρ and $\|u\|_{E^\rho_{N_+}}$ for $\tau\in]0,1/2]$ fixed. Similarly it follows, using Theorem A.1 with L^∞-norms, that

$$\|(\delta(t))^{3/2+2(1-\rho)}(1+\lambda_1(t))^{k/2}Q_j\big(Y_1,\dots,Y_j;i_1,\dots,i_j;(t,\cdot)\big) \tag{6.258c}$$
$$\big(P_\varepsilon(-i\partial)(\xi^D_{Y_{j+1}Z}\psi_0^{(i_{j+1})})(t)-\phi^{(i_{j+1})}_{(\varepsilon)Y_{j+1}Z}(t)\big)\|_{L^\infty}$$
$$\le t^{-(2\rho-1)}C_{|Y|+|Z|+k+l}\overline{\mathcal{R}}^{(l)}_{N_+,N_++|Y|+|Z|+k+l}(u;u_1,\dots,u_l),\quad t>0.$$

Since

$$\big(\vartheta(H^{(l)})-\vartheta^\infty(H^{(l)})\big)(y)=-\int_1^\infty y^\mu H^{(l)}_\mu(sy)ds,$$

for $y^\mu y_\mu>0$, $y_0>0$, since $\xi_Y\ \chi_1$ is uniformly bounded in the half space $t\ge 0$ for $Y\in\Pi'$ and since $(1+\lambda_1(t))(x)\le C(1+t/(t-|x|))$ inside the forward light cone, it follows, using the $sl(2,\mathbb{C})$ covariance of ϑ and ϑ^∞ and using inequality (6.248a) with $j=0$, that

$$\big|\big(\xi_Y\chi_1(\vartheta(H^{(l)})-\vartheta^\infty(H^{(l)}))\big)(t,x)\big| \tag{6.259}$$
$$\le(1+t)^{-1+\rho-\tau}(1+t/(t-|x|))^\tau C_{|Y|+l}\overline{\mathcal{R}}^{(l)}_{N_+,N_++|Y|+l}(u;u_1,\dots,u_l),$$

for $(t,x) \in \text{supp}\ \chi_1$, $l \in \mathbb{N}$, $Y \in \Pi' \cap \cup(sl(2,\mathbb{C}))$, $0 \leq \tau \leq 1-\rho$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_1,\ldots,u_l \in E_\infty^{\circ\rho}$, where $C_{|Y|+l}$ depends only on $\|u\|_{E^\rho_{N_+}}$ and ρ. It follows from inequality (6.247), from the fact that $\vartheta^\infty(H^{(l)})$ is homogeneous of degree zero inside the forward light cone and from inequality (6.249) that

$$\begin{aligned} &\big|\big(\xi_Y(1-\chi_1)\vartheta^\infty(H^{(l)})\big)(t,x)\big| + \big|\big(\xi_Y(1-\chi_1)\vartheta(A^{(l)}-A_0^{(l)})\big)(t,x)\big| \\ &\quad \leq (1+t/(t-|x|))^\tau C_{|Y|+l}\overline{\mathcal{R}}^{(l)}_{N_+,N_++|Y|+l}(u;u_1,\ldots,u_l), \end{aligned} \tag{6.260}$$

for $(t,x) \in \text{supp}\ (1-\chi_1) \cap \{(t,x)\big| t > 0 \text{ and } t > |x|\}$, $Y \in \Pi' \cap U(sl(2,\mathbb{C}))$, $l \in \mathbb{N}$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_1,\ldots,u_l \in E_\infty^{\circ\rho}$, $\tau > 0$, where $C_{|Y|+l}$ depends only on τ, ρ and $\|u\|_{E^\rho_{N_+}}$. Since, inside the forward light cone

$$\begin{aligned} &\big(D^l(\varphi - \vartheta(A) - \vartheta^\infty(A^{(+)}-A))\big)(u;u_1,\cdots,u_l) \\ &\quad = \chi_1\big(\vartheta(H^{(l)}) - \vartheta^\infty(H^{(l)})\big) + (1-\chi_1)\big(\vartheta^\infty(H^{(l)}) + \vartheta(A_0^{(l)} - A^{(l)})\big) \end{aligned} \tag{6.261}$$

it follows from inequalities (6.259) and (6.260) that if e is the characteristic function of the support of χ_1 and $\overline{e}$ the characteristic function of supp $(1-\chi_1) \cap \{(t,x)\big| t > 0, |x| < t\}$, then inequalities (6.259) and (6.260) give that

$$\begin{aligned} &\big|\big(\xi_Y(D^l(\varphi - \vartheta(A) - \vartheta^\infty(A^{(+)}-A)))(u;u_1,\ldots,u_l)\big)(t,x)\big| \\ &\quad \leq \big(e(t,x)(1+t)^{-1+\rho-\tau_1}(1+t/(t-|x|))^{\tau_1} + \overline{e}(t,x)(1+t/(t-|x|))^{\tau_2}\big) \\ &\quad\quad C_{|Y|+l}\overline{\mathcal{R}}^{(l)}_{N_+,N_++|Y|+l}(u;u_1,\ldots,u_l), \end{aligned} \tag{6.262}$$

for $0 \leq |x| < t$, $t > 0$, $l \in \mathbb{N}$, $Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_1,\ldots,u_l \in E_\infty^{\circ\rho}$, $0 \leq \tau_1 \leq 1-\rho$, $\tau_2 > 0$, where $C_{|Y|+l}$ depends only on τ_2, ρ and $\|u\|_{E^\rho_{N_+}}$. Let

$$Q_j^\infty(Y_1,\ldots,Y_j;i_1,\ldots,i_j;(t,x)) = \big((\xi_{Y_1}\vartheta^\infty(H^{(i_1)}))\cdots(\xi_{Y_j}\vartheta^\infty(H^{(i_j)}))\big)(t,x), \tag{6.263}$$

where $Y_1,\ldots,Y_j$, $i_1,\ldots,i_j$ are as in expression (6.239). Let a_q (resp. b_q) be the q^{th} factor in the product defining $Q_j(Y_1,\ldots,Y_j;i_1,\ldots,i_j)$ (resp. $Q_j^\infty(Y_1,\ldots,Y_j;i_1,\ldots,i_j)$) in expression (6.258a) (resp. (6.263)). Then

$$|(Q_j - Q_j^\infty)(Y_1,\ldots,Y_j;i_1,\ldots,i_j)| \leq C_j \sum_{1\leq q\leq j} |a_q - b_q| M_q,$$

where M_q is a monomial of degree $j-1$ given by

$$M_q = \sum c_1 \cdots c_{q-1}c_{q+1}\cdots c_q,$$

where the domain of summation is defined by $c_p \in \{|a_p|,|b_p|\}$ for $p \in \{1,\ldots,q-1\} \cup \{q+1,\ldots,j\}$. We estimate $M_q(t,x)$ for $t > 0$, $|x| < t$ by using inequality (6.252), with $0 < \tau \leq 1/2$, and by using inequality (6.249), with $Z = \mathbb{I}$, together with the fact that

$\xi_Y \vartheta^\infty(H^{(l)})$ is homogeneous of degree zero. We estimate $|a_q - b_q|(t,x)$, for $t>0$, $|x|<t$, by using inequality (6.262) with $\tau_1 = 1-\rho$ and $0<\tau_2 \le 1/2$ and by observing that $t^{2(1-\kappa)} \le 4t/(t-|x|)$ in the support of $\overline{e}$ and that

$$\begin{aligned}&e(t,x)(1+t)^{-2(1-\rho)}(1+t/(t-|x|))^{1-\rho} + \overline{e}(t,x)(1+t/(t-|x|))^{\tau_2}\\ &\quad \le C(1+t)^{-2(1-\rho)}(1+t/(t-|x|))^{\tau_2+(1-\rho)/(1-\kappa)},\end{aligned}$$

for $t>0$, $|x|<t$, where C is independent of (t,x). Since $t/(t-|x|) \le 2t^2/(t^2-|x|^2)$ for $t>0$, $|x|<t$, it then follows from Theorem A.1, after redefinition of N_+, that

$$\begin{aligned}&\|(1+\lambda_1(t))^{k/2}(Q_j - Q_j^\infty)(Y_1,\ldots,Y_j;i_1,\ldots,i_j;(t,\cdot))\phi^{(i_{j+1})}_{(\varepsilon)Y_{j+1}Z}(t)\|_D \\ &\quad \le t^{-2(1-\rho)}C_{|Y|+|Z|+k+l}\overline{\mathcal{R}}^{(l)}_{N_+,N_++|Y|+|Z|+k+l}(u;u_1,\ldots,u_l)\end{aligned} \tag{6.264a}$$

and that

$$\begin{aligned}&\|(1+\lambda_1(t))^{k/2}(Q_j - Q_j^\infty)(Y_1,\ldots,Y_j;i_1,\ldots,i_j;(t,\cdot))\phi^{(i_{j+1})}_{(\varepsilon)Y_{j+1}Z}(t)\|_{L^\infty} \\ &\quad \le t^{-2(1-\rho)-3/2}C_{|Y|+|Z|+k+l}\;\overline{\mathcal{R}}^{(l)}_{N_+,N_++|Y|+|Z|+k+l}(u;u_1,\ldots,u_l)\end{aligned} \tag{6.264b}$$

$t>0$, where $Y_1,\ldots,Y_{j+1};i_1,\ldots,i_{j+1}$ are as in expression (6.239), where $C_{|Y|+|Z|+k+l}$ depends only on ρ and $\|u\|_{E^\rho_{N_+}}$ for τ and τ_2 fixed and where we have used that $|Y_q|+i_q \ge 1$ for $q \le j$. $\xi^D_{YZ}((D^l\psi_0^{(+)})(u;u_1,\ldots,u_l))$, $Y \in \Pi' \cap U(sl(2,\mathbb{C}))$, $Z \in \Pi' \cap U(\mathbb{R}^4)$ is a solution of the Dirac equation with initial condition $T^{D1}_{YZ}((D^l\alpha_+)(u;u_1,\ldots,u_l))$. Since we have already proved statement i) of the proposition we can apply Theorem A.1 to this solution. Hence there exists two functions $\beta^{(+)(l)}_{(\varepsilon)YZ} \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^3 - \{0\})$, $\varepsilon = \pm$, given by formula (A.1), homogeneous of degree $-3/2$ and with support contained in the forward light cone such that, if

$$\begin{aligned}(\phi^{(+)(l)}_{(\varepsilon)YZ}(t))(x) &= e^{i\varepsilon m(t^2-|x|^2)^{1/2}}\beta^{(+)(l)}_{(\varepsilon)YZ}(t,x),\\ \alpha^{(+)(l)} &= (D^l\alpha_+)(u;u_1,\ldots,u_l),\end{aligned}$$

then there exists $N \in \mathbf{N}$ such that

$$\begin{aligned}&\|(1+\lambda_1(t))^{k/2}\big(P_\varepsilon(-i\partial)U^{D1}_{\exp(tP_0)}T^{D1}_{YZ}\alpha^{(+)(l)} - \phi^{(+)(l)}_{(\varepsilon)YZ}(t)\big)\|_D \\ &\quad \le t^{-1}C_k\|T^{D1}_{YZ}\alpha^{(+)(l)}\|_{D_{N+k}},\end{aligned}$$

for $t>0$, $k \in \mathbb{N}$, where C_k is independent of t. Theorem A.1 also gives a similar statement for the L^∞-norm. Hence according to statement i) of this proposition

$$\begin{aligned}&t\|(1+\lambda_1(t))^{k/2}\big(P_\varepsilon(-i\partial)U^{D1}_{\exp(tP_0)}T^{D1}_{YZ}\alpha^{(+)(l)} - \phi^{(+)(l)}_{(\varepsilon)YZ}(t)\big)\|_D \\ &\quad + t^{-1+2\rho}\|(\delta(t))^{3/2+2(1-\rho)}(1+\lambda_1(t))^{k/2}\big(P_\varepsilon(-i\partial)U^{D1}_{\exp(tP_0)}T^{D1}_{YZ}\alpha^{(+)(l)} - \phi^{(+)(l)}_{(\varepsilon)YZ}(t)\big)\|_{L^\infty} \\ &\quad \le C_{|Y|+|Z|+k+l}\overline{\mathcal{R}}^{(l)}_{N_+,N_++|Y|+|Z|+l+k}(u;u_1,\ldots,u_l),\end{aligned} \tag{6.265}$$

for $t \geq 1$, $\varepsilon = \pm$, $Y \in \Pi' \cap U(\mathfrak{sl}(2,\mathbb{C}))$, $Z \in \Pi' \cap U(\mathbb{R}^4)$, $k \in \mathbb{N}$, $l \in \mathbb{N}$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_1, \ldots, u_l \in E_\infty^{\circ\rho}$, where $C_{|Y|+|Z|+k+l}$ depends only on ρ and $\|u\|_{E^\rho_{N_+}}$. We have here suitably redefined N_+. It follows from (A.1a) that $\beta^{(+)(0)}_{(\varepsilon)Z} = e^{i\vartheta^\infty(A^{(+)}-A)}\beta^{(0)}_{(\varepsilon)Z}$. The construction (A.1a)–(A.3) is $\mathrm{sl}(2,\mathbb{C})$ covariant, which shows that

$$\beta^{(+)(0)}_{(\varepsilon)YZ} = \xi^D_Y\big((D^l\beta^{(+)(0)}_{(\varepsilon)Z})(u; u_1, \ldots, u_l)\big), \quad Y \in \Pi' \cap U(sl(2,\mathbb{C})).$$

Since $\xi_X\Lambda = 0$ for $\Lambda(t,x) = \sqrt{t^2 - |x|^2}$, $t > 0$, $|x| < t$, $X \in {}_fraksl(2,\mathbb{C})$, it follows that $\sum_\varepsilon \phi^{(+)(l)}_{(\varepsilon)YZ}$ is the sum of the expressions (6.257) with $\varphi^{(i_q)} - \vartheta(A^{(i_q)})$ replaced by $\vartheta^\infty(H^{(i_q)})$ and $\xi^D_{Y_{j+1}Z}\psi_0^{(i_{j+1})}$ replaced by $\sum_\varepsilon \phi^{(i_{j+1})}_{(\varepsilon)Y_{j+1}Z}$. Hence, inequality (6.256), development (6.257) and inequalities (6.258b), (6.258c), (6.264a), (6.264b) and (6.265) give that

$$\begin{aligned}(1+t)^{2(1-\rho)}&\|(1+\lambda_1(t))^{k/2}\big(\xi^D_Y((D^l(e^{i(\varphi-\vartheta(A))}\psi_0 - \psi_0^{(+)}))(u; u_1, \ldots, u_l))\big)(t)\|_D \\ &+ \|(\delta(t))^{3/2+2(1-\rho)}(1+\lambda_1(t))^{k/2}\big(\xi^D_Y((D^l(e^{i(\varphi-\vartheta(A))}\psi_0 - \psi_0^{(+)}))(u; u_1, \ldots, u_l))\big)(t)\|_{L^\infty} \\ &\leq C_{|Y|+k+l}\overline{\mathcal{R}}^{(l)}_{N_+,N_++|Y|+k+l}(u; u_1, \ldots, u_l),\end{aligned} \tag{6.266}$$

for $t \geq 0$, $Y \in \Pi'$, k, $l \in \mathbb{N}$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_1, \ldots, u_l \in E_\infty^{\circ\rho}$, where $C_{|Y|+k+l}$ depends only on ρ and $\|u\|_{E^\rho_{N_+}}$. Since

$$e^{i\varphi}\psi - \psi_0^{(+)} = e^{i(\varphi-\vartheta(A))}(e^{i\vartheta(A)}\psi - \psi_0) + e^{i(\varphi-\vartheta(A))}\psi_0 - \psi_0^{(+)},$$

inequalities (6.254) and (6.266) and theorem 6.10 for A give that

$$\begin{aligned}&\|\!|(D^l(A - A_0^{(+)}, e^{i\varphi}\psi - \psi_0^{(+)}))(u; u_1, \ldots, u_l)\|\!|_{\rho',\varepsilon,L} \\ &\qquad \leq C_{L+l}\overline{\mathcal{R}}^{(l)}_{N_+,N_++L+l}(u; u_1, \ldots, u_l),\end{aligned} \tag{6.267}$$

for L, $l \in \mathbb{N}$, $1/2 < \rho' \leq 1$, $\varepsilon(0) > 0$, $\varepsilon(1) \geq \rho$, $u \in \mathcal{O}_{1,\infty(+)}$, $u_1, \ldots, u_l \in E_\infty^{\circ\rho}$, where C_{L+l} depends only on ρ', ε, ρ and $\|u\|_{E^\rho_{N_+}}$, where $u_+ = (f_+, \dot{f}_+, \alpha_+) = E_+(u)$, $(A_0^{(+)}, \dot{A}_0^{(+)}, \psi_0^{(+)})(t) = U^1_{\exp(tP_0)}u_+$ and where $(A, \dot{A}, \psi) = h(v)$ is the solution of the M-D equations given by theorem 6.15 with $v = \Omega_1^{(+)}(u)$.

To prove that $E_+ \colon \mathcal{O}_{1,\infty(+)} \to E_\infty^{\circ\rho}$ has a differentiable inverse $E_+^{-1} \colon \mathcal{O}_{\infty(+)} \to \mathcal{O}_{1,\infty(+)}$, where $\mathcal{O}_{\infty(+)}$ is an open neighbourhood of zero in $E_\infty^{\circ\rho}$, we shall use the implicit function theorem for Fréchet spaces. We therefore have to prove that the linear continuous operator $DE_+(u) \in L(E_\infty^{\circ\rho}, E_\infty^{\circ\rho})$ has a right inverse $w(u) \in L(E_\infty^{\circ\rho}, E_\infty^{\circ\rho})$, i.e. $(DE_+)(u; w(u)V_+) = V_+$ for $u \in \mathcal{O}_{1,\infty(+)}$ and $V_+ \in E_\infty^{\circ\rho}$. Since $\Omega_1^{(+)} \colon \mathcal{O}_{1,\infty(+)} \to \mathcal{U}_{\infty(0)}$ is a diffeormorphism and if $V'(u)$ is a tangent vector of $\mathcal{U}_{\infty(0)}$ at the point $\Omega_1^{(+)}(u)$ satisfying the equation $(DG)(u; V'(u)) = V_+$ for $V_+ \in E_\infty^{\circ\rho}$, where $G = E_+ \circ (\Omega_1^{(+)})^{-1}$ then it follows by the chain rule that $V(u) = (D(\Omega_1^{(+)})^{-1}(\Omega_1^{(+)}(u); V'(u))$ is a solution of the equation

$(DE_+)(u;V(u)) = V_+$. Let $h(\Omega_1^{(+)}(u)) = (A,\dot{A},\psi)$ be the solution of the M-D equations, with initial condition $\Omega_1^{(+)}(u) = (A(0),\dot{A}(0),\psi(0))$ at $t=0$, given by Theorem 6.15. If $V(u) \in E_\infty^{\circ\rho}$ and $(D(h\circ\Omega_1^{(+)}))(u;V(u)) = (a,\dot{a},\Psi)$, then $(a,\dot{a},\Psi)$ is the solution of the equations given by the derivative of the M-D equations, i.e.

$$\frac{d}{dt}(a(t),\dot{a}(t),\Psi(t)) = (DT_{P_0})\big((A(t),\dot{A}(t),\psi(t));(a(t),\dot{a}(t),\Psi(t))\big), \tag{6.268a}$$

with initial data at $t=0$ given by

$$(a(0),\dot{a}(0),\Psi(0)) = V'(u) = (D\Omega_1^{(+)})(u;V(u)). \tag{6.268b}$$

Let $\tilde{h}(u) = (A,\dot{A},e^{i\varphi}\psi), \tilde{\psi} = e^{i\varphi}\psi$ and $(D\tilde{h})(u;V(u)) = (a,\dot{a},\tilde{\Psi})$. Since $\varphi(t)=0$ for $0\le t\le 2$ it follows from (6.268a) and (6.268b) that

$$\big(i\gamma^\mu\partial_\mu + m - \gamma^\mu(A_\mu - \partial_\mu\varphi)\big)\tilde{\Psi} = \gamma^\mu\big(a_\mu - \partial_\mu(D\varphi)(u;V(u))\big)\tilde{\psi}, \tag{6.269a}$$

$$\Box a_\mu = \tilde{\Psi}^+\gamma_0\gamma_\mu\tilde{\psi} + \tilde{\psi}^+\gamma_0\gamma_\mu\tilde{\Psi}, \tag{6.269b}$$

and

$$(a(0),\dot{a}(0),\tilde{\Psi}(0)) = V'(u). \tag{6.269c}$$

Moreover defining $V_+(u) = (DG)(u;V'(u))$ then $V_+(u) = (DE_+(u;V(u))$ by the definition of G and so it follows from inequality (6.267) that

$$\|(a(t),\dot{a}(t),\tilde{\Psi}(t)) - U^1_{\exp(tP_0)}V_+(u)\|_D \to 0, \tag{6.269d}$$

when $t\to\infty$. Therefore, if for fixed $V_+ \in E_\infty^{\circ\rho}$ equations (6.269a) and (6.269b) have, for each $u \in \mathcal{O}_{1,\infty(+)}$, a solution $(a,\dot{a},\tilde{\Psi})$, in a suitable space, satisfying the asymptotic condition (6.269d), then $V'(u)$ defined by (6.269c) is a solution of the equation $(DG)(u;V'(u)) = V_+$. Let $V_+ = (g_+,\dot{g}_+,\beta_+) \in E_\infty^{\circ\rho}$, $\phi'(t) = e^{i(\varphi-\vartheta(A))}U^{D1}_{\exp(tP_0)}\beta_+$, $\psi' = e^{i\vartheta(A)}\psi$, $\Psi' = e^{i(\vartheta(A)-\varphi)}(\tilde{\Psi} - U^{D1}_{\exp(tP_0)}\beta_+)$ and let $b_\mu = \partial_\mu((D\varphi)(u;V(u)))$. Equations (6.269a) and (6.269b) give that

$$\big(i\gamma^\mu\partial_\mu + m - \gamma^\mu(A_\mu - \partial_\mu\vartheta(A))\big)\Psi' = \gamma^\mu\big(a_\mu - b_\mu\big)\psi' + \gamma^\mu(A_\mu - \partial_\mu\varphi)\phi', \tag{6.270a}$$

$$\Box a_\mu = (\Psi'+\phi')^+\gamma_0\gamma_\mu\psi' + (\psi')^+\gamma_0\gamma_\mu(\Psi'+\phi') \tag{6.270b}$$

and the asymptotic condition (6.269d) gives that

$$\|\Psi'(t)\|_D + \|(a(t),\dot{a}(t)) - U^{M1}_{\exp(tP_0)}(g_+,\dot{g}_+)\|_{M_0^\rho} \to 0, \tag{6.270c}$$

when $t\to\infty$. Denoting for the moment by $\alpha\mapsto J^{(+)}(\alpha)$ the function defined in (1.21). Since $\hat{\alpha}(k) = c(k)\hat{\alpha}_+(k)$, where $c(k)\in\mathbb{C}$ and $|c(k)|=1$, it follows that $J^{(+)}(\alpha) = J^{(+)}(\alpha_+)$. According to the definition of φ, we therefore obtain that

$$b_\mu = \partial_\mu((D\varphi)(u_+;V_+)), \tag{6.271}$$

where $u_+ = E_+(u)$. We shall prove that system (6.270a)–(6.270b) have a solution (a,Ψ') with finite norms $|||(a-a_0,\Psi')|||_{\rho',r,L}$ for $L\in\mathbb{N}$, $1/2<\rho'\le 1$ and $r=(r(0),r(1))$, $r(0)>0$, $r(1)\ge\rho$. This is done by using Theorem 5.14 with $t_0=\infty$ and by using inequalities (6.176) and (6.177). Since the proof is standard at this point, though tedious, we only state the result, which after taking N_+ sufficiently large and $\mathcal{O}_{1,\infty(+)}$ sufficiently small, reads:

$$|||(D^l(a-a_0,\Psi'))(u;u_1,\ldots,u_l)|||_{\rho',r,L}\le C_{L+l}\overline{\mathcal{R}}^{(l+1)}_{N_+,N_++L+l}(u;u_1,\ldots,u_l,V_+), \tag{6.273}$$

for L, $l\in\mathbb{N}$, $1/2<\rho'<1$, $r=(r(0),r(1))$, $r(0)>0$, $r(1)\ge\rho$, $u\in\mathcal{O}_{1,\infty(+)}$, $u_1,\ldots,u_l\in E^{\circ\rho}_\infty$, where C_{L+l} depends only on ρ', r, ρ and $\|u\|_{E^\rho_{N_+}}$ and where $(a_0(t),\dot a_0(t)) = U^{M1}_{\exp(tP_0)}(g,\dot g)$. It then follows using Lemma 2.19 that, if $V'(u)$ is defined by (6.269), then

$$\|(D^lV')(u;u_1,\ldots,u_l)\|_{D_n}\le C_{n+l}\overline{\mathcal{R}}^{(l+1)}_{N_+,N_++n+l}(u;u_1,\ldots,u_l,V_+), \tag{6.274}$$

for n, $l\in\mathbb{N}$, $V_+\in E^{\circ\rho}_\infty$, $u\in\mathcal{O}_{1,\infty(+)}$, $u_1,\ldots,u_l\in E^{\circ\rho}_\infty$, where C_{n+l} depends only on ρ and $\|u\|_{E^\rho_{N_+}}$. Defining $V(u)=(D(\Omega^{(+)}_1)^{-1})(\Omega^{(+)}_1(u);V'(u))$, Theorem 6.13 gives, after redefining N_+, that

$$\|(D^lV)(u;u_1,\ldots,u_l)\|_{D_n}\le C_{n+l}\overline{\mathcal{R}}^{(l+1)}_{N_+,N_++l+n}(u;u_1,\ldots,u_l,V_+), \tag{6.275}$$

for n, $l\in\mathbb{N}$, $V_+\in E^{\circ\rho}_\infty$, $u\in\mathcal{O}_{1,\infty(+)}$, $u_1,\ldots,u_l\in E^{\circ\rho}_\infty$, where C_{n+l} depends only on ρ and $\|u\|_{E^\rho_{N_+}}$. It follows from inequality (6.275) that $w(u)$, defined by $V(u)=w(u)V_+$, is a right inverse of $DE_+(u)$, i.e. $(DE_+)(u;V(u))=V_+$, satisfying the hypotheses of the implicit function theorem in Fréchet spaces (Theorem 4.1.1. of [17]), so there exists an integer M_+, an open neighbourhood $\mathcal{O}_{M_+(+)}$ of zero in $E^{\circ\rho}$ and a C^∞ map $H\colon\mathcal{O}_{\infty(+)}=\mathcal{O}_{M_+(+)}\cap E^{\circ\rho}_\infty\to\mathcal{O}_{1,\infty(+)}$ such that $E_+(H(u))=u$ for $u\in\mathcal{O}_{\infty(+)}$. A similar discussion as in the end of the proof of Theorem 6.13 shows that $\mathcal{O}_{\infty(+)}$ and $\mathcal{O}_{1,\infty(+)}$ can be chosen such that $E_+\colon\mathcal{O}_{1,\infty(+)}\to\mathcal{O}_{\infty(+)}$ is a diffeomorphism and such that the inequality of statement ii) of the proposition is satisfied. This proves statement ii).

Statement iii) follows from statement ii) and inequality (6.267). Statement iv) is a particular case of statement iii). This proves the proposition.

We could, of course, have defined a map E_- which satisfies the suitably modified statements in Proposition 6.16, when $t\to-\infty$. We can therefore define two new wave operators $\Omega^{(\varepsilon)}\colon\mathcal{O}_{\infty(\varepsilon)}\to\mathcal{U}_{\infty(0)}$, where

$$\Omega^{(+)}=\Omega^{(+)}_1\circ E_+^{-1},\quad \Omega^{(-)}=\Omega^{(-)}_1\circ E_-^{-1}. \tag{6.276}$$

We shall extend the diffeomorphisms $\Omega^{(\varepsilon)}\colon\mathcal{O}_{\infty(\varepsilon)}\to\mathcal{U}_{\infty(0)}$, $\varepsilon=\pm$, to a diffeomorphism $\Omega_\varepsilon\colon\mathcal{O}^{(\varepsilon)}_\infty\to\mathcal{U}_\infty$, where $\mathcal{U}_{\infty(0)}\subset\mathcal{U}_\infty$ (resp. $\mathcal{O}_{\infty(\varepsilon)}\subset\mathcal{O}^{(\varepsilon)}_\infty$) is an open neighbourhood of zero in V^ρ_∞ (resp. $E^{\circ\rho}_\infty$), such that the nonlinear representation $X\mapsto T_X$ of the Poincaré Lie algebra is integrable to a *nonlinear representation* $g\mapsto U_g$ *of the Poincaré group* $\mathcal{P}_0$ on $\mathcal{U}_\infty$ and such that Ω_ε, $\varepsilon=\pm$, are modified wave operators. We shall do this in two

steps. First we construct, using the explicit covariance of the M-D equations, an invariant set S of solutions of the M-D equations and prove that each solution in S has an initial condition at $t=0$ in the manifold V^{ρ}_{∞} and satisfies asymptotic conditions analog to those in statement iv) of Proposition 6.16 when $\varepsilon t \to \infty, \varepsilon = \pm$. Second U_g is defined by the action of $\mathcal{P}_0$ on S, the extension Ω_ε is defined by considering $U_g \circ \Omega^{(\varepsilon)} \circ U^{(\varepsilon)}_{g-1}$ and Ω_ε is proved to be a diffeomorphism by using the implicit function theorem in Fréchet spaces. With suitably chosen finite-dimensional matrix representations $\Lambda \mapsto V(\Lambda)$, $\Lambda \mapsto V'(\Lambda)$ of $SL(2,\mathbb{C})$, solutions (A,ψ) of the M-D equations transform under $g=(a,\Lambda)\in\mathcal{P}_0$ as

$$A_g(y) = V(\Lambda)A(V(\Lambda)^{-1}(y-a)) \tag{6.277a}$$

and

$$\psi_g(y) = V'(\Lambda)\psi(V(\Lambda)^{-1}(y-a)), \quad y \in \mathbb{R}^4. \tag{6.277b}$$

For $v \in \mathcal{U}_{\infty(0)}$ and $h(v) = (A, \dot{A}, \psi)$ as in Theorem 6.15, we introduce the notation, $h_g(v) = (A_g, \dot{A}_g, \psi_g)$,

$$S_0 = \{h_e(v) \big| v \in \mathcal{U}_{\infty(0)}\}, \quad S = \{h_g(v) \big| v \in \mathcal{U}_{\infty(0)}, g \in \mathcal{P}_0\}, \tag{6.277c}$$

$$\mathcal{U}_\infty = \{(h_g(v))(t,.) \big| h_g(v) \in S, t=0\}, \tag{6.277d}$$

where e is the identity element in $\mathcal{P}_0$. It follows from Theorem 6.15 and the definitions of S and $\mathcal{U}_\infty$ that $S \subset C^\infty(\mathbb{R}^4, \mathbb{R}^4 \oplus \mathbb{R}^4 \oplus \mathbb{C}^4)$ and $\mathcal{U}_\infty \subset C^\infty(\mathbb{R}^3, \mathbb{R}^4 \oplus \mathbb{R}^4 \oplus \mathbb{C}^4)$ respectively.

Lemma 6.17. *Let $\Gamma_g(A,\psi) = (A_g, \psi_g)$ be defined by (6.277a) and (6.277b). In the situation of statement iii) of proposition 6.16 for $t \to \infty$ and its analog for $t \to -\infty$, there exists N_+ such that*

$$\begin{aligned}&\|\!|\Gamma_g\big((D^l(A-A_0^{(+)}, e^{i\varphi}\psi - \psi_0^{(+)}))(u_+; u_{+1}, \dots, u_{+l})\big)\|\!|^0_{\rho',r,L}\\ &\quad \le C_{L+l}\big(\mathcal{R}^l_{N_+,L+l}(u_{+1},\dots,u_{+l}) + \|u_+\|_{E^\rho_{N_++L+l}} \|u_{+1}\|_{E^\rho_{N_+}} \cdots \|u_{+l}\|_{E^\rho_{N_+}}\big),\end{aligned}$$

for $L,\ l \in \mathbb{N}$, $1/2 < \rho' \le 1$, $r = (r(0), r(1))$, $r(0) > 0$, $r(1) \ge \rho$, $u_+ \in \mathcal{O}_{\infty(+)}$, $u_{+1},\dots,u_{+l} \in E^{\circ\rho}_\infty$, where C_{L+l} depends only on ρ', r, ρ and $\|u_+\|_{E^\rho_{N_+}}$ and where

$$\begin{aligned}\|\!|(a,\Phi)\|\!|^0_{\rho',r,L} = \sum_{i\in\{0,1\}} \sum_{\substack{Y\in\boldsymbol{\sigma}^i\\ k+|Y|\le L}} \sup_{t\ge 0} \Big(&(1+t)^{\rho'-1/2}\|(\xi^M_Y a, \xi^M_{P_0Y} a)(t)\|_{M^{\rho'}_0}\\ &+ (1+t)^{2(1-\rho)}\|(1+\lambda_1(t))^{k/2}(\xi^D_Y\Phi)(t)\|_{D_0}\\ &+ \|(\delta(t))^{1+i-r(i)}(\xi^M_Y a)(t)\|_{L^\infty}\\ &+ \|(\delta(t))^{3/2+2(1-\rho)}(1+\lambda_1(t))^{k/2}(\xi^D_Y\Phi)(t)\|_{L^\infty}\Big).\end{aligned}$$

Proof. For a moment we write $\lambda_1(y)$ (resp. $\delta(y)$) instead of $(\lambda_1(t))(x)$ (resp. $(\delta(t))(x)$) for $y=(t,x)\in\mathbb{R}^4$. It follows from the definition of λ_1 and δ that there exists a constant $C_g > 0$ such that if $g = (a,\Lambda) \in \mathcal{P}_0$ then

$$C_g^{-1}(1+\lambda_1(y)) \le 1 + \lambda_1(\Lambda^{-1}(y-a)) \le C_g(1+\lambda_1(y)) \tag{6.278a}$$

and

$$C_g^{-1}\delta(y) \leq \delta(\Lambda^{-1}(y-a)) \leq C_g\delta(y), \tag{6.278b}$$

for each $y \in \mathbb{R}^4$. It follows from these two inequalities that, if

$$\Delta_Y^{D(l)} = \xi_Y^D(D^l(e^{i\varphi}\psi - \psi_0^{(+)}))(u_+; u_{+1}, \ldots, u_{+l}), \quad Y \in \Pi', \tag{6.279a}$$

then

$$\begin{aligned} &\sup_{y\in\mathbb{R}^+\times\mathbb{R}^3} \Big((\delta(y))^{3/2+2(1-\rho)}(1+\lambda_1(y))^{k/2}\big|V'(\Lambda)\Delta_Y^{D(l)}(V(\Lambda)^{-1}(y-a))\big|\Big) \\ &\quad\leq C_g\Big(\sup_{y\in H_1}\big((\delta(y))^{3/2+2(1-\rho)}(1+\lambda_1(y))^{k/2}\big|\Delta_Y^{D(l)}(y)\big|\big) \\ &\qquad + \sup_{y\in H_2}\big((\delta(y))^{3/2+2(1-\rho)}(1+\lambda_1(y))^{k/2}\big|\Delta_Y^{D(l)}(y)\big|\big)\Big), \quad g \in \mathcal{P}_0, \end{aligned} \tag{6.279b}$$

where $H_1 = \{y \in \mathbb{R}^4 \big| y_0 \geq 0 \text{ and } V(\Lambda)y + a \in \mathbb{R}^+ \times \mathbb{R}^3\}$ and $H_2 = \{y \in \mathbb{R}^4 \big| y_0 \leq 0$ and $V(\Lambda)y + a \in \mathbb{R}^+ \times \mathbb{R}^3\}$. Since $H_2 \cap \{y \in \mathbb{R}^4 \big| y^\mu y_\mu \geq 0\}$ is a bounded region of $\mathbb{R}^4$, it follows that $\delta(y) \leq C_g(1+\lambda_1(y))$ for $y \in H_2$. Hence the second term on the right-hand side is, according to the analog of Proposition 6.16 for $t \to -\infty$ and Theorem 5.7 applied to a free field, majorized by

$$\begin{aligned} &C_g \sup_{y\in H_2}\big((\delta(y))^{3/2}(1+\lambda_1(y))^{k/2+2(1-\rho)}\big|\Delta_Y^{D(l)}(y)\big|\big) \\ &\quad\leq C_{g,|Y|+k+l}\big(\mathcal{R}^l_{N_+,|Y|+k+l}(u_{+1},\ldots,u_{+l}) + \|u_+\|_{E^\rho_{N_++|Y|+k+l}}\|u_{+1}\|_{E^\rho_{N_+}}\cdots\|u_{+l}\|_{E^\rho_{N_+}}\big), \end{aligned}$$

where N_+ has been suitably chosen. Application of Theorem 6.16 to the first term on the right-hand side of inequality (6.279b) then gives that

$$\begin{aligned} &\sup_{y\in\mathbb{R}^+\times\mathbb{R}^3}\big((\delta(y))^{3/2+2(1-\rho)}(1+\lambda_1(y))^{k/2}\big|V'(\Lambda)\Delta_Y^{D(l)}(V(\Lambda)^{-1}(y-a))\big|\big) \\ &\quad\leq C_{g,|Y|+k+l}\big(\mathcal{R}^l_{N_+,|Y|+k+l}(u_{+1},\ldots,u_{+l}) + \|u_+\|_{E^\rho_{N_++|Y|+k+l}}\|u_{+1}\|_{E^\rho_{N_+}}\cdots\|u_{+l}\|_{E^\rho_{N_+}}\big), \end{aligned} \tag{6.280}$$

where $C_{g,|Y|+k+l}$ depends only on ρ and $\|u_+\|_{E^\rho_{N_+}}$. Let

$$\Delta_Y^{M(l)} = \xi_Y^M\big((D^l(A - A_0^{(+)}))(u_+; u_{+1}, \ldots, u_{+l})\big). \tag{6.281a}$$

Then it follows from inequalities (6.278a) and (6.278b) that

$$\begin{aligned} &\sup_{y\in\mathbb{R}^+\times\mathbb{R}^3}\big((\delta(y))^{1+i-r(i)}\big|V(\Lambda)\Delta_Y^{M(l)}(V(\Lambda)^{-1}(y-a))\big|\big) \\ &\quad\leq C_g\Big(\sup_{y\in H_1}\big((\delta(y))^{1+i-r(i)}\big|\Delta_Y^{M(l)}(y)\big|\big) + \sup_{y\in H_2}\big((\delta(y))^{1+i-r(i)}\big|\Delta_Y^{M(l)}(y)\big|\big)\Big), \end{aligned} \tag{6.281b}$$

where $Y \in \boldsymbol{\sigma}^i$, $i \in \{0,1\}$. Let $F(t) = A_0^{(+)}(t) - A_0^{(-)}(t)$, $t \in \mathbb{R}$, where $A_0^{(+)}$ is given by Proposition 6.16 and $A_0^{(-)}$ by the analog of Proposition 6.16 for $t \to -\infty$. By the

definition of $A_0^{(\varepsilon)}$, $\Box A_0^{(\varepsilon)} = 0$ for $\varepsilon = \pm$, so $\Box\xi_Y^M F = 0$ for $Y \in \Pi'$. Since $(\xi_Y^M F)(0) = (\xi_Y^M(A_0^{(+)} - A))(0) - (\xi_Y^M(A_0^{(-)} - A))(0)$, it follows from Proposition 6.16 and its analog for $t \to -\infty$ that $((\xi_Y^M F^{(l)})(0), (\xi_{P_0 Y}^M F^{(l)})(0)) = T_Y^{M1}((F^{(l)}(0), \dot F^{(l)}(0)) \in M_0^{\rho'}$ for $Y \in \Pi'$, $1/2 < \rho' \le 1$, where $\dot F^{(l)}(0) = (\xi_{P_0}^M F^{(l)})(0)$ and $F^{(l)}$ is the l^{th} derivative of F. Hence, by the definition of the spaces $M_n^{\rho'}$, $n \ge 0$, $(F^{(l)}(0), \dot F^{(l)}(0)) \in M_n^{\rho'}$ for $n \ge 0$ and $1/2 < \rho' \le 1$. Proposition 2.15 then gives that

$$(1+|t|+|x|)^{3/2-\rho'}|(\xi_Y^M F^{(l)})(t,x)| + (1+|t|+|x|)(1+\big||t|-|x|\big|)^{3/2-\rho'}|(\xi_{P_\mu Y}^M F^{(l)})(t,x)| \le C_{|Y|,\rho'}\|(F^{(l)}(0), \dot F^{(l)}(0))\|_{M_{|Y|+2}^{\rho'}},$$

for $Y \in \Pi'$, $0 \le \mu \le 3$ and $1/2 < \rho' < 1$. Since $\delta(y) \le C_g(1+\big||y_0|-|\vec y|\big|)$ for $y = (y_0, \vec y) \in H_2$ it follows, using the bounds in Proposition 6.16 and its analog when $t \to -\infty$ that

$$\sup_{y\in H_2}\big((\delta(y))^{1+i-r(i)}|(\xi_Y^M F^{(l)})(y)|\big) \le C_{g,|Y|+l}\big(\mathcal{R}^l_{N_+,|Y|+l}(u_{+1},\dots,u_{+l}) + \|u_+\|_{E^\rho_{N_++|Y|+l}}\|u_{+1}\|_{E^\rho_{N_+}}\cdots\|u_{+l}\|_{E^\rho_{N_+}}\big), \tag{6.282}$$

where $Y \in \boldsymbol{\sigma}^i$, $i \in \{0,1\}$. The definition of F gives that $A - A_0^{(+)} = A - A_0^{(-)} - F$. Inequality (6.282) and the analog of Proposition 6.16 for $t \to -\infty$ give that

$$\sup_{y\in H_2}\big((\delta(y))^{1+i-r(i)}|\Delta_Y^{M(l)}(y)|\big) \le C_{g,|Y|+l}\big(\mathcal{R}^l_{N_+,|Y|+l}(u_{+1},\dots,u_{+l}) + \|u_+\|_{E^\rho_{N_++|Y|+l}}\|u_{+1}\|_{E^\rho_{N_+}}\cdots\|u_{+l}\|_{E^\rho_{N_+}}\big), \tag{6.283a}$$

where $Y \in \boldsymbol{\sigma}^i$, $i \in \{0,1\}$. This inequality and Proposition 6.16 or the first term on the right-hand side of inequality (6.181b) give that

$$\sup_{y\in\mathbb{R}^+\times\mathbb{R}^3}\big((\delta(y))^{1+i-r(i)}\big|V(\Lambda)\Delta_Y^{M(l)}(V(\Lambda)^{-1}(y-a))\big|\big) \le C_{g,|Y|+l}\big(\mathcal{R}^l_{N_+,|Y|+l}(u_{+1},\dots,u_{+l}) + \|u_+\|_{E^\rho_{N_++|Y|+l}}\|u_{+1}\|_{E^\rho_{N_+}}\cdots u_{+l}\|_{E^\rho_{N_+}}\big), \tag{6.283b}$$

for $Y \in \boldsymbol{\sigma}^i$, $i \in \{0,1\}$, $l \in \mathbb{N}$, where $C_{g,|Y|+l}$ depends only on r, ρ and $\|u_+\|_{E^\rho_{N_+}}$.

We now go back to the usual notations $(\lambda_1(t))(x)$ and $(\delta(t))(x)$ for $(t,x) \in \mathbb{R}\times\mathbb{R}^3$. Since $(\lambda_1(t))(x) \ge |x|$ for $|x| \ge |t|$, it follows from inequality (6.280) that

$$(1+t)^{2(1-\rho)}\|(1+\lambda_1(t))^{k/2}V'(\Lambda)\Delta_Y^{D(l)}(V(\Lambda)^{-1}((t,\cdot)-a))\|_{D_0} \le C_{g,|Y|+k+l}\big(\mathcal{R}^l_{N_+,|Y|+k+l}(u_{+1},\dots,u_{+l}) + \|u_+\|_{E^\rho_{N_++|Y|+k+l}}\|u_{+1}\|_{E^\rho_{N_+}}\cdots\|u_{+l}\|_{E^\rho_{N_+}}\big), \tag{6.284}$$

for $Y \in \Pi'$, k, $l \in \mathbb{N}$, where $C_{g,|Y|+k+l}$ depends only on ρ and $\|u_+\|_{E^\rho_{N_+}}$ and where we have redefined N_+. By the invariance of the current under local phase transformations, it

follows that $\Box A_\mu = (e^{i\varphi}\psi)^+\gamma_0\gamma_\mu e^{i\varphi}\psi$. Inequality (2.67), then gives that

$$\begin{aligned}&\|\big(V(\Lambda)\Delta_Y^{M(l)}(V(\Lambda)^{-1}((t,\cdot)-a)), V(\Lambda)\Delta_{P_\nu Y}^{M(l)}(V(\Lambda)^{-1}((t,\cdot)-a))\big)\|_{M^{\rho'}}\\&\quad\le C_{\rho'}\sum_{0\le\mu\le3}\int_t^\infty\sum_{Y_1,Y_2}^{Y}\|(V'(\Lambda)(\xi_{Y_1}^D(e^{i\varphi}\psi))(V(\Lambda)^{-1}((s,\cdot)-a)))^+\\&\qquad\gamma_0\gamma_\mu V'(\Lambda)(\xi_{Y_2}^D(e^{i\varphi}\psi))(V(\Lambda)^{-1}((s,\cdot)-a))\|_{L^p}ds,\end{aligned}$$

where $p=6/(5-2\rho)$, $0\le\nu\le3$, $Y\in\Pi'$, $l\in\mathbb{N}$. The inequality

$$\|fg\|_{L^p}\le\|f\|_{L^2}\|g\|_{L^2}^{2(1-\rho')/3}\|g\|_{L^\infty}^{(1+2\rho')/3},$$

inequalities (6.281b) and (6.284), the use of Theorem 5.5 and Theorem 5.7 for the free field $\psi_0^{(+)}$ show together with the above inequality that

$$\begin{aligned}&\|\big(V(\Lambda)\Delta_Y^{M(l)}(V(\Lambda)^{-1}((t,\cdot)-a)), V(\Lambda)\Delta_{P_\nu Y}^{M(l)}(V(\Lambda)^{-1}((t,\cdot)-a))\big)\|_{M^{\rho'}}\\&\quad\le C_{\rho',|Y|+l}(1+t)^{-\rho'+1/2}\\&\quad\big(\mathcal{R}^l_{N_+,|Y|+l}(u_{+1},\dots,u_{+l})+\|u_+\|_{E^\rho_{N_++|Y|+l}}\|u_{+1}\|_{E^\rho_{N_+}}\cdots\|u_{+l}\|_{E^\rho_{N_+}}\big),\end{aligned}\tag{6.285}$$

for $Y\in\Pi'$, $l\in\mathbb{N}$, $1/2<\rho'\le1$, where $C_{\rho',|Y|+l}$ depends only on ρ and $\|u_+\|_{E^\rho_{N_+}}$.

Inequalities (6.280), (6.283b), (6.284) and (6.285) together with the fact that natural action of $\mathcal{P}_0$ on $U(\mathfrak{p})$ leaves invariant the ideal spanned by $\boldsymbol{\sigma}^1$, prove the lemma.

Proposition 6.18. *Let the group action Γ, the set S and the set $\mathcal{U}_\infty$ be defined by (6.277a) and (6.277b), (6.277c) and (6.277d). Then $\mathcal{U}_\infty\subset V^\rho_\infty$, the group action $\Gamma\colon\mathcal{P}_0\times S\to S$ defines a group action $U\colon\mathcal{P}_0\times\mathcal{U}_\infty\to\mathcal{U}_\infty$ and $U\colon\mathcal{P}_0\times\mathcal{U}_{\infty(0)}\to\mathcal{U}_\infty$ is C^∞. One can choose $\mathcal{O}_{\infty(+)}$ such that if $u_+\in\mathcal{O}_{\infty(+)}$ and $g\in\mathcal{P}_0$ are such that $U_g^{(+)}(u_+)\in\mathcal{O}_{\infty(+)}$, then $U_g(\Omega^{(+)}(u_+))=\Omega^{(+)}(U_g^{(+)}(u_+))$. Moreover there exists $N_0\in\mathbb{N}$ such that if $F\colon V^\rho_\infty\to E^{\circ\rho}_\infty$ is as in Theorem 6.11 then*

$$\begin{aligned}&\|(D^l(F\circ U_g\circ F^{-1}))(u;u_1,\dots,u_l)\|_{E^\rho_n}\\&\quad\le C_{g,n+l}\big(\mathcal{R}^l_{N_0,n+l}(u_1,\dots,u_l)+\|u\|_{E^\rho_{N_0+n+l}}\|u_1\|_{E^\rho_{N_0}}\cdots\|u_l\|_{E^\rho_{N_0}}\big),\end{aligned}$$

for $g\in\mathcal{P}_0$, n, $l\in\mathbb{N}$, $u\in F(\mathcal{U}_{\infty(0)})$, $u_1,\dots,u_l\in E^{\circ\rho}_\infty$, where $C_{g,n+l}$ depends only on ρ and $\|u\|_{E^\rho_{N_0}}$.

Proof. The proof that $\mathcal{U}_\infty\subset V^\rho_\infty$ which is based on Lemma 6.17 is so similar to the proof of Theorem 6.12 based on Theorem 6.10, that we omit it.

It follows by the definition of Γ that if $t\mapsto s(t)$ is an element of S and $a_0\in\mathbb{R}$, then $t\mapsto s(t-a_0)$ is also an element of S. Hence $s(-a_0)\in\mathcal{U}_\infty\subset V^\rho_\infty$. Moreover $(d/dt)^n s(t)=T_{P_0^n}(s(t))\in E^\rho_\infty$ according to statement i) of Corollary 2.21, so $s\in C^\infty(\mathbb{R},V^\rho_\infty)$. The uniqueness of the local solutions in a larger topological space than V^ρ_∞, given by [10] then

proves that the map $p\colon s\mapsto s(0)$ of S onto $\mathcal{U}_\infty$ is a bijection. Since $U_g=p\circ\Gamma_g\circ p^{-1}$, it follows that $U\colon\mathcal{P}_0\times\mathcal{U}_\infty\to\mathcal{U}_\infty$ is a group action. The fact that $U\colon\mathcal{P}_0\times\mathcal{U}_{\infty(0)}\to\mathcal{U}_\infty$ is C^∞ and the inequality of the proposition follow from Lemma 2.19 and Lemma 6.17. We omit the details.

To prove that $\Omega^{(+)}$ intertwines U_g and $U_g^{(+)}$, let $u_+=(f,\dot f,\alpha)\in\mathcal{O}_{\infty(+)}$. Let (A,ψ) be the solution in S with initial condition $\Omega^{(+)}(u_+)=v$, let $(A_g,\psi_g)=\Gamma_g(A,\psi)$ be the transformed solution in S and let $\dot A(t)=\frac{d}{dt}A(t)$, $\dot A_g(t)=\frac{d}{dt}A_g(t)$ and $((\varphi_g(u_+))(t))(x)=((\varphi(u_+))(y_0))(\vec y)$, where $(y_0,\vec y)=y=V(\Lambda)^{-1}((t,x)-a)$, $g=(a,\Lambda)$ and where φ is given by statement iii) of Proposition 6.16. Lemma 6.17 then gives that

$$\|(A_g,\dot A_g,e^{i\varphi_g(u_+)}\psi_g)(t)-U^1_{\exp(tP_0)}U^1_g u_+\|_{E_0^\rho}\to 0, \tag{6.286}$$

when $t\to\infty$. Let us study the limit of

$$I(t)=\|e^{-i(\varphi_g(u_+))(t)}U^{D1}_{\exp(tP_0)}U^{D1}_g\alpha-e^{-i\left(\varphi(U_g^{(+)}(u_+))\right)(t)}U^{D1}_{\exp(tP_0)}\alpha_g\|_D,$$

when $t\to\infty$. Let $\theta_g(t,x)=\big((\varphi_g(u_+)-\varphi(U_g^{(+)}(u_+)))(t)\big)(x)$. Since $P_\varepsilon(-i\partial)U^{D1}_g\alpha\in D_\infty$ (resp. $P_\varepsilon(-i\partial)\alpha_g\in D_\infty$) we can apply Theorem A.1 to $e^{i\varepsilon\omega(-i\partial)t}P_\varepsilon(-i\partial)U_g^{D1}\alpha$ (resp. $e^{i\varepsilon\omega(-i\partial)}P_\varepsilon(-i\partial)\alpha_g$) and let β_ε (resp. β'_ε)$\in C^\infty(\mathbb{R}^+\times\mathbb{R}^3-\{0\})$ be the corresponding function homogeneous of degree $-3/2$ defined by (A.1) to (A.3). This gives that

$$I_\infty=\lim_{t\to\infty}I(t)=\sum_{\varepsilon=\pm}\lim_{t\to\infty}\|\beta_\varepsilon(t,\cdot)-e^{i\theta_g(t,\cdot)}\beta'_\varepsilon(t,\cdot)\|_D.$$

The variable transformation $x\mapsto p_\varepsilon(t,x)$ gives, since $x/t=-\varepsilon p_\varepsilon(t,x)/\omega(p_\varepsilon(t,x))$ and since the support of β_ε and β'_ε is contained in the forward light cone:

$$I_\infty=\sum_\varepsilon\lim_{t\to\infty}\|P_\varepsilon\hat\alpha_g-e^{-i\theta_g(t,tK_\varepsilon)}P_\varepsilon(U_g^{D1}\alpha)^\wedge\|_D, \tag{6.287}$$

where $K_\varepsilon(k)=-\varepsilon k/\omega(k)$, $k\in\mathbb{R}^3$. It follows from the definition of χ_1 and φ in statement iii) of Proposition 6.16, from the definition of $A^{(+)}$ in (1.22a) (with $\chi_0=1$) and from the fact that the function $\alpha\mapsto J^{(+)}(\alpha)$ defined in (1.21) satisfies $J^{(+)}(U^{D1}_{(a,I)}\alpha)=J^{(+)}(\alpha)$ that $I_\infty=0$. It then follows from (6.286) that

$$\|(A_g(t),\dot A_g(t),\psi_g(t))-(I\oplus I\oplus e^{-i\varphi(U_g^{(+)}(u_+),t)})U^1_{\exp(tP_0)}U_g^{(+)}(u_+)\|_{E_0^\rho}\to 0, \tag{6.288}$$

when $t\to\infty$. We now choose $\mathcal{O}_{\infty(+)}$ such that if $u_+\in\mathcal{O}_{\infty(+)}$ and $U_g^{(+)}(u_+)\in\mathcal{O}_{\infty(+)}$ then there is a product of one parameter subgroups $g_i(s)$ such that $U^{(+)}_{g_i(s)}(u_+)\in\mathcal{O}_{\infty(+)}$ for $0\le s\le 1$ and $g=\Pi_i g_i(1)$, where the product is finite. Replacing g by $g_i(s)$ in (6.288), it follows from statement iv) of Proposition 6.16 that $U_{g_i(s)}(\Omega^{(+)}(u_+))=\Omega^{(+)}(U^{(+)}_{g_i(s)}(u_+))$, for $0\le s\le 1$ since $U_{g_i(s)}(\Omega^{(+)}(u_+))\in\mathcal{U}_{\infty(0)}$ for s sufficiently small by continuity. This shows that $U_g(\Omega^{(+)}(u_+))=\Omega^{(+)}(U_g^{(+)}(u_+))$, which proves the proposition.

We shall now extend $\Omega^{(+)}\colon \mathcal{O}_{\infty(+)} \to \mathcal{U}_{\infty(0)}$ to a Poincaré invariant domain. Let $\mathcal{O}_\infty^{(+)} = \cup_{g\in\mathcal{P}_0} U_g^{(+)}(\mathcal{O}_{\infty(+)})$. $\mathcal{O}_\infty^{(+)}$ is an open neighbourhood of zero in $E_\infty^{\circ\rho}$, since this is the case for $\mathcal{O}_{\infty(+)}$ and since $U_g^{(+)} = (U_{g^{-1}}^{(+)})^{-1}$ and $u \mapsto U_{g^{-1}}^{(+)}(u)$ is continuous on $E_\infty^{\circ\rho}$. If $u \in \mathcal{O}_\infty^{(+)}$, then there exists g such that $U_{g^{-1}}^{(+)}(u) \in \mathcal{O}_{\infty(+)}$ and we define

$$\Omega_+(u) = U_g\bigl(\Omega^{(+)}(U_{g-1}^{(+)}(u))\bigr), \tag{6.289}$$

which is an element of $\mathcal{U}_\infty$ since $U_{g-1}^{(+)}(u) \in \mathcal{O}_{\infty(+)}$. Next theorem shows that Ω_+ is a map from $\mathcal{O}_\infty^{(+)}$ to $\mathcal{U}_\infty$.

Theorem 6.19. *If $u_+ \in \mathcal{O}_\infty^{(+)}$, g_1, $g_2 \in \mathcal{P}_0$, $U_{g_1^{-1}}^{(+)}(u_+) \in \mathcal{O}_{\infty(+)}$ and $U_{g_2^{-1}}^{(+)}(u_+) \in \mathcal{O}_{\infty(+)}$, then $U_{g_1}(\Omega^{(+)}(U_{g_1^{-1}}^{(+)}(u_+))) = U_{g_2}(\Omega^{(+)}(U_{g_2^{-1}}^{(+)}(u_+)))$. The set $\mathcal{U}_\infty$ is an open neighbourhood of zero in V_∞^ρ, $\Omega_+\colon \mathcal{O}_\infty^{(+)} \to \mathcal{U}_\infty$ is a diffeomorphism, the nonlinear group representation $U\colon \mathcal{P}_0 \times \mathcal{U}_\infty \to \mathcal{U}_\infty$ is C^∞, $\Omega_+ \circ U_g^{(+)} = U_g \circ \Omega_+$ for each $g \in \mathcal{P}_0$ and*

$$\begin{aligned}&\|U^M_{\exp(tP_0)}(\Omega_+(u)) - U^{M1}_{\exp(tP_0)}(f,\dot f)\|_{M_0^\rho}\\&\quad + \Bigl\|U^D_{\exp(tP_0)}(\Omega_+(u)) - \sum_{\varepsilon=\pm} e^{is_\varepsilon^{(+)}(u,t,-i\partial)} P_\varepsilon(-i\partial) U^{D1}_{\exp(tP_0)}\alpha\Bigr\|_D \to 0,\end{aligned}$$

when $t\to\infty$, for $u = (f,\dot f,\alpha) \in \mathcal{O}_\infty^{(+)}$, where $s_\varepsilon^{(+)}$ is defined by formula (1.18).

Proof. Since $U_{g_i^{-1}}^{(+)}(u_+) \in \mathcal{O}_{\infty(+)}$ for $i \in \{0,1\}$, it follows from Proposition 6.18 that

$$\begin{aligned}U_{g_1}(\Omega^{(+)}(U_{g_1^{-1}}^{(+)}(u_+))) &= U_{g_2}\bigl(U_{(g_1^{-1}g_2)^{-1}}\bigl(\Omega^{(+)}(U_{g_1^{-1}g_2}^{(+)}(U_{g_2^{-1}}^{(+)}(u_+)))\bigr)\bigr)\\&= U_{g_2}(\Omega^{(+)}(U_{g_2^{-1}}^{(+)}(u_+)))\end{aligned}$$

and it follows that $\Omega^{(+)}(U_{g_1^{-1}}^{(+)}(u_+)) \in \mathcal{U}_{\infty(0)}$. This proves that $\Omega_+\colon \mathcal{O}_\infty^{(+)} \to \mathcal{U}_\infty$ is a map.

If $u_+ \in \mathcal{O}_\infty^{(+)}$, then there exists $h \in \mathcal{P}_0$ such that $U_{h^{-1}}^{(+)}(u_+) \in \mathcal{O}_{\infty(+)}$ so $U_{(gh)^{-1}}^{(+)}(U_g^{(+)}(u_+)) \in \mathcal{O}_{\infty(0)}$ also. Definition (6.289) then gives that

$$\begin{aligned}\Omega^{(+)}(U_{g_1}^{(+)}(u_+)) &= U_{gh}\bigl(\Omega^{(+)}(U_{(gh)^{-1}}^{(+)}(U_g^{(+)}(u_+)))\bigr)\\&= U_g\bigl(U_h(\Omega^{(+)}(U_{h^{-1}}^{(+)}(u_+)))\bigr) = U_g(\Omega_+(u_+)),\end{aligned}$$

which proves the intertwining property. If $u \in \mathcal{U}_{\infty(0)}$, then $u_+ = (\Omega^{(+)})^{-1}(u) \in \mathcal{O}_{\infty(+)}$ and $U_g(u) = \Omega_+(U_g^{(+)}(u_+))$, which shows that $\cup_{g\in\mathcal{P}_0} U_g(\mathcal{U}_{\infty(0)})$ is a subset of the image of Ω_+. Hence according to the definition $\mathcal{U}_\infty$, the map Ω_+ is onto. If $u \in \mathcal{U}_\infty$ and $u = \Omega_+(u_+) = U_g(\Omega^{(+)}(U_{g^{-1}}^{(+)}(u_+)))$, where $U_{g^{-1}}^{(+)}(u_+) \in \mathcal{O}_{\infty(+)}$, then $u_+ = U_g^{(+)}((\Omega^{(+)})^{-1}(U_{g^{-1}}(u)))$ is

independent of the choice of g and shows, since $\Omega^{(+)}: \mathcal{O}_{\infty(+)} \to \mathcal{U}_{\infty(0)}$ is a bijection, that Ω_+ is one-to-one. Hence $\Omega_+: \mathcal{O}_\infty^{(+)} \to \mathcal{U}_\infty$ is a bijection.

It follows from Theorem 3.12, Theorem 6.12, statement ii) of Proposition 6.16 and Proposition 6.18 that $\Omega_+: \mathcal{O}_\infty^{(+)} \to \mathcal{U}_\infty$ is C^∞ and that there exists an integer N_+ such that

$$\begin{aligned}&\|(D^l\Omega_+)(u; u_1, \ldots, u_l)\|_{E_n^\rho} \\ &\quad \leq C_{n+l}\big(\mathcal{R}^l_{N_+,n+l}(u_1,\ldots,u_l) + \|u\|_{E^\rho_{N_++n+l}}\|u_1\|_{E^\rho_{N_+}}\cdots\|u_l\|_{E^\rho_{N_+}}\big),\end{aligned} \tag{6.290}$$

for $u \in \mathcal{O}_\infty^{(+)}$, $u_1, \ldots, u_l \in E_\infty^{\circ\rho}$, $n, k \in \mathbb{N}$, where C_{n+l} depends only on ρ and $\|u\|_{E^\rho_{N_+}}$. Let $\overline{u} \in \mathcal{O}_\infty^{(+)}$. Since $\mathcal{O}_\infty^{(+)} \subset E_\infty^{\circ\rho}$ is an open set there exists a open neighbourhood $\mathcal{O}$ of zero in $E_\infty^{\circ\rho}$ such that $\overline{u} + \mathcal{O} \subset \mathcal{O}_\infty^{(+)}$. The function $F: v \mapsto \Omega_+(\overline{u}+v)$ from $\mathcal{O}$ to $\mathcal{U}_\infty$ satisfies estimate (6.290) with $\overline{u}+v$ instead of u and with C_{n+l} depending only on v, for fixed $\overline{u}$. To prove that F has a C^∞ local inverse it is therefore sufficient to prove that $DF(v)$ has a right inverse, for v in a neighbourhood of zero, satisfying the hypotheses of the implicit function theorem in Fréchet spaces. The existence of such a right inverse is proved using Lemma 6.17 and following the proof of Theorem 6.13. We leave out the details since the proofs are so similar. Since Ω_+ is a bijection this proves that $\Omega_+: \mathcal{O}_\infty^{(+)} \to \mathcal{U}_\infty$ is a diffeomorphism and that $\mathcal{U}_\infty$ is open.

Since Ω_+ is a diffeomorphism and $U_g = \Omega_+ \circ U_g^{(+)} \circ \Omega_+^{-1}$, it follows that $U: \mathcal{P}_0 \times \mathcal{U}_\infty \to \mathcal{U}_\infty$ is C^∞.

To prove the statement concerning the limit let $u_+ = (f, \dot{f}, \alpha) \in \mathcal{O}_\infty^{(+)}$, let $g \in \mathcal{P}_0$ be such that $U^{(+)}_{g^{-1}}(u_+) \in \mathcal{O}_{\infty(+)}$, let (A, ψ) be the solution in S with initial condition $\Omega_+(u_+)$ and let (A', ψ') be the solution in S with initial condition $U_{g^{-1}}(\Omega_+(u_+))$. According to the already proved intertwining property and the definition of Ω_+, it follows that $U_{g^{-1}}(\Omega_+(u_+)) = \Omega^{(+)}(U^{(+)}_{g^{-1}}(u_+)) \in \mathcal{U}_{\infty(0)}$. Defining $\Gamma_g(A', \psi') = (A'_g, \psi'_g)$ it follows from the definition of U_g that $(A'_g, \psi'_g) = (A, \psi)$. Therefore applying (6.288) with $(A'_g, \dot{A}'_g, \psi_g)$ instead of $(A_g, \dot{A}_g, \psi_g)$ and $U^{(+)}_{g^{-1}}(u_+)$ instead of u_+, we obtain that

$$\|(A(t), \dot{A}(t), \psi(t)) - (I \otimes I \otimes e^{-i\varphi(u_+,t)})U^1_{\exp(tP_0)}u_+\|_D \to 0,$$

when $t \to \infty$. This shows that we only have to prove that

$$\|e^{-i\varphi(u_+,t)}U^{D1}_{\exp(tP_0)}\alpha - \sum_{\varepsilon=\pm} e^{is_\varepsilon(u_+,t,-i\partial)}P_\varepsilon(-i\partial)U^{D1}_{\exp(tP_0)}\alpha\|_D \to 0, \tag{6.291}$$

when $t \to \infty$. Using that

$$\omega(k)\frac{\partial}{\partial k_i}h(t, tk/\omega(k)) = h_{M_{0i}}(t, tk/\omega(k)) - (k_i/\omega(k))h_D(t, tk/\omega(k)),$$

where

$$h_{M_{0i}}(t,x) = x_i(\partial/\partial t)h(t,x) + t(\partial/\partial x_i)h(t,x)$$

and

$$h_D(t,x) = t(\partial/\partial t)h(t,x) + \sum_{1\le i\le 3} x_i(\partial/\partial x_i)h(t,x),$$

using Lemma 4.4 and Corollary 4.2 and using formula (1.20b) it follows that there exists $N \in \mathbb{N}$ such that

$$\Big|\frac{\partial^n}{\partial t^n}\partial_k^\alpha s_\varepsilon(u_+,t,k)\Big| \le C_{|\alpha|+n}\omega(k)^{-|\alpha|}(1+t)^{\rho-1/2}(1+\|u_+\|_{E_N^\rho})\|u_+\|_{E^\rho_{N+|\alpha|+n}}, \qquad (6.292)$$

for some constants $C_{|\alpha|+n}$ depending only on ρ. Let $B(r)$, $r>0$, be the open ball of radius r in $\mathbb{R}^3$. If $0<R<R'$, then it follows from inequality (6.292), formula (A.2) and from the inverse function theorem in $\mathbb{R}^3$ there exists $T \ge 0$ such that the equation

$$\varepsilon\frac{q_\varepsilon(t,x)}{\omega(q_\varepsilon(t,x))} + t^{-1}F_\varepsilon(u_+,t,q_\varepsilon(t,x)) + \frac{x}{t} = 0, \qquad (6.293)$$

where $F_\varepsilon(u_+,t,k) = \nabla_k s_\varepsilon(u_+,t,k)$, has a unique solution $q_\varepsilon(t,x) \in B(mR')$ for $(t,x) \in \{(s,y)\big|s>T, y/s < R(1+R^2)^{1/2}\} = Q(T,R)$. The function $q_\varepsilon \in C^\infty(Q(T,R),\mathbb{R}^3)$ and the inverse $y_\varepsilon(t,\cdot)$ of $x \mapsto q_\varepsilon(t,x)$ is an element of $C^\infty(]T,\infty[\times B(mR'),\mathbb{R}^3)$. Moreover if $p_\varepsilon(t,x)$ is given by formula (A.2) then

$$|q_\varepsilon(t,x) - p_\varepsilon(t,x)| \le C(1+t)^{-3/2+\rho}, \quad (t,x) \in Q(T,R), \qquad (6.294)$$

where C is independent of (t,x). Since $|(\varphi(u_+,t))(x) + \vartheta(A^{(+)},(t,x))|$ converges to zero uniformly inside every conic neighbourhood which is included in the interior of the forward light cone when $t\to\infty$, it follows from (6.294) and Theorem 7.7.5 of [11] that if supp $\hat\alpha_1 \subset B(mR)$, then

$$\|e^{-i\varphi(u_+,t)}U^{D1}_{\exp(tP_0)}\alpha_1 - \sum_{\varepsilon=\pm} e^{is_\varepsilon(u_+,t,-i\partial)}P_\varepsilon(-i\partial)U^{D1}_{\exp(tP_0)}\alpha_1\|_D \to 0. \qquad (6.295)$$

Let $\alpha \in D_\infty$ and $\nu > 0$. Then there exists $R>0$ and $\alpha_1 \in D_\infty$ such that supp $\hat\alpha_1 \subset B(mR)$ and $\|\alpha-\alpha_1\|_D \le \nu/4$. According to inequality (6.294) there exists $T_0 \ge 0$ such that for this α_1 the norm in (6.295) is majorized by $\nu/2$ for $t \ge T_0$. The norm in expression (6.291) is therefore majorized by ν for $t \ge T_0$. This proves the theorem.

Appendix

We shall prove here auxiliary L^2- and L^∞-estimates for approximate solutions of the linear homogeneous Klein-Gordon equation. This can be done by a direct application of the method of stationary phase. However here we shall combine it with Theorem 5.5 for $G = 0$, Theorem 5.7 and with the method of symbolic calculus developed in [12] which is easier and leads to the same result.

For $f \in D_\infty$ ($\simeq S(\mathbb{R}^3, \mathbb{C}^4)$) and $\varepsilon = \pm 1$, we introduce the sequence $g_l \in C^\infty((\mathbb{R}^+ \times \mathbb{R}^3) - \{0\})$ defined by $g_l(t,x) = 0$ for $0 \le t \le |x|$, and

$$g_0(t,x) = e^{i3\varepsilon\pi/4}(D(t,x))^{-1/2}\hat{f}(p_\varepsilon(t,x)), \quad \varepsilon = \pm, \tag{A.1a}$$

$$g_l = \frac{\rho}{2i\varepsilon l m}\Box g_{l-1}, \quad l \ge 1, \tag{A.1b}$$

(i.e. $g = e^{(2i\varepsilon m)^{-1}\rho\Box} g_0$ as formal power series in the invariable m^{-1}), for $0 \le |x| < t$, where

$$\rho(t,x) = (t^2 - |x|^2)^{1/2}, \quad p_\varepsilon(t,x) = -\varepsilon m x/\rho(t,x), \tag{A.2}$$

and

$$D(t,x) = m^2(\omega(p_\varepsilon(t,x)))^{-5} = (t/m)^3(\rho(t,x)/t)^5 \tag{A.3}$$

is the Jacobian of the transformation $x \mapsto p_\varepsilon(t,x)$ for fixed t. The support of g_l is contained in the set $\{y \in \mathbb{R}^4 \big| y_\mu y^\mu \ge 0, y^0 \ge 0\}$ and g_l is homogeneous of degree $-3/2 - l$.

We introduce the representation $X \mapsto \xi_X$ of the Poincaré Lie algebra $\mathfrak{p}$ by:

$$\begin{aligned}
\xi_{P_0} &= \frac{\partial}{\partial t}, \\
\xi_{P_i} &= \frac{\partial}{\partial x_i}, \quad 1 \le i \le 3, \\
\xi_{M_{0i}} &= x_i \frac{\partial}{\partial t} + t\frac{\partial}{\partial x_i}, \quad 1 \le i \le 3, \\
\xi_{M_{ij}} &= -x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i}, \quad 1 \le i < j \le 3.
\end{aligned}$$

We recall that Π is an ordered basis of $\mathfrak{p}$ and that Π' is the corresponding standard basis of the enveloping algebra $U(\mathfrak{p})$ of $\mathfrak{p}$. Given an ordering on the basis $Q = \{M_{\mu\nu} \big| 0 \le \mu < \nu \le 3\}$ of $\mathfrak{so}(3,1)$, let Q' be the corresponding standard basis of the enveloping algebra $U(\mathfrak{so}(3,1))$ of $\mathfrak{so}(3,1)$.

To state the results on decrease properties of the Klein-Gordon equation, we introduce

$$\begin{aligned}
\varphi_0 &= e^{i\varepsilon\omega(-i\partial)t} f, \\
\varphi_n &= \varphi_0 - e^{i\varepsilon m \rho(t)} \sum_{0 \le l \le n-1} g_l(t), \quad n \ge 1,
\end{aligned}$$

and if $X \mapsto a_X$ is a map from Π' to D_∞, then we introduce

$$E_j^{(p)}(a) = \sum_{\substack{Y \in \Pi' \\ |Y| \le j}} \Big(m\|a_Y\|_{L^p} + \sum_{0 \le \mu \le 3} \|a_{P_\mu Y}\|_{L^p} \Big),$$

for $j \in \mathbb{N}$ and $1 \leq p \leq \infty$. We also introduce $\lambda(t)$ and $\delta(t)$ for $t \geq 0$ by

$$(\lambda(t))(x) = t/(1+t-|x|) \quad \text{for } 0 \leq |x| \leq t,$$
$$(\lambda(t))(x) = |x| \quad \text{for } 0 \leq t \leq |x|,$$
$$(\delta(t))(x) = 1+t+|x|.$$

If $\mu\colon \mathbb{R}_+ \to D$, we define

$$(\tilde{\mu}(t))_Y = (\xi_Y \mu)(t), Y \in \Pi', t \in \mathbb{R}_+$$

Theorem A.1. *There exist positive numbers $C_i \in \mathbb{R}^+$, $i \geq 0$, such that*

$$E_j^{(2)}\big((1+\lambda(t))^{k/2}\tilde{\varphi}_0(t)\big) \leq C_{j+k}\Big(\|f\|_{D_{j+k}} + \sum_{1\leq i\leq 3} \|\partial_i f\|_{D_{j+k}}\Big),$$

for $j,k \in \mathbb{N}$, $t \geq 0$, $f \in D_\infty$,

$$E_j^{(2)}\big((1+\lambda(t))^{k/2}\tilde{\varphi}_{n+1}(t)\big) \leq C_{j+k+n}\ t^{-n-1}\|f\|_{D_{3(j+k+n)+4}}, \tag{A.4}$$

for $j,k,n \in \mathbb{N}$, $t \geq 1$, $f \in D_\infty$,

$$E_j^{(\infty)}\big(\delta(t)^{3/2}(1+\lambda(t))^{k/2}\tilde{\varphi}_0(t)\big) \leq C_{j+k}\Big(\|f\|_{D_{j+k+8}} + \sum_{1\leq i\leq 3} \|\partial_i f\|_{D_{j+k+8}}\Big),$$

for $j,k \in \mathbb{N}$, $t \geq 0$, $f \in D_\infty$ and

$$E_j^{(\infty)}\big(\delta(t)^{3/2}(1+\lambda(t))^{k/2}\tilde{\varphi}_{n+1}(t)\big) \leq C_{j+k+n}\ t^{-n-1}\|f\|_{D_{3(j+k+n)+28}}, \tag{A.5}$$

for $j,k,n \in \mathbb{N}$, $t \geq 1$, $f \in D_\infty$. Moreover supp $g_l(t) \subset \{x \in \mathbb{R}\big||x| \leq t\}$ *for $t > 0$,*

$$\|(\rho^{-j}\xi_{XY}g_l)(t)\|_{L^2} \leq C_{j+|X|+l}\ t^{-j-l-|X|}\|f\|_{D_{3|X|+3l+|Y|+j}}, \tag{A.6}$$

and

$$\|(\rho^{-j}\xi_{XY}g_l)(t)\|_{L^\infty} \leq C_{j+|X|+l}\ t^{-3/2-j-l-|X|}\|f\|_{D_{3|X|+3l+|Y|+j+5}}, \tag{A.7}$$

for $t > 0$, $j,l \in \mathbb{N}$, $X \in \Pi' \cap U(\mathbb{R}^4)$ and $Y \in Q'$.

This theorem will be proved at the end of the appendix.

The development in Theorem A.1 can be inverted. Given a homogeneous function $g \in C^\infty((\mathbb{R}^+ \times \mathbb{R}^3) - \{0\})$ of degree $-3/2$ such that supp $g(1,\cdot) \subset \{x \in \mathbb{R}^3\big||x| \leq 1\}$, we construct by iteration $f_0, \ldots, f_n \in D_\infty$:

$$\hat{f}_l(k) = e^{-i3\varepsilon\pi/4}(m^2/\omega(k)^5)^{1/2}g_{l,0}(1, -\varepsilon k/\omega(k)), \quad g_{0,0} = g, \quad 0 \leq l \leq n, \tag{A.8a}$$

$$g_{l,0}(t,x) = -\sum_{1\leq j\leq l} t^j g_{l-j,j}(t,x), \quad 1 \leq l \leq n, \tag{A.8b}$$

$$g_{l,j} = \frac{\rho}{2ij\varepsilon m}\Box g_{l,j-1}, \quad 1 \leq j \leq n-l. \tag{A.8c}$$

By this construction $g_{l,j} \in C^\infty((\mathbb{R}^+ \times \mathbb{R}^3) - \{0\})$ is homogenous of degree $-3/2 - j$ with support in the forward light cone.

Theorem A.2. *Let $g \in C^\infty((\mathbb{R}^+ \times \mathbb{R}^3) - \{0\})$ be a homogeneous function of degree $-3/2$ with* supp $g(1, \cdot) \subset \{x \in \mathbb{R}^3 \big| |x| \leq 1\}$. *If $f_0, \dots, f_n$ is given by the construction (A.8a)–(A.8c) and*

$$u_n(t) = e^{i\varepsilon m\rho(t)} g(t) - \sum_{0 \leq l \leq n} t^{-l} e^{i\varepsilon\omega(-i\partial)t} f_l,$$

then

$$\begin{aligned} &E_j^{(2)}\big((1+\lambda(t))^{k/2}\tilde{u}_n(t)\big) && \text{(A.9)} \\ &\quad \leq C_{j+k+n} \sum_{\substack{Y \in Q' \\ q+|Y| \leq 3(j+k+n)+4}} \|(m/\rho(1,\cdot))^q (\xi_Y g)(1,\cdot)\|_{L^2} t^{-n-1}, \quad t \geq 1, \end{aligned}$$

and

$$\begin{aligned} &E_j^{(\infty)}\big(\delta(t)^{3/2}(1+\lambda(t))^{k/2}\tilde{u}_n(t)\big) && \text{(A.10)} \\ &\quad \leq C_{j+k+n} \sum_{\substack{Y \in Q' \\ q+|Y| \leq 3(j+k+n)+28}} \|(m/\rho(1,\cdot))^q (\xi_Y g)(1,\cdot)\|_{L^2} t^{-n-1}, \quad t \geq 1, \end{aligned}$$

for $j, k, n \in \mathbb{N}$. Moreover

$$\|f_l\|_{D_j} \leq C_{l+j} \sum_{\substack{Y \in Q' \\ q+|Y| \leq j+2l}} \|(m/\rho(1,\cdot))^{q+l} (\xi_Y g)(1,\cdot)\|_{L^2}, \quad j, l \in \mathbb{N}. \tag{A.11}$$

Proof. Since the proofs of the statements are so similar for the L^2 case and the L^∞ case, we only prove the L^2 case.

Application of Theorem A.1 to the functions f_l, $0 \leq l \leq n$, with a development up to order $n - l$ gives, if $v_n(t) = e^{i\varepsilon m\rho(t)}\big(g(t) - \sum_{0\leq l\leq n} t^{-l} \sum_{0 \leq j \leq n-l} g_{l,j}(t)\big)$:

$$\begin{aligned} E_j^{(2)}\big((1+\lambda(t))^{k/2}\tilde{u}_n(t)\big) &\leq E_j^{(2)}\big((1+\lambda(t))^{k/2}\tilde{v}_n(t)\big) && \text{(A.12)} \\ &\leq C_{j+k+n}\, t^{-n-1} \sum_{0 \leq l \leq n} \|f_l\|_{D_{3(j+k+n-l)+4}}, \quad t \geq 1, \end{aligned}$$

where $g_{l,j}$ is given by (A.1) with f_l instead of f i.e.:

$$g_{l,0}(t,x) = e^{i3\varepsilon\pi/4}(D(t,x))^{-1/2} \hat{f}_l(p_\varepsilon(t,x)), \quad 0 \leq l \leq n, \tag{A.13a}$$

$$g_{l,j} = \frac{\rho}{2i\varepsilon jm} \Box g_{l,j-1}, \quad 1 \leq j \leq n - l. \tag{A.13b}$$

According to definition (A.8) and formulas (A.13a), (A.13b) we have $g_{0,0} = g$ and

$$\sum_{0 \leq j \leq r} t^j g_{r-j,j} = 0, \quad 1 \leq r \leq n,$$

which proves that

$$\sum_{0\le l\le n}\sum_{0\le j\le n-l} t^{-l} g_{l,j}(t) = \sum_{0\le r\le n} t^{-r} \sum_{0\le j\le r} t^j g_{r-j,j} = g, \tag{A.14}$$

so $v_n = 0$. By (A.12) we get

$$E_j^{(2)}\big((1+\lambda(t))^{k/2}\tilde{u}_n(t)\big) \le C_{j+k+n}\ t^{-n-1} \sum_{0\le l\le n} \|f_l\|_{D_{3(j+k+n-l)+4}}, \quad t\ge 1. \tag{A.15}$$

Inequality (A.9) follows from (A.15) and inequality (A.11), which we now prove.

Introduce the linear representation $X\mapsto \eta_X$ of the Poincaré Lie algebra $\mathfrak{p}$ by

$$(\eta_{P_0}h)(k) = -\varepsilon i\omega(k)h(k), \quad (\eta_{P_j}h)(k) = ik_jh(k), \quad j=1,2,3, \tag{A.16a}$$

$$(\eta_{M_{ij}}h)(k) = -(k_i\frac{\partial}{\partial k_j} - k_j\frac{\partial}{\partial k_i})h(k), \quad 1\le i<j\le 3, \tag{A.16b}$$

$$(\eta_{M_{0j}}h)(k) = -\varepsilon\frac{\partial}{\partial k_j}(\omega(k)h(k)), \quad 1\le j\le 3, \tag{A.16c}$$

where $h\in D_\infty$ and $\varepsilon=1$ or $\varepsilon=-1$.

The norms of f:

$$\|f\|_{D_s}, \quad \|\hat{f}\|_{D_s}, \quad \Big(\sum_{\substack{Y\in\Pi'\\ |Y|\le s}} \|\eta_Y\hat{f}\|^2_{L^2}\Big)^{1/2}, \tag{A.17}$$

are then equivalent. Let $X\mapsto \xi_X$ be the representation of the Lorentz Lie algebra $\mathfrak{so}(3,1)$ on functions on $\mathbb{R}^4$ given by

$$(\xi_{M_{ij}}h)(t,x) = -\big(x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}\big)h(t,x), \quad 1\le i<j\le 3, \tag{A.18a}$$

$$(\xi_{M_{0i}}h)(t,x) = \big(x_i\frac{\partial}{\partial t} + t\frac{\partial}{\partial x_i}\big)h(t,x), \quad 1\le i\le 3. \tag{A.18b}$$

Since $g_{l,0}$ is homogeneous of degree $-3/2$ it follows from (A.8a) that

$$\hat{f}_l(k) = e^{-i3\varepsilon\pi/4}(m/\omega(k))g_{l,0}(\omega(k),-\varepsilon k). \tag{A.19}$$

Since $(\eta_X\hat{f}_l)(k) = e^{-i3\varepsilon\pi/4}(m/\omega(k))(\xi_Xg_{l,0})(\omega(k),-\varepsilon k)$ for $X\in\mathfrak{so}(3,1)$, we get

$$(\eta_Y\hat{f}_l)(k) = e^{-i3\varepsilon\pi/4}(m/\omega(k))(\xi_Yg_{l,0})(\omega(k),-\varepsilon k),$$

for $Y\in U(\mathfrak{so}(3,1))$, the enveloping algebra of $\mathfrak{so}(3,1)$. $\xi_Yg_{l,0}$ and $g_{l,0}$ have the same degree of homogeneity, so

$$(\eta_Y\hat{f}_l)(k) = e^{-i3\varepsilon\pi/4}(m/(\omega(k))^{5/2})(\xi_Yg_{l,0})(1,-\varepsilon k/\omega(k)), \quad Y\in U(\mathfrak{so}(3,1)). \tag{A.20}$$

Choosing an ordering on Π such that

$$P_0 < P_1 < P_2 < P_3 < M_{\mu\nu}, \quad 0 \leq \mu < \nu \leq 3,$$

we obtain, because of the equivalence of the norms in (A.17) and because of the explicit expression (A.16a) of η_{P_μ}, that

$$\|f\|_{D_s} \leq C_s \Big(\sum_{q+|Z| \leq s} \sum_{Z \in Q'} \|\omega^q \eta_Z \hat{f}\|^2_{L^2} \Big)^{1/2}, \tag{A.21}$$

where Q' is a basis of $U(\mathfrak{so}(3,1))$. Since $m(\omega(k))^{-5/2}$ is the Jacobian of the transformation $k \mapsto -\varepsilon k/\omega(k)$, it follows from formula (A.20) that

$$\|\omega^q \eta_Z \hat{f}_l\|_{L^2} = \|(m\rho(1,\cdot))^q (\xi_Z g_{l,0})(1,\cdot)\|_{L^2}, \quad Z \in U(\mathfrak{so}(3,1)), \tag{A.22}$$

and then from (A.21) that

$$\|f_l\|_{D_s} \leq C'_s \sum_{q+|Z| \leq s} \sum_{Z \in Q'} \|(m/\rho(1,\cdot))^q (\xi_Z g_{l,0})(1,\cdot)\|_{L^2}, \tag{A.23}$$

where $0 \leq l \leq n$, $s \geq 0$.

It follows from definition (A.8c) of $g_{l,j}$, $1 \leq j \leq n-l$, that

$$g_{l,j} = (j!(2i\varepsilon m)^j)^{-1} (\rho \Box)^j g_{l,0}. \tag{A.24}$$

Since ξ_X, $X \in \mathfrak{so}(3,1)$, commutes with $\Box$ and with the multiplication by ρ, we get

$$\xi_Z g_{l,j} = (j!(2i\varepsilon m)^j)^{-1} (\rho \Box)^j \xi_Z g_{l,0}, \quad Z \in U(\mathfrak{so}(3,1)). \tag{A.25}$$

Using that inside the light cone

$$\Box = \rho^{-2} \Big(L^2 + 2L - \sum_{0 \leq \mu < \nu \leq 3} \xi_{M_{\mu\nu} M^{\mu\nu}} \Big), \tag{A.26}$$

where $L = t\frac{\partial}{\partial t} + \sum_{i=1}^3 x_i \frac{\partial}{\partial x_i}$ and $M^{0i} = -M_{0i}$, $1 \leq i \leq 3$, and $M^{ij} = M_{ij}$, $1 \leq i < j \leq 3$, we obtain for h being a homogeneous function of degree χ on $\mathbb{R}^4$:

$$\Box h = \rho^{-2} \Big((\chi+1)^2 - 1 - \sum_{0 \leq \mu < \nu \leq 3} \xi_{M_{\mu\nu} M^{\mu\nu}} \Big), \tag{A.27}$$

which is homogeneous of degree $\chi - 2$. It follows from (A.24) and (A.27) that

$$\xi_Z g_{l,j} = (j!(2i\varepsilon m)^j)^{-1} \rho^{-j} \Big(\prod_{q=1}^{j} \big((q-1/2)^2 - 1 - \sum_{0 \leq \mu < \nu \leq 3} \xi_{M_{\mu\nu} M^{\mu\nu}} \big) \Big) \xi_Z g_{l,0}, \tag{A.28}$$

$Z \in U(\mathfrak{so}(3,1))$, $1 \leq j \leq n-l$, $0 \leq l \leq n$, since $\xi_Z g_{l,0}$ is homogeneous of degree $-3/2$ and since ξ_X, $X \in \mathfrak{so}(3,1)$, commutes with the multiplication by ρ. We get from (A.28) that for $Z \in U(\mathfrak{so}(3,1))$:

$$\begin{aligned} &\|(m/\rho(1,\cdot))^q(\xi_Z g_{l,j})(1,\cdot)\|_{L^2} \\ &\quad \leq C_j \sum_{\substack{Y\in Q' \\ |Y|\leq 2j}} \|(m/\rho(1,\cdot))^{q+j}(\xi_Y \xi_Z g_{l,0})(1,\cdot)\|_{L^2}, \quad 1 \leq j \leq n-l. \end{aligned} \tag{A.29}$$

Definition (A.8b) shows that

$$\begin{aligned} &\|(m/\rho(1,\cdot))^q(\xi_Z g_{l,0})(1,\cdot)\|_{L^2} \\ &\quad \leq \sum_{1\leq i\leq l} C_{|Z|} \sum_{\substack{Y\in Q' \\ |Y|\leq |Z|}} \|(m/\rho(1,\cdot))^{q}(\xi_Y g_{l-j,j})(1,\cdot)\|_{L^2}, \quad Z \in Q', 1 \leq l \leq n, \end{aligned} \tag{A.30}$$

since $|\xi_{M_{0j}} t| \leq t$ inside the light cone. Let $1 \leq l \leq n$, it then follows from (A.29) and (A.30) that

$$\begin{aligned} &\|(m/\rho(1,\cdot))^q(\xi_Z g_{l,0})(1,\cdot)\|_{L^2} \\ &\quad \leq C_{|Z|,l} \sum_{\substack{Y\in Q' \\ |Y|\leq |Z|}} \sum_{1\leq j\leq l} \sum_{\substack{Y\in Q' \\ |X|\leq 2j}} \|(m/\rho(1,\cdot))^{q+j}(\xi_X \xi_Y g_{l-j,0})(1,\cdot)\|_{L^2}. \end{aligned} \tag{A.31}$$

Using that $\xi_X \xi_Y = \xi_{XY}$ and inequality (A.31) we get

$$\begin{aligned} &\|(m/\rho(1,\cdot))^q(\xi_Z g_{l,0})(1,\cdot)\|_{L^2} \\ &\quad \leq C_{|Z|,l} \sum_{1\leq j\leq l} \sum_{\substack{Y\in Q' \\ 0\leq |Y|\leq |Z|+2j}} \|(m/\rho(1,\cdot))^{q+j}(\xi_Y g_{l-j,0})(1,\cdot)\|_{L^2}, \quad l \geq 1. \end{aligned} \tag{A.32}$$

Inequality (A.32) shows that, (the case $l=0$ being trivial):

$$\begin{aligned} &\|(m/\rho(1,\cdot))^q(\xi_Z g_{l,0})(1,\cdot)\|_{L^2} \\ &\quad \leq C_{|Z|,l} \sum_{\substack{Y\in Q' \\ |Y|\leq |Z|+2l}} \|(m/\rho(1,\cdot))^{q+l}(\xi_Y g_{0,0})(1,\cdot)\|_{L^2}, \quad 0 \leq l \leq n. \end{aligned} \tag{A.33}$$

Since $g_{0,0} = g$ we obtain according to inequalities (A.23) and (A.33)

$$\|f_l\|_{D_s} \leq C'_{l,s} \sum_{\substack{Y\in Q' \\ q+|Y|\leq s+2l}} \|(m/\rho(1,\cdot))^{q+l}(\xi_Y g)(1,\cdot)\|_{L^2}, \quad 0 \leq l \leq n. \tag{A.34}$$

This proves the theorem.

Theorem A.3. *If $F \in C^\infty((\mathbb{R}^+ - \{0\}) \times \mathbb{R}^3)$ is homogeneous of degree 0 and if $f \in D_\infty$, then there exists a unique sequence of functions $g_l \in D_\infty$, $l \geq 0$, such that for every $n \geq 0$, $j \geq 0$ and $L \geq 0$*

$$\|\big(\frac{d}{dt}\big)^j \big(e^{-i\varepsilon\omega(-i\partial)t}F(t)e^{i\varepsilon\omega(-i\partial)t}f - \sum_{0\leq l\leq n} t^{-l}g_l\big)\|_{D_L} \tag{A.35}$$
$$\leq C_{n+L+j}\; t^{-n-1-j}\Big(\sum_{\substack{X\in\Pi'\cap U(\mathbb{R}^4)\\ |X|\leq L+j}} \|\big(\xi_X F(t)\|_{L^\infty(\mathbb{R}^3)} + \sum_{\substack{Y\in Q'\\ |Y|\leq N}} \|(\xi_Y F)(1,\cdot)\|_{L^\infty(B)}\Big)\|f\|_{D_N},$$

$t \geq 1$, where B is the unit ball in $\mathbb{R}^3$, N depends on n, j and L. Moreover

$$\hat{g}_0(k) = F(1, -\varepsilon k/\omega(k))\hat{f}(k) \tag{A.36}$$

and g_l is a bilinear function of F and f satisfying

$$\|g_l\|_{D_i} \leq C_{i,l} \sum_{\substack{Y\in Q'\\ |Y|\leq M}} \|(\xi_Y F)(1,\cdot)\|_{L^\infty(B)}\|f\|_{D_M}, \quad l, i \geq 0, \tag{A.37}$$

where M is an integer depending on l and i.

Before proving the theorem we remark that formula (A.36) can be generalized to give explicit expressions for g_n, $n \geq 0$. As a matter of fact

$$\hat{g}_n(k) = \frac{i^n}{n!}\sum_j \frac{\partial}{\partial k_{j_1}} \cdots \frac{\partial}{\partial k_{j_n}}\big((\partial_{j_1}\cdots\partial_{j_n}F)(1, -\varepsilon k/\omega(k))\hat{f}(k)\big). \tag{A.38}$$

Since we shall not use this result for $n \geq 1$, we shall only prove the particular case $n = 0$ in (A.36).

Proof. To prove the the uniqueness of the sequence g_l, $l \geq 0$, let g'_l, $l \geq 0$ be another sequence satisfying (A.35) and let $n \in \mathbb{N}$ be such that $g_l = g'_l$ for $0 \leq l \leq n-1$. Then

$$\begin{aligned}
\|g_n - g'_n\|_{D_L} &\leq t^n \|\sum_{0\leq l\leq n} t^{-l}(g_l - g'_l)\|_{D_L} \\
&\leq t^n\Big(\|e^{-i\varepsilon\omega(-i\partial)t}F(t)e^{i\varepsilon\omega(-i\partial)t}f - \sum_{0\leq l\leq n} t^{-l}g_l\|_{D_L} \\
&\quad + \|e^{-i\varepsilon\omega(-i\partial)t}F(t)e^{i\varepsilon\omega(-i\partial)t}f - \sum_{0\leq l\leq n} t^{-l}g'_l\|_{D_L}\Big) \\
&\leq t^{-1}2C_{N,L}\Big(\|F(1,\cdot)\|_{L^\infty(\mathbb{R}^3)} + \sum_{Y\in D(N)} \|(\xi_Y F)(1,\cdot)\|_{L^\infty(B)}\Big)\|f\|_{D_N}.
\end{aligned}$$

Taking the limit $t \to \infty$ now proves that $g_n = g'_n$, $n \geq 0$. Hence by induction $g_l = g'_l$ for $l \geq 0$.

To construct the sequence g_n, $n \geq 0$, we define $r \in C^\infty((\mathbb{R}^+ - \{0\}) \times \mathbb{R}^3)$, homogeneous of degree $-3/2$ by formula (A.1a),

$$r_0(t,x) = e^{i3\varepsilon\pi/4}\big(m/t\big)^{3/2}\big(t/\rho(t,x)\big)^{5/2}\hat{f}\big(-\varepsilon mx/\rho(t,x)\big), \tag{A.39a}$$

we define the sequence $r_n \in C^\infty((\mathbb{R}^+ - \{0\}) \times \mathbb{R}^3)$, $n \geq 0$, of homogeneous functions of degree $-n-3/2$ by formula (A.1b), i.e.

$$r_n = \frac{1}{n!}\Big(\frac{\rho\Box}{2i\varepsilon m}\Big)^n r_0, \quad n \geq 0. \tag{A.39b}$$

It now follows from Theorem A.1 and Leibniz rule that if $X \in \Pi' \cap U(\mathbb{R}^4)$, then

$$\begin{aligned}
&\|(1+\lambda(t))^{k/2}\xi_X\Big(F(t)e^{i\varepsilon\omega(-i\partial)t}f - e^{im\varepsilon\rho(t)}F(t)\sum_{0\leq l\leq K} r_l(t)\Big)\|_{L^2} \\
&\leq C_{|X|}\sum_{|X_1|+|X_2|=|X|}\|(\xi_{X_1}F)(t)\xi_{X_2}\Big(e^{i\varepsilon\omega(-i\partial)t}f - e^{im\varepsilon\rho(t)}\sum_{0\leq l\leq K} r_l(t)\Big)\|_{L^2} \\
&\leq C_{K,|X|,k}\Big(\sum_{|X_1|\leq|X|}\|\xi_{X_1}F(t)\|_{L^\infty}\Big)\|f\|_{D_{N'}}t^{-K-1}, \quad t \geq 1,
\end{aligned} \tag{A.40}$$

where $X_1, X_2 \in \Pi' \cap U(\mathbb{R}^4)$ in the summation domains, where N' depends on K, X, k and where

$$\|\rho(t)^{-j}\xi_X r_l(t)\|_{L^2} \leq C_{l,|X|,j}\|f\|_{D_{N'}}t^{-(l+|X|+j)}, \quad t > 0. \tag{A.41}$$

The support of r_l is contained in the light cone. Since the function $(t,x) \mapsto F(t,x)r_l(t,x)$ is homogeneous of degree $-l-3/2$, we can apply Theorem A.2 to the function $(t,x) \mapsto t^l F(t,x)r_l(t,x) = q_l(t,x)$, which shows that there are functions $g_{l,j} \in D_\infty$, $l \geq 0$, $j \geq 0$, given by the construction (A.8) satisfying

$$\begin{aligned}
&\|(1+\lambda(t))^{k/2}\xi_X\Big(e^{im\varepsilon\rho(t)}q_l(t,x) - \sum_{0\leq j\leq L} t^{-j}e^{im\varepsilon\rho(t)}g_{l,j}\Big)\|_{L^2} \\
&\qquad \leq C_{L,|X|,k}\sum_{\substack{Y\in Q' \\ s+|Y|\leq|Z|\leq N'}}\|(m/\rho(1,\cdot))^s(\xi_Y g_l)(1,\cdot)\|_{L^2}\, t^{-L-1}, \quad t \geq 1,
\end{aligned}$$

and

$$\|g_{l,j}\|_{D_i} \leq C_{j,i}\sum_{\substack{Y\in Q' \\ s+|Y|\leq i+2j}}\|(m/\rho(1,\cdot))^{s+j}(\xi_Y q_l)(1,\cdot)\|_{L^2}.$$

These two inequalities, the definition of q_l and inequality (A.41) give

$$\begin{aligned}
&\|(1+\lambda(t))^{k/2}\xi_X\Big(e^{im\varepsilon\rho(t)}F(t)r_l(t) - \sum_{0\leq j\leq L} t^{-j-l}e^{i\varepsilon\omega(-i\partial)t}g_{l,j}\Big)\|_{L^2} \\
&\qquad \leq C_{L,|X|,k}\sum_{\substack{Y\in Q' \\ |Y|\leq N''}}\|(\xi_Y F)(1,\cdot)\|_{L^\infty(B)}\|f\|_{D_{N''}}t^{-L-1-l}, \quad t > 0,
\end{aligned} \tag{A.42a}$$

and

$$\|g_{l,j}\|_{D_i} \leq C_{l,j,i} \sum_{\substack{Y \in Q' \\ |Y| \leq M}} \|(\xi_Y F)(1,\cdot)\|_{L^\infty(B)} \|f\|_{D_M}, \tag{A.42b}$$

where N'' depends on L, $|X|$, k and M on l, j, i. We have used repeatedly here that

$$\begin{aligned} &\xi_{M_{\mu\nu}} t^l F(t,x) r_l(t,x) \\ &\quad = (\xi_{M_{\mu\nu}} F)(t,x) t^l r_l(t,x) + F(t,x)[\xi_{M_{\mu\nu}}, t^l] r_l(t,x) + F(t,x) t^l \xi_{M_{\mu\nu}} r(t,x) \end{aligned}$$

and that the commutator of a monomial of degree l in (t,x) with $\xi_{M_{\mu\nu}}$ is a monomial of degree l.

We now define

$$g_n = \sum_{l+j=n} g_{l,j}, \quad n \geq 0, \tag{A.43}$$

which, according to (A.42b), proves the inequality (A.37) and which, together with (A.40), (A.42a) and (A.42b) gives, choosing L and K sufficiently large,

$$\begin{aligned} &\|(1+\lambda(t))^{k/2} \xi_X \Big(F(t) e^{i\varepsilon\omega(-i\partial)t} f - \sum_{0 \leq l \leq n} t^{-l} e^{i\varepsilon\omega(-i\partial)t} g_l \Big)\|_{L^2} \\ &\quad \leq C_{n,|X|,k} \Big(\sum_{\substack{X_1 \in \Pi' \cap U(\mathbb{R}^4) \\ |X_1| \leq |X|}} \|\xi_{X_1} F(t)\|_{L^\infty(\mathbb{R}^3)} + \sum_{\substack{Y \in Q' \\ |Y| \leq N}} \|(\xi_Y F)(1,\cdot)\|_{L^\infty(B)} \Big) \|f\|_{D_N} t^{-n-1}, \end{aligned} \tag{A.44}$$

$t \geq 1$, where N depends on n, $|X|$, k. If

$$h_n(t) = e^{-i\varepsilon\omega(-i\partial)t} F(t) e^{i\varepsilon\omega(-i\partial)t} - \sum_{0 \leq l \leq n} t^{-l} g_l,$$

then

$$\|\big(\frac{d}{dt}\big)^l h_n(t)\|_{D_L} \leq \sum_{\substack{|\alpha| \leq L \\ |\beta| \leq L}} \|x^\beta \partial^\alpha \big(\frac{d}{dt}\big)^l h_n(t)\|_{L^2},$$

according to Theorem 2.9. Since $(i\varepsilon x_j \omega(-i\partial) - t\partial_j) e^{-i\varepsilon\omega(-i\partial)t} = e^{-i\varepsilon\omega(-i\partial)t} x_j$, it follows that

$$\|\big(\frac{d}{dt}\big)^l h_n(t)\|_{D_L} \leq C_{L,l} \sum_{\substack{s+|\beta| \leq L \\ j+|\alpha| \leq L+l}} t^s \|x^\beta \partial^\alpha \big(\frac{d}{dt}\big)^j e^{i\varepsilon\omega(-i\partial)t} h_n(t)\|_{L^2}.$$

This shows together with (A.44) that

$$\begin{aligned} &\|\big(\frac{d}{dt}\big)^l h_n(t)\|_{D_L} \\ &\quad \leq C_{n,L,l} \Big(\sum_{\substack{X \in \Pi' \cap U(\mathbb{R}^4) \\ |X| \leq L+l}} \|\xi_X F(t)\|_{L^\infty(\mathbb{R}^3)} + \sum_{\substack{Y \in Q' \\ |Y| \leq N}} \|(\xi_Y F)(1,\cdot)\|_{L^\infty(B)} \Big) \|f\|_{D_N} t^{-n-1+L}, \end{aligned} \tag{A.45}$$

$t \geq 1, n \geq L$, where N is redefined. Let $n' = n + l + L$. Then, it follows from (A.37) and (A.45) that

$$\begin{aligned} &\|(\frac{d}{dt})^l h_n(t)\|_{D_L} \\ &\leq \|(\frac{d}{dt})^l h_{n'}(t)\|_{D_L} + C_l \sum_{0 \leq j \leq L-1} t^{-(n-1+l+j)} \|g_{n+1+j}\|_{D_L} \\ &\leq C_{n,L,l}\Big(\sum_{\substack{X \in \Pi' \cap U(\mathbb{R}^4) \\ |X| \leq L+l}} \|\xi_X F(1)\|_{L^\infty(\mathbb{R}^3)} + \sum_{\substack{Y \in Q' \\ |Y| \leq N}} \|(\xi_Y F)(1, \cdot)\|_{L^\infty(B)}\Big) \|f\|_{D_N} t^{-n-1-l}, \end{aligned} \tag{A.46}$$

$t \geq 1$, $L, l \geq 0$, where we have redefined N. This proves (A.35).

To prove (A.36) we note that $g_0 = g_{0,0}$ according to (A.43) and that $\hat{g}_{0,0}$ is obtained from q_0 by formula (A.8a)

$$\hat{g}_{0,0}(k) = e^{-i\varepsilon\pi/4}(m^2/\omega(k)^5)^{1/2} q_0(1, -\varepsilon k/\omega(k)).$$

Since $q_0(t,x) = F(t,x) r_0(t,x)$ formula (A.39a) now gives

$$\hat{g}_0(k) = F(1, -\varepsilon k/\omega(k)) \hat{f}(k),$$

which proves the theorem.

Proof of Theorem A.1 We recall that

$$\varphi_0(t) = e^{i\varepsilon\omega(-i\partial)t} f, \tag{A.47a}$$

and

$$\varphi_n(t) = \varphi_0(t) - e^{i\varepsilon m \rho(t)} \sum_{0 \leq l \leq n-1} g_l(t), \quad n \geq 1. \tag{A.47b}$$

The construction of g_l by (A.1a) and (A.1b) then gives that

$$(\Box + m^2)\varphi_0 = 0, \quad (\Box + m^2)\varphi_{n+1} = -e^{i\varepsilon m\rho} \Box g_n, \quad n \geq 0. \tag{A.48}$$

Let Γ^A, $A \in \{1, \ldots, 16\}$ be the matrices I, γ^μ, $0 \leq \mu \leq 3$, $\gamma^\mu\gamma^\nu$, $0 \leq \mu < \nu \leq 3$, $\gamma^\mu\gamma^\nu\gamma^\tau$, $0 \leq \mu < \nu < \tau \leq 3$, $\gamma^0\gamma^1\gamma^2\gamma^3$, in a given order. The set $\{\Gamma^A \big| 1 \leq A \leq 16\}$ is then a basis of the complex vector space of 4×4 complex matrices, and Γ^A is invertible for $1 \leq A \leq 16$. If $a_\mu \in \mathbb{C}$ for $0 \leq \mu \leq 3$ and $b \in \mathbb{C}$, and if

$$M = a_0 I - \sum_{1 \leq j \leq 3} a_j \gamma^0 \gamma^j + ib\gamma^0, \tag{A.49a}$$

then there exist complex numbers $c_\mu(A)$ and $d(A)$, $0 \leq \mu \leq 3$, $1 \leq A \leq 16$, independent of a_μ and b, such that

$$a_\mu I = \gamma_\mu \gamma^0 \sum_A c_\mu(A) (\Gamma^A)^{-1} M \Gamma^A, \tag{A.49b}$$

(where there is no summation over μ on the right-hand side) and

$$bI = \gamma^0 \sum_A d(A)(\Gamma^A)^{-1} M \Gamma^A. \tag{A.49c}$$

As a matter of fact, conjugation of M with $\gamma^1\gamma^2\gamma^3$ (resp. $\gamma^0\gamma^1\gamma^2\gamma^3$, $\gamma^0\gamma^i\gamma^j$) corresponds to replace (a_1, a_2, a_3, b) (resp. b, a_k) by $(-a_1, -a_2, -a_3, -b)$ (resp. $-b, -a_k$), in (A.49a), where (i, j, k) is a permutation of $(1, 2, 3)$. Let

$$h_n^{(A)} = \big(\frac{\partial}{\partial t} + \mathcal{D}\big)\Gamma^A \varphi_n, \tag{A.50a}$$

where $\mathcal{D}$ is as in equation (1.2b), and let

$$r_0^{(A)} = 0, \quad r_n^{(A)} = -i\gamma^0 \Gamma^A e^{i\varepsilon m\rho} \Box g_{n-1}, \quad n \geq 1. \tag{A.50b}$$

Using that $\Box = (d/dt - \mathcal{D})(d/dt + \mathcal{D})$, it then follows from (A.48) that

$$(i\gamma^\mu \partial_\mu + m) h_n^{(A)} = r_n^{(A)}, \quad n \geq 0, 1 \leq A \leq 16. \tag{A.51}$$

The representation ξ of the Poincaré Lie algebra $\mathfrak{p}$ satisfies

$$\xi_{P_0} = t\rho^{-2}\Big(L - \sum_{j=1}^{3} (x_j/t)\xi_{M_{0j}}\Big) \tag{A.52a}$$

$$\xi_{P_i} = t^{-1}\xi_{M_{0i}} - x_i \rho^{-2}\Big(L - \sum_{j=1}^{3} (x_j/t)\xi_{M_{0j}}\Big), \quad 1 \leq i \leq 3, \tag{A.52b}$$

where $\rho^2 = t^2 - |x|^2 \neq 0$,

$$L = t\frac{\partial}{\partial t} + \sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i}. \tag{A.52c}$$

Let $H \in C^\infty((\mathbb{R}^+ \times \mathbb{R}^3) - \{0\})$ be a homogeneous function of degree $\chi \in \mathbb{R}$ and let $\operatorname{supp} H \subset \{y \in \mathbb{R}^+ \times \mathbb{R}^3 \big| y^\mu y_\mu \geq 0\}$. It follows from (A.52a)–(A.52c) that

$$|(\xi_{P_\mu} H)(t, x)| \leq (2 + |\chi|) t\rho^{-2} \sum_{\substack{Y \in Q' \\ |Y| \leq 1}} |(\xi_Y H)(t, x)|, \quad t - |x| > 0.$$

Using that $\mathbb{R}^4$ is an ideal of $\mathfrak{p}$, we obtain by induction that

$$|(\xi_Y H)(t, x)| \leq C_{|Y|, |\chi|} (t\rho^{-2})^{|Y|} \sum_{\substack{Z \in Q' \\ |Z| \leq |Y|}} |(\xi_Z H)(t, x)|, \tag{A.53}$$

where $Y \in \Pi' \cap U(\mathbb{R}^4)$ and $t - |x| > 0$.

The operator $\Box$ and the multiplication by ρ commute with ξ_Z when $Z \in Q'$. Since $(\rho\Box)^j \xi_X g_0$, $X \in Q'$, is homogeneous of degree $-j-3/2$, it follows from (A.53) that

$$|(\xi_{YX}(\rho\Box)^j g_0)(t,x)| \leq C_{|Y|,j+3/2}(t\rho^{-2})^{|Y|} \sum_{\substack{Z\in Q' \\ |Z|\leq|Y|}} |((\rho\Box)^j \xi_{ZX} g_0)(t,x)|, \tag{A.54}$$

where $j \geq 0$, $X \in Q'$, $Y \in \Pi' \cap U(\mathbb{R}^4)$, $t-|x|>0$. Using, as in (A.28), expression (A.27) for the operator $\Box$, we obtain from (A.54) that

$$|(\xi_{YX}(\rho\Box)^j g_0)(t,x)| \leq C_{|Y|,j}(t\rho^{-2})^{|Y|}\rho^{-j} \sum_{\substack{Z\in Q' \\ |Z|\leq|Y|+2j}} |(\xi_{ZX} g_0)(t,x)|, \tag{A.55a}$$

where $j \geq 0$, $X \in Q'$, $Y \in \Pi' \cap U(\mathbb{R}^4)$, $t-|x|>0$ and where $C_{|Y|,j}$ is a new constant. Similarly it follows that

$$|(\xi_{YX}\Box(\rho\Box)^j g_0)(t,x)| \leq C_{|Y|,j}(t\rho^{-2})^{|Y|}\rho^{-2-j} \sum_{\substack{Z\in Q' \\ |Z|\leq|Y|+2j+2}} |(\xi_{ZX} g_0)(t,x)|, \tag{A.55b}$$

where $j \geq 0$, $X \in Q'$, $Y \in \Pi' \cap U(\mathbb{R}^4)$, $t-|x|>0$. Formulas (A.55a) and (A.55b), and definition (A.1b) of g_l give that

$$|(\xi_{YX} g_l)(t,x)| \leq C_{|Y|,l}(t\rho^{-2})^{|Y|}\rho^{-l} \sum_{\substack{Z\in Q' \\ |Z|\leq|Y|+2l}} |(\xi_{ZX} g_0)(t,x)| \tag{A.56a}$$

and

$$|(\xi_{YX}\Box g_l)(t,x)| \leq C_{|Y|,l}(t\rho^{-2})^{|Y|}\rho^{-l-2} \sum_{\substack{Z\in Q' \\ |Z|\leq|Y|+2l+2}} |(\xi_{ZX} g_0)(t,x)|, \tag{A.56b}$$

where $l \in \mathbb{N}$, $X \in Q'$, $Y \in \Pi' \cap U(\mathbb{R}^4)$, $t-|x|>0$. Similarly as we obtained equality (A.20), it follows from definition (A.1a) of g_0 that

$$(\eta_Z \hat{f})(k) = e^{-3\varepsilon\pi/4} m(\omega(k))^{-5/2} t^{3/2} (\xi_Z g_0)(t, -t\varepsilon k/\omega(k)), \tag{A.57}$$

where $t>0$, $k \in \mathbb{R}^3$ and $Z \in U(\mathfrak{so}(3,1))$. After multiplication with $\omega(k)^j$ and integration, we obtain using the variable substitution (A.2) that

$$\|\omega^j \eta_Z \hat{f}\|^2_{L^2} = \|(mt/\rho(t))^j (\xi_Z g_0)(t)\|^2_{L^2}, \tag{A.58}$$

where $t>0$ and $Z \in U(\mathfrak{so}(3,1))$. Inequalities (A.56a) and (A.56b), and inequality (A.58) give that

$$\|\rho(t)^{-j}(\xi_{XY} g_l)(t)\|_{L^2} \leq C_{|X|+j+l}\, t^{-j-l-|X|} \sum_{\substack{Z\in Q' \\ |Z|\leq|X|+2l}} \|\omega^{2|X|+l+j}\eta_{ZY}\hat{f}\|_{L^2} \tag{A.59a}$$

and

$$\|\rho(t)^{-j}(\xi_{XY}\Box g_l)(t)\|_{L^2} \leq C_{|X|+j+l}\ t^{-j-l-|X|-2} \sum_{\substack{Z\in Q' \\ |Z|\leq|X|+2l+2}} \|\omega^{2|X|+l+j+2}\eta_{ZY}\hat{f}\|_{L^2}, \tag{A.59b}$$

where $t>0$, $l,j\in\mathbb{N}$, $X\in\Pi'\cap U(\mathbb{R}^4)$ and $Y\in Q'$. The equivalence of the norms in (A.17) and inequalities (A.59a) and (A.59b) give

$$\|\rho(t)^{-j}(\xi_{XY}g_l)(t)\|_{L^2} \leq C_{|X|+j+l}\ t^{-j-l-|X|}\|f\|_{D_{3|X|+3l+|Y|+j}} \tag{A.60a}$$

and

$$\|\rho(t)^{-j}(\xi_{XY}\Box g_l)(t)\|_{L^2} \leq C_{|X|+j+l}\ t^{-j-l-|X|-2}\|f\|_{D_{3|X|+3l+|Y|+j+4}}, \tag{A.60b}$$

where $t>0$, $l,j\in\mathbb{N}$, $X\in\Pi'\cap U(\mathbb{R}^4)$ and $Y\in Q'$. Inequality (A.60a) proves inequality (A.6) of the theorem. It follows from inequality (A.56a) and equality (A.57) that

$$\|\rho(t)^{-j}(\xi_{XY}g_l)(t)\|_{L^\infty} \leq C_{|X|+j+l}\ t^{-3/2-|X|-j-l} \sum_{\substack{Z\in Q' \\ |Z|\leq|X|+2l}} \|\omega^{5/2+2|X|+j+l}\eta_{ZY}\hat{f}\|_{L^\infty}, \tag{A.61}$$

where $t>0$, $l,j\in\mathbb{N}$, $X\in\Pi'\cap U(\mathbb{R}^4)$ and $Y\in Q'$. If $h\in S(\mathbb{R}^3,\mathbb{C}^4)$, then $\|\hat{h}\|_{L^\infty}\leq \|h\|_{L^1}\leq C\|h\|_{D_2}\leq C'\|\hat{h}\|_{D_2}$, so it follows from (A.61) and from the equivalence of norms in (A.17) that

$$\|\rho(t)^{-j}(\xi_{XY}g_l)(t)\|_{L^\infty} \leq C_{|X|+j+l}\ t^{-3/2-|X|-j-l}\|f\|_{D_{3|X|+3l+|Y|+j+5}}, \tag{A.62}$$

where $t>0$, $l,j\in\mathbb{N}$, $X\in\Pi'\cap U(\mathbb{R}^4)$ and $Y\in Q'$. This proves inequality (A.7) of the theorem.

It follows by induction that there exist polynomials $R^{(k)}_{X,Y}$ of degree k such that

$$e^{-i\varepsilon m\rho}\xi_X e^{i\varepsilon m\rho} = \xi_X + \sum_{\substack{Y\in\Pi'\cap U(\mathbb{R}^4) \\ |Y|\leq|X|-1}} \sum_{0\leq i\leq|X|-|Y|-1} \rho^{-i}R_{X,Y}^{(|X|-|Y|+i)}(y/\rho)\xi_Y, \tag{A.63}$$

for $X\in\Pi'\cap U(\mathbb{R}^4)$, $y\in\mathbb{R}^+\times\mathbb{R}^3$, $y^\mu y_\mu>0$. This equality and inequalities (A.60a) and (A.60b) give that

$$\|(\rho(t)^{-j}\xi_{XY}e^{i\varepsilon m\rho}g_l)(t)\|_{L^2} \leq C_{|X|+j+l}\ t^{-j-l}(1+t^{-|X|})\|f\|_{D_{3|X|+3l+|Y|+j}}, \tag{A.64a}$$

and

$$\|(\rho(t)^{-j}\xi_{XY}e^{im\rho}\Box g_l)(t)\|_{L^2} \leq C_{|X|+j+l}\ t^{-j-l-2}(1+t^{-|X|})\|f\|_{D_{3|X|+3l+|Y|+j+4}}, \tag{A.64b}$$

where $t>0$, $l,j\in\mathbb{N}$, $X\in\Pi'\cap U(\mathbb{R}^4)$ and $Y\in Q'$.

Since $\|(\xi^D_X h)(t)\|_D\le\|(\xi_X h)(t)\|_D+C_X\|h(t)\|_D$, for $X\in\mathfrak{p}$, it follows that

$$\begin{aligned}\wp^D_n(h(t))&\le C_n\Big(\sum_{\substack{Y\in\Pi'\\|Y|\le n}}\|(\xi_Y h)(t)\|^2_D\Big)^{1/2}\\&\le C'_n\wp^D_n(h(t)),\quad n\ge 0,\end{aligned}\tag{A.65}$$

for some constants C_n and C'_n. If λ_1 is defined as in Theorem 5.5, then $(\lambda_1(t))(x)=(1+t)(1+t-|x|)^{-1}$ for $0\le|x|<t$, so $(\lambda_1(t))(x)\le Ct^2\rho^{-2}$ for $0\le|x|<t$. Therefore it follows from (A.50b), (A.64b) and (A.65) that

$$\wp^D_j\big((1+\lambda_1(t))^{k/2}r^{(A)}_n(t)\big)\le nC_{j+k+n}\;t^{-n-1}(1+t^{-j})\|f\|_{D_{3j+3n+k+1}},\tag{A.66}$$

for $t>0$, $j,k,n\in\mathbb{N}$.

Let $Y\in U(\mathfrak{p})$. It then follows from definition (A.47b) of φ_n and from theorem 1 of [12] (and [13] for the details) that there exists an integer $N\ge 1$ such that $\|\delta(t)^2(\xi_Y\varphi_N)(t)\|_{L^\infty}\le C$ for $t\ge 1$, where $C\in\mathbb{R}^+$ is independent of t and where $(\delta(t)(x)=1+t+|x|$. Therefore $\lim_{t\to\infty}\|(\xi_Y\varphi_N)(t)\|_{L^2}=0$. It then follows from (A.47b) and (A.64a) with $j=0$ that $\lim_{t\to\infty}\|(\xi_Y\varphi_n)(t)\|_{L^2}=0$ for $Y\in U(\mathfrak{p})$ and $n\ge 1$. Definition (A.50a) and inequality (A.65) now give that

$$\lim_{t\to\infty}\wp^D_j(h^{(A)}_n(t))=0\quad\text{for } j\ge 0, n\ge 1.\tag{A.67}$$

It follows from equality (A.51), inequality (A.66) with $k=0$ and limit (A.67) that

$$\wp^D_j(h^{(A)}_n(t))\le C_{j+n}\;t^{-n}\|f\|_{D_{3j+3n+1}},\quad t\ge 1, j\ge 0, n\ge 1.\tag{A.68a}$$

We also note that, since $r_0=0$, definition (A.50a) give that

$$\wp^D_j(h^{(A)}_0(t))=\|h^{(A)}_0(0)\|_{D_j}\le C\Big(\|f\|_{D_j}+\sum_{1\le i\le 3}\|\partial_j f\|_{D_j}\Big),\quad j\ge 0.\tag{A.68b}$$

Theorem 5.5, with $G=0$, $g=0$, and inequality (A.68b) give that

$$\wp^D_j\big((1+\lambda_1(t))^{k/2}h^{(A)}_0(t)\big)\le C_{j+k}\Big(\|f\|_{D_{j+k}}+\sum_{1\le i\le 3}\|\partial_j f\|_{D_{j+k}}\Big),\tag{A.69}$$

fot $t\ge 0$ and $j,k\in\mathbb{N}$. Since $(1+\lambda_1(t))(x)\le C(1+t)$ in the support of $r^{(A)}_n$, it follows from Theorem 5.5, with $G=0$, that

$$\begin{aligned}&\wp^D_j\big((1+\lambda_1(t))^{k/2}h^{(A)}_n(t)\big)\\&\quad\le C_{j+k}\Big(\wp^D_{j+k}(h^{(A)}_n(t))+t\sum_{0\le i\le k-1}\wp^D_{j+i}\big((1+\lambda_1(t))^{(k-1-i)/2}r^{(A)}_n(t)\big)\Big),\quad t>0,\end{aligned}$$

Inequalities (A.66) and (A.68a) then give that

$$\wp_j^D\big((1+\lambda_1(t))^{k/2}h_n^{(A)}(t)\big) \leq C_{j+k+n}\ t^{-n}\|f\|_{D_{3(j+k+n)+1}}, \tag{A.70}$$

for $t \geq 1$, $j,k \in \mathbb{N}$, $n \geq 1$.

Applying the operator ξ_Y^D, $Y \in \Pi'$, on both sides of equation (A.51), it follows from Theorem 5.7 and inequality (A.68b) that

$$\|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2}(\xi_Y^D h_0^{(A)})(t)\|_{L^\infty} \leq C\Big(\|f\|_{D_{|Y|+k+8}} + \sum_{1\leq i\leq 3}\|\partial_j f\|_{D_{|Y|+k+8}}\Big), \tag{A.71}$$

for $t \geq 0$, $k \in \mathbb{N}$ and $Y \in \Pi'$. Similarly, it follows from Theorem 5.7 and inequalities (A.66) and (A.68a) that

$$\|(\delta(t))^{3/2}(1+\lambda_1(t))^{k/2}(\xi_Y^D h_n^{(A)})(t)\|_{L^\infty} \leq C_{k+|Y|+n}\ t^{-n}\|f\|_{D_{3(k+|Y|+n)+25}}, \tag{A.72}$$

for $t \geq 1$, $k,n \in \mathbb{N}$, $n \geq 1$ and $Y \in \Pi'$.

According to equalities (A.49b), (A.49c) and (A.50a), $\partial_\mu\varphi_n$, $0 \leq \mu \leq 3$, and $m\varphi_n$ are linear combinations of $h_n^{(A)}$, $1 \leq A \leq 16$. This fact and inequalities (A.70) and (A.72), prove inequalities (A.4) and (A.5). This proves the theorem.

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Contents